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Relative Error Issues in Sampled Data Models

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Abstract: Most real world systems operate in continuous time. However, to store, analyze or transmit data from such systems the signals must first be sampled. Consequently there has been on-going interest in sampled data models for continuous time systems. The emphasis in the literature to-date has been on three main issues namely the impact of folding, sampled zero dynamics and the associated model error quantification. Existing error analyses have almost exclusively focused on unnormalized performance. However, in many applications relative errors are more important. For example, high performance controllers tend to invert the system dynamics and consequently relative errors underpin closed loop performance issues including robustness and stability. This motivates us to examine the relative errors associated with several common sampled data model types. This analysis reveals that the inclusion of appropriate zero dynamics is essential to ensure that the relative error converges to zero as the sampling period is reduced.

1. INTRODUCTION

Real systems are usually modeled by using laws of physics and are consequently described by ordinary differential equation models. However, we typically interact with these systems by digital devices that utilize sampled data. Thus, the issue of sampling and sampled-data models has attracted on-going interest. There is extensive literature on sampling and sampled-data models for linear systems – see, for example, Feuer and Goodwin [1996], Kuo [1992], Franklin and Powel [1990], Middleton and Goodwin [1990], Åström and Wittenmark [1997]. Moreover, when dealing with sampled signals, a well known consequence of the sampling process can be characterized in terms of aliasing of the continuous-time spectrum [Oppenheim and Schafer, 1999].

When dealing with sampled-data models, it is known that extra zeros appear in the discrete-time transfer function which have no counterpart in the underlying continuous-time system. Åström et al. [1984] showed that these zeros can be asymptotically characterized, as the sampling period goes to zero, in terms of precise polynomials which depend only on the system relative degree. These polynomials are, in fact, the Euler-Fröbenius polynomials [Weller et al., 2001]. Moreover, the asymptotic location of the zeros depends on the hold device used to generate the continuous-time input signal to the system [Hagiwara et al., 1993, Yuz et al., 2004]

Similar results hold for stochastic systems. In this case, extra zeros appear in the sampled output spectrum [Wahlberg, 1988]. These *stochastic* sampling zeros can also been asymptotically characterized in terms of the Euler-Fröbenius polynomials. The precise asymptotic location of the extra zeros in the stochastic case depends on the prefilter used prior to sampling the output [Wahlberg, 1988, Yuz and Goodwin, 2005a].

The presence of *sampling zeros* has also been described in sampled-data models for nonlinear systems. In this case, extra *zero dynamics* due to the sampling process have been characterized by Monaco and Normand-Cyrot [1988]. Also, simple sampled-data models have been proposed that have the same zeros as in the asymptotic linear case [Yuz and Goodwin, 2005b]. Sampled-data models for stochastic nonlinear systems have also been considered in Goodwin et al. [2007], Yuz and Goodwin [2006], Yuz [2005].

It is well known that the sampling zeros that appear in a sampled-data model correspond to *artifacts* of the sampling process. In fact, when expressing the discrete-time models in the δ-operator framework [Middleton and Goodwin, 1990], these zeros go asymptotically to infinity when the sampling period is reduced. This becomes apparent in the frequency domain, where it is observed that the sampling zeros play a role only near the Nyquist frequency $\omega_n = \frac{\pi}{h}$, where h is the sampling period.

Further insights into the nature of sampling zeros for linear systems has been provided by Blachuta [1999a]. In that work a closed form expression has been given for the principal term of the Taylor expansion of the difference between the true zeros and their asymptotic values as a function of the sample period.

It is interesting to note that essentially all of the existing literature on sampled data models deals with absolute errors. However, in practice, one is often more interested in *relative* errors. For example, it is typically more useful to know that a model has accuracy 1% as opposed to knowing that the absolute error is 0.1 which leaves open the question of whether the true value is 1 (corresponding to 10% error) or 0.01 (corresponding to a 1000% error). The only previous paper we are aware of that has briefly mentioned relative errors in the context of sampling is [Blachuta, 1999b]. However, no analysis is provided in that paper of the impact of different model types.

Of course, in the linear case, one can always calculate the sampled-data model to any desired degree of accuracy. However, our goal here is to study the degree of model approximation required to achieve certain relative error convergence properties as the sampling period is reduced. This has the advantage of giving insight into different simple models and is in the same spirit as the work by Åström et al. [1984], Hagiwara et al. [1993], Wahlberg [1988], Blachuta [1999a] and our own earlier work on nonlinear systems [Yuz and Goodwin, 2005b, 2006, Goodwin et al., 2007]. Of particular interest are the so called *sampling zeros* since they can have a profound effect in the estimation and control applications. The location of these zeros are highlighted in the approximated models, but their presence is blurred in the exact representation. These are also potential benefits for nonlinear systems where exact models are in general unobtainable.

2. SAMPLED-DATA MODELS FOR LINEAR SYSTEMS

In this paper we focus on sampled-data models for linear deterministic systems, although it seems reasonable that corresponding results hold in the stochastic case.

Let us consider a general continuous-time system described by the transfer function:

$$G(s) = \frac{F(s)}{E(s)} = \frac{f_m s^m + \ldots + f_0}{s^n + e_{n-1}s^{n-1} + \ldots + e_0},\ f_m \neq 0 \quad (1)$$

The system is of order n, and relative degree $r = n-m > 0$. Such system can be equivalently expressed in state-space form as:

$$\dot{x}(t) = A\,x(t) + B\,u(t) \quad (2)$$
$$y(t) = C\,x(t) \quad (3)$$

Exact discrete-time representation of the system can be obtained under appropriate assumptions. Usually we assume that the input signal is generated by a zero-order hold (ZOH), i.e.

$$u(t) = u_k \qquad ; t \in [kh, kh+h[\quad (4)$$

and that the output is sampled at the same uniform distributed instants, i.e,

$$y_k = y(kh) \quad (5)$$

where h is the sampling period.

The input sequence u_k in (4) and the output sequence y_k are related by the following discrete-time model [Åström and Wittenmark, 1997, Middleton and Goodwin, 1990]:

$$G_q(z) = \frac{F_q(z)}{E_q(z)} = \frac{\bar{f}_{n-1}z^{n-1} + \ldots + \bar{f}_0}{z^n + \bar{e}_{n-1}z^{n-1} + \ldots + \bar{e}_0} \quad (6)$$

In state space form, a direct link can be established between the continuous-time model matrices in (2)–(3) and those of the following sampled data model:

$$q\,x_k = x_{k+1} = A_q\,x_k + B_q\,u_k \quad (7)$$
$$y_k = C\,x_k \quad (8)$$

where q is the (forward) shift operator. The matrices in (7), (8) are given by:

$$A_q = e^{Ah} \qquad B_q = \int_0^h e^{A\eta}\,B\,d\eta \quad (9)$$

An important observation is that the sampled-data system (6) will have, in general, relative degree 1. This means that there are *extra zeros* in the discrete-time model with no continuous-time counterpart. These *sampling zeros* can be precisely characterized asymptotically as the sampling period h goes to 0 [Åström et al., 1984]:

$$G_q(z) \xrightarrow{h\approx 0} \frac{h^r\,B_r(z)\,(z-1)^m}{r!\,(z-1)^n} \quad (10)$$

where $r = n-m$ is the system relative degree and $B_r(z)$ is the Euler-Fröbenius polynomial of order r [Weller et al., 2001].

The sampled-data model previously obtained can be equivalently represented in terms of the delta operator, and its associated complex variable γ [Middleton and Goodwin, 1990, Feuer and Goodwin, 1996]:

$$\delta = \frac{q-1}{h} \quad \Longleftrightarrow \quad \gamma = \frac{z-1}{h} \quad (11)$$

Use of this operator makes the sampling period h explicit, and shows that the discrete-time results converge to their continuous-time counterpart when $h \to 0$.

If we rewrite the sampled-data model (7) in terms of the δ-operator (11), we obtain:

$$\frac{x_{k+1} - x_k}{h} = \delta\,x_k = A_\delta\,x_k + B_\delta\,u_k \quad (12)$$

where the matrices will converge to their continuous-time counterpart as $h \to 0$:

$$A_\delta = \frac{A_q - I_n}{h} \xrightarrow{h\to 0} A \quad (13)$$
$$B_\delta = \frac{B_q}{h} \xrightarrow{h\to 0} B \quad (14)$$

In transfer function form, we have that (6) can be rewritten as:

$$G_\delta(\gamma) = \frac{F_\delta(\gamma)}{E_\delta(\gamma)} = \frac{\tilde{f}_{n-1}\gamma^{n-1} + \ldots + \tilde{f}_0}{\gamma^n + \tilde{e}_{n-1}\gamma^{n-1} + \ldots + \tilde{e}_0} \quad (15)$$

In this form we have

$$F_\delta(\gamma) \xrightarrow{h\to 0} F(\gamma) \quad (16)$$
$$E_\delta(\gamma) \xrightarrow{h\to 0} E(\gamma) \quad (17)$$

A more accurate model is obtained by including the asymptotic sampling zeros to give

$$F_\delta(\gamma) \xrightarrow{h\approx 0} p_r(h\gamma)\,F(\gamma) \quad (18)$$

where $p_r(h\gamma)$ is the equivalent to the Euler-Fröbenius polynomial in the γ-domain that can be expressed as (see Yuz and Goodwin [2005b], Yuz [2005] for more details):

$$p_r(h\gamma) = \det \begin{bmatrix} 1 & \frac{h}{2} & \cdots & \frac{h^{r-1}}{(r-2)!} \\ -\gamma & 1 & \cdots & \frac{h^{r-2}}{(r-3)!} \\ 0 & \ddots & \ddots & \vdots \\ 0 & \cdots & -\gamma & 1 \end{bmatrix} = \frac{B_r(1+h\gamma)}{r!} \tag{19}$$

where $B_r(\cdot)$ is the Euler-Fröbenius polynomial in (10). From (19) it is straightforward to see that

$$p_r(0) = 1 \quad \Longleftrightarrow \quad B_r(1) = r! \tag{20}$$

We see from (17), (18) that the discrete poles converge to their continuous counterparts as $h \to 0$. Also, from (18), we see that the discrete zeros split into the continuous counterparts plus $r-1$ extra zeros arising from the sampling process.

We will call the latter zeros, *sampling zeros* and the other poles and zeros, *intrinsic* poles and zeros.

Our principal interest here will be in various simplified models and their associated relative error properties. The models that we will compare are:

(1) **Exact sampled-data model** (ESD-model): This is not an approximate model, but the exact model that can be obtained for a linear deterministic system assuming a ZOH input. This model has been already given in (6), or, in state-space form, by (7)–(8). We can write the model as (see e.g. [Goodwin et al., 2001, page 339]):

$$G_q^{ESD}(z) = \mathcal{Z}\left\{\frac{1-e^{sh}}{s}G(s)\right\} \tag{21}$$

where $\mathcal{Z}\{H(s)\}$ denotes the zeta transform of the sampled impulse response of the transfer function $H(s)$. In the sequel we consider $G(s)$ given by:

$$G(s) = \frac{\prod_{i=1}^{m}(s-c_i)}{\prod_{i=1}^{n}(s-p_i)} \tag{22}$$

Notice that, without loss of generality, we have not use an extra gain K in the transfer function $G(s)$. This choice will not affect the relative error analysis in the sequel, provided we adjust the gain of the approximated models such that d.c. gain match the d.c. gain of the continuous time system.

(2) **Simple derivative replacement model** (SDR-model): Here, we simply replace derivatives by divided differences. Note that this corresponds to simple Euler integration or, equivalently, the use of a first order Taylor expansion:

$$G_q^{SDR}(z) = \frac{\prod_{i=1}^{m}(\frac{z-1}{h}-c_i)}{\prod_{i=1}^{n}(\frac{z-1}{h}-p_i)} \tag{23}$$

Note that this model does not include sampling zeros.

(3) **Asymptotic sampling zeros model** (ASZ-model): In this case we use a discrete-time transfer function with sampling zeros located at their asymptotic location, and the intrinsic poles and zeros are placed corresponding to their location given by Euler integration:

$$G_q^{ASZ}(z) = \frac{B_r(z)\prod_{i=1}^{m}(\frac{z-1}{h}-c_i)}{r!\prod_{i=1}^{n}(\frac{z-1}{h}-p_i)} \tag{24}$$

Note that by using the fact that $B_r(1) = r!$, we adjusted the d.c. gain of this model in order to match the continuous-time d.c. gain.

(4) **Corrected sampling zero model** (CSZ-model): In this case we place the sampling zero near -1 (if one exist) at the locations such that errors are of $\mathcal{O}(h^2)$ whilst other sampling zeros and the intrinsic poles and zeros are located at the values given by Euler integration and the d.c. gain is matched to that of the continuous model. There exists a sampling zero at $z=-1$ when the relative degree r is even, when we use the sampled-data model

$$G_q^{CSZ}(z) = \frac{\tilde{B}_r(z)\prod_{i=1}^{m}(\frac{z-1}{h}-c_i)}{r!\prod_{i=1}^{n}(\frac{z-1}{h}-p_i)} \tag{25}$$

where

$$\tilde{B}_r(z) = \frac{B_r(z)}{z+1}(z+1+\sigma_1 h) \tag{26}$$

For r even $B_r(z)$ has a root at -1 which is canceled in $\tilde{B}_r(z)$ which remains bounded at that point.

When r is odd, we use the previous model, i.e., $G_q^{CSZ}(z) = G_q^{ASZ}(z)$.

Of course, there are many other possibilities. For example, one could use Taylor series of order $r+1$ to approximate the intrinsic poles and zeros. It turns out that this leads to the adjusted sampling zeros used in CSZ and slightly different values for the intrinsic poles and zeros. We will restrict our analysis to the four cases listed above since this will suffice to illustrate our main claim.

Remark 1. We wish to clarify a point of confusion that exists in some areas. The use of the *delta* operator is simply a way of re-parameterizing discrete models via the transformation $q = \delta h + 1$ or $\delta = (q-1)/h$. This has the advantage of highlighting the link between discrete and continuous time domain and also achieving improved numerical properties [Goodwin et al., 1992]. Of course, any shift domain model including (ESD, SDR, ASZ, and CSZ) can be converted to delta form. This is totally different to the use of Euler integration which, by chance, happens to have the property that continuous poles and zeros appear in the same location in the corresponding delta domain discrete model.

Remark 2. To characterize the sampling zero near $z=-1$ (as used in the CSZ-model (25)) we utilize the result of [Blachuta, 1999a] who gives the following expression for this zero (when the relative degree is even):

$$\sigma^h = -1 + \frac{h}{r+1}\left\{\sum_{i=1}^{n} p_i - \sum_{i=1}^{n} z_i\right\} + \mathcal{O}(h^2) \tag{27}$$

where r is the relative degree and p_i, c_i denote the continuous poles and zeros. For example, for systems having relative degree r an even number, the zero at $z=-1$ can be expressed as:

$$(r=2) \qquad \sigma^h(h) = -1 + \frac{h}{3}\left(\sum_{i=1}^{m} c_i - \sum_{i=1}^{n} p_i\right) \tag{28}$$

$$(r=4) \qquad \sigma^h(h) = -1 + \frac{h}{5}\left(\sum_{i=1}^{m} c_i - \sum_{i=1}^{n} p_i\right) \tag{29}$$

$$(r=6) \qquad \sigma^h(h) = -1 + \frac{h}{7}\left(\sum_{i=1}^{m} c_i - \sum_{i=1}^{n} p_i\right) \tag{30}$$

3. RELATIVE ERRORS FOR APPROXIMATE SAMPLED-DATA MODELS

In this section we compare the relative errors between the various sampled data models presented in the previous section.

There are two choices for the normalizing transfer function, namely to use ESD or to use the approximate model. This leads to two relative error functions

$$R_1^i(h) = \left\| \frac{G_q^i(z) - G_q^{ESD}(z)}{G_q^{ESD}(z)} \right\|_\infty \tag{31}$$

$$R_2^i(h) = \left\| \frac{G_q^i(z) - G_q^{ESD}(z)}{G_q^i(z)} \right\|_\infty \tag{32}$$

where the super-script i refers to the model types SDR, ASZ and CSZ. Note that $R_1(h)$ has been mentioned in [Blachuta, 1999b]. The error function $R_2(h)$ is closely related to control where relative errors of this type appear in robustness analysis [Goodwin et al., 2001].

Our key result is described in the following Theorem.

Theorem 3. The relative error performance of the different discrete models are as follows:

	Relative error	r: odd	r: even
1.i)	$R_1^{SDR}(h)$	$\mathcal{O}(1)$	$\mathcal{O}(1/h)$
1.ii)	$R_1^{ASZ}(h)$	$\mathcal{O}(h)$	$\mathcal{O}(1)$
1.iii)	$R_1^{CSZ}(h)$	$\mathcal{O}(h)$	$\mathcal{O}(h)$
2.i)	$R_2^{SDR}(h)$	$\mathcal{O}(1)$	$\mathcal{O}(1)$
2.ii)	$R_2^{ASZ}(h)$	$\mathcal{O}(h)$	∞
2.iii)	$R_2^{CSZ}(h)$	$\mathcal{O}(h)$	$\mathcal{O}(h)$

Proof. The main issue in the proof arises when $G_q^{ESD}(z)$ has a zero near the unit circle. This, occurs only for continuous-time models having even relative degree, in which case there is a zero near -1. Specifically, there exists an asymptotic sampling zero at -1 for even relative degree [Åström et al., 1984, Weller et al., 2001]. Note that, in the frequency domain, $z=-1$ corresponds in fact to the Nyquist frequency, since $e^{j\pi}=-1$.

From (10), for h small, we have that:

$$G_q^{ESD}(z) = \frac{h^r\, B_r(z)}{r!\,(z-1)} + \mathcal{O}(h^{r+1}) \tag{33}$$

Thus, at the Nyquist frequency we have that

$$G_q^{ESD}(-1) = \begin{cases} \mathcal{O}(h^r) & r: \text{ odd} \\ \mathcal{O}(h^{r+1}) & r: \text{ even} \end{cases} \tag{34}$$

The SDR-model in (23) can be rewritten as:

$$\begin{aligned} G_q^{SDR}(z) &= \frac{h^r}{(z-1)^r} \frac{\prod_{i=1}^m (1-\frac{h\,c_i}{z-1})}{\prod_{i=1}^n (1-\frac{h\,p_i}{z-1})} \\ &= \frac{h^r}{(z-1)^r} + \mathcal{O}(h^{r+1}) \end{aligned} \tag{35}$$

Cases *1.i)* and *2.i)* of the theorem follows from (34) and (35).

When we include the asymptotic sampling zero we obtain the following model:

$$\begin{aligned} G_q^{ASZ}(z) &= \frac{h^r\, B_r(z)}{r!(z-1)^r} \frac{\prod_{i=1}^m (1-\frac{h\,c_i}{z-1})}{\prod_{i=1}^n (1-\frac{h\,p_i}{z-1})} \\ &= \frac{h^r\, B_r(z)}{r!\,(z-1)^r}(1+\mathcal{O}(h)) \end{aligned} \tag{36}$$

Thus, recalling that $B_r(-1)=0$ for r even, at the Nyquist frequency, (36) yields

$$G_q^{ASZ}(-1) = \begin{cases} \dfrac{h^r\, B_r(-1)}{r!\,(-2)^r} + \mathcal{O}(h^{r+1}) & r: \text{ odd} \\ 0 & r: \text{ even} \end{cases} \tag{37}$$

Cases *1.ii)* and *2.ii)* for the case of r even follow from (34) and (37).

For r odd the result follows from (33) and (36), noting that

$$G_q^{ESD}(z) - G_q^{ASZ}(z) = \mathcal{O}(h^{r+1}) \tag{38}$$

Finally, for $G_q^{CSZ}(z)$, two different models are used:

(1) When r is odd, we have that $G_q^{CSZ}(z) = G_q^{ASZ}(z)$ and, thus, the previous analysis can be applied
(2) When r is even, $B_r(-1)=0$ and we thus utilize the result in (27). We then consider the sampled-data model (25), for which, using (27), we obtain

$$\tilde{B}_r(z) = \frac{B_r(z)}{z+1}(z+1+\sigma_1 h) = B_r(z) + \mathcal{O}(h) \tag{39}$$

Thus, (25) can be written as

$$\begin{aligned} G_q^{CSZ}(z) &= \frac{h^r\,(B_r(z)+\mathcal{O}(h))}{r!(z-1)^r} \frac{\prod_{i=1}^m (1-\frac{h\,c_i}{z-1})}{\prod_{i=1}^n (1-\frac{h\,p_i}{z-1})} \\ &= \frac{h^r\, B_r(z)}{r!(z-1)^r} + \mathcal{O}(h^{r+1}) \end{aligned} \tag{40}$$

Finally, *1.iii)* and *2.iii)* follows from

$$G_q^{ESD}(z) - G_q^{CSZ}(z) = \mathcal{O}(h^{r+1}) \tag{41}$$

for r even.

4. EXAMPLES

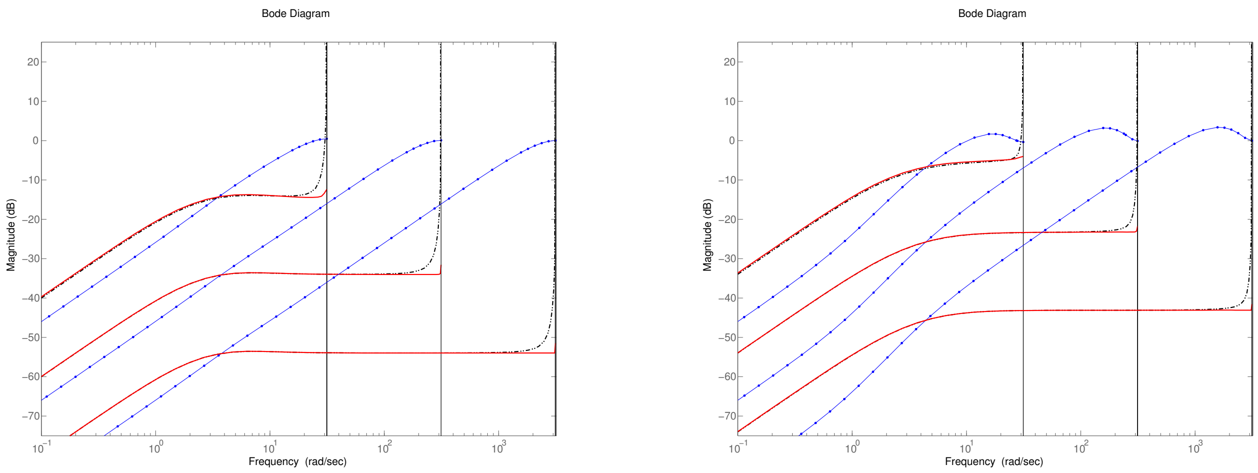
4.1 Third order system with zero

We consider the following third order system, having relative degree 2:

$$G(s) = \frac{K(s-c_1)}{(s-p_1)(s-p_2)(s-p_3)} \tag{42}$$

Under the ZOH-input assumption we discretize (42) to obtain the exact sampled data model:

$$G_q^{ESD}(z) = K\frac{b_2(h)z^2 + b_1(h)z + b_0(h)}{(z-e^{p_1 h})(z-e^{p_2 h})(z-e^{p_3 h})} \tag{43}$$



(a) Third order system with relative degree 2

(b) Fouth order system with relative degree 4.

Fig. 1. Relative error for sampled-data models SDR ('.-'), ASZ (':'), and CSZ (solid) compared to the exact model for third and fourth order systems.

where the coefficients $b_\ell(h)$ depend on the system coefficients and the sampling period.

The transfer function (43) can also be expressed in the γ-domain, corresponding to the use of operator δ, as:

$$G_\delta^{ESD}(\gamma) = K\frac{\beta_0(h)+\beta_1(h)\gamma++\beta_2(h)\gamma^2}{\left(\gamma-\frac{e^{p_1h}-1}{h}\right)\left(\gamma-\frac{e^{p_2h}-1}{h}\right)\left(\gamma-\frac{e^{p_3h}-1}{h}\right)} \tag{44}$$

We also have

$$G_q^{SDR}(z) = \frac{K\left(\frac{z-1}{h}-c_1\right)}{\left(\frac{z-1}{h}-p_1\right)\left(\frac{z-1}{h}-p_2\right)\left(\frac{z-1}{h}-p_3\right)} \tag{45}$$

This model is equivalent to directly to directly replace derivatives by the forward Euler operator in (42), and thus, in δ domain can be rewritten as:

$$G_\delta^{SDR}(\gamma) = \frac{K(\gamma-c_1)}{(\gamma-p_1)(\gamma-p_2)(\gamma-p_3)} \tag{46}$$

Also,

$$G_q^{ASZ}(z) = \frac{K(z+1)\left(\frac{z-1}{h}-c_1\right)}{2\left(\frac{z-1}{h}-p_1\right)\left(\frac{z-1}{h}-p_2\right)\left(\frac{z-1}{h}-p_3\right)} \tag{47}$$

By using the γ-variable, associated with the δ-operator, we obtain:

$$G_\delta^{ASZ}(\gamma) = \frac{K(\gamma-c_1)(1+\gamma\frac{h}{2})}{(\gamma-p_1)(\gamma-p_2)(\gamma-p_3)} \tag{48}$$

Finally, we have

$$G_q^{CSZ}(z) = \frac{N^{CSZ}(z)}{D^{CSZ}(z)} \tag{49}$$

where

$$N^{CSZ}(z) = K\left(z+1+\frac{h(-c_1+p_1+p_2+p_3)}{3}\right)\left(\frac{z-1}{h}-c_1\right) \tag{50}$$

$$D^{CSZ}(z) = \left(\frac{z-1}{h}-p_1\right)\left(\frac{z-1}{h}-p_2\right)\left(\frac{z-1}{h}-p_3\right) \times\left(2+\frac{h(-c_1+p_1+p_2+p_3)}{3}\right) \tag{51}$$

Using the γ-variable, this can be rewritten as:

$$G_\delta^{CSZ}(\gamma) = \frac{K\left(1+\gamma\frac{h}{2+\frac{h(-c_1+p_1+p_2+p_3)}{3}}\right)(\gamma-c_1)}{(\gamma-p_1)(\gamma-p_2)(\gamma-p_3)} \tag{52}$$

We compute the relative error between the exact sampled-data model (44) and the three **approximate** sampled-data models (46), (48), and (52) via $R_2^i(h)$. We choose $K=6$, $c_1=-5$, $p_1=-2$, $p_2=-3$,$p_3=-4$.

The relative errors are shown in Figure 4, for three different sampling periods: $h=0.1$, 0.01, and 0.001. From that figure, we can clearly see the relative error of Euler model (46) is of the order of 1, whereas for models that include the corrected sampling zero (models in (48) and (52)) the relative error decreases as the the sampling period decreases (a factor of 0.1 is equivalent to -20dB).

However, we notice a clear difference near the Nyquist frequency $\omega_c=\frac{\pi}{h}$, where the relative error corresponding to the asymptotic sampling zero tends to infinity.

4.2 *Fourth order system*

In this case, we consider a system having transfer function:

$$G(s) = \frac{K}{(s-p_1)(s-p_2)(s-p_3)(s-p_4)} \tag{53}$$

where $K=120$, $p_1=-2$, $p_2=-3$, $p_3=-4$ and $p_4=-5$.

The sampled-data model corresponding to this system has, in general, relative degree 1. Thus, it has 3 sampling zeros. These are asymptotically located at the roots of the Euler-Fröbenius polynomial

$$B_4(z) = z^3+11z^2+11z+1 = (z+1)(z^2+10z+1) \tag{54}$$

The sampled-data models that are similar to (44), (46), and (48) in the case of the third order system with one zero, are readily obtained.

The approximate sampled-data model similar to (49), includes the sampling zero that converges to $z = -1$, but in the form (29). Note that the other two sampling zeros are included at their asymptotic location, i.e.

$$G_q^{CSZ}(z) = \frac{N^{CSZ}(z)}{D^{CSZ}(z)} \tag{55}$$

where

$$N^{CSZ}(z) = K\left(z + 1 + \tfrac{h}{5}\sum p_\ell\right)(z^2 + 10z + 1) \tag{56}$$

$$D^{CSZ}(z) = 12\left(\tfrac{z-1}{h} - p_1\right)\left(\tfrac{z-1}{h} - p_2\right)\left(\tfrac{z-1}{h} - p_3\right) \times \left(\tfrac{z-1}{h} - p_4\right)\left(2 + \tfrac{h}{5}\sum p_\ell\right) \tag{57}$$

The relative errors are shown in Figure 4, where we see that the relative error corresponding to (55), i.e., the model with the *corrected-asymptotic* sampling zero near $z = -1$ clearly exhibits the best behavior.

4.3 Observations

We see from the above results that, surprisingly, the Euler model gives the smallest relative errors up to a frequency which is about ten times the open loop poles and zeros. Thus, provided one samples quickly but restricts the bandwidth (e.g. in control or identification) to about 10 times the location of open loop poles and zeros then one can use simple Euler models with confidence. At higher frequencies, the relative error of Euler models converges to order 1 when the relative degree is even. On the other hand, the model using asymptotic sampling zeros gives good performance up to the vicinity of the folding frequency at which time the relative error diverges to ∞. The model with corrected asymptotic sampling zero has relative errors that are or order h.

5. CONCLUSIONS

This paper has presented novel results related to relative errors associated with various simple sampled-data models. A key conclusion is that to ensure that the relative error approaches zero, one needs to include the sampling zeros adjusting the zero near $z = -1$ to a corrected value which is of order h. However, this is easily done. The zero is in fact a linear function of the continuous poles and zeros. Further work could include the extension of these results to nonlinear systems. However, this seems non-trivial in view of the dependence of the corrected sampling zeros on the continuous poles and zeros in the linear case.

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