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Robust static output feedback sliding mode control design via an artificial stabilizing delay

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Abstract: It is well known that for uncertain linear systems, a static output feedback sliding mode controller can only be determined if a particular triple associated with the reduced order dynamics in the sliding mode is stabilizable. This paper shows that the static output feedback sliding mode control design problem can be solved for a broader class of systems if a known delay term is deliberately introduced into the switching function. Effectively the reduced order sliding mode dynamics are stabilized by the introduction of this artificial delay.

Keywords: Sliding mode control, output feedback, exponential stability, discretized Lyapunov-Krasovskii functionals, stabilizing delay, Robustness.

1. INTRODUCTION

In many practical situations, all the states are not available to the controller. In some circumstances it is impossible or prohibitively expensive to measure all of the process variables. With this in mind, many authors have developed methods to control systems only using output feedback, of which one approach is the output feedback sliding mode control paradigm Edwards et al. [2003].

The idea developed in this paper is to broaden the class of systems for which a *static output feedback* based sliding mode controller can be developed based on a recent result from time delay systems. In Gu [2001], Fridman [2006], the authors show that for some systems, the presence of delay can have a stabilizing effect. This affords the possibility of taking a system which is not stabilizable by static output feedback without delay and finding a constant delay τ strictly greater than 0 such that the system is stable. In this case, a *stabilizing delay* is introduced into the dynamics to effect output feedback stability.

This design concept is not new. Several authors have considered this possibility Niculescu and Abdallah [2000], Niculescu et al. [2003], Michiels et al. [2004] and it has been shown that introducing a delay in an output feedback controller can stabilize a system which cannot be stabilized without delay. The novelty in this paper is in overcoming the output feedback stabilizability assumption of Edwards and Spurgeon [1995] in the design of sliding mode controllers by static output feedback. The authors propose a new switching function for robust control which contains an additional term which is linear in the delayed output developed in Seuret et al. [2007]. This is shown to be constructive in stabilizing the reduced order sliding mode dynamics. It is then shown that a sliding motion can be reached in finite time. Note that in this article the control law involves only a delay term and does not involve an amalgamation of delayed and non delayed terms.

The article is organized as follows. The second section presents the problem formulation. Section three formulates the definition of a new sliding function which contains an artificial delay. Section four deals with the exponential stabilization of non-delayed systems by a sliding mode controller including delay. In the last section, a numerical example demonstrates the design of the gains and the effect of the choice of the delay in the sliding mode controller.

Throughout the article, the notation $P > 0$ for $P \in \mathbb{R}^{n\times n}$ means that P is a symmetric and positive definite matrix. For the sake of simplicity, a time-varying matrix $D(t)$ will be written as D^t.The symbol I_n represents the $n \times n$ identity matrix. The notations $|.|$ and $\|.\|$ refer to the Euclidean vector norm and its induced matrix norm, respectively. For any function ϕ from $\mathcal{C}^1([-\tau;\ 0], R^n)$, we denote $|\phi|_\tau = \sup_{s\in[-\tau,\ 0]}(|\phi(s)|)$. When there is no confusion, we write $x(t)$ as x.

2. PRELIMINARIES AND PROBLEM FORMULATION

Consider the linear uncertain system :

$$\begin{aligned}\dot{x}(t) &= A^t x(t) + B^t(u(t) + \psi(t, y)) \\ y(t) &= Cx(t)\end{aligned} \tag{1}$$

where $x \in R^n$, $u \in R^m$ and $y \in R^p$ with $m < p < n$, corresponds to the state, control and output variables. The

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function $\psi \in R^m$ represents the matched disturbances. It is assumed there exists a known function Ψ_2 such that :

$$\|\psi(t,y)\| \leq \Psi_2(t,y) \tag{2}$$

The matrices $A^t \in R^{n\times n}$, $B^t \in R^{n\times m}$, $C \in R^{p\times n}$ have appropriate dimension. It is also assumed that the input and the output matrices B^t and C are full rank and it is assumed $rank(CB^t) = m$, for all $t \in R$. In addition and based on Edwards and Spurgeon [1995, 1998a], it is also assumed that there exists a change of coordinates T such that, for all $t > 0$,

$$TB^t = \begin{bmatrix} 0 \\ B_2^t \end{bmatrix}$$

Then the system has the following representation:

$$\begin{aligned} \dot{x}(t) &= \begin{bmatrix} A_{11}^t & A_{12}^t \\ A_{21}^t & A_{22}^t \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ B_2^t \end{bmatrix} (u(t) + \psi(t,y)) \\ y(t) &= \begin{bmatrix} 0 & T \end{bmatrix} x(t) \end{aligned}$$

where it is assumed that the time-varying matrices have the following form for all $k, l = 1, 2$:

$$\begin{aligned} A_{kl}^t &= A_{kl}^0 + \sum{}_{i=1}^{N} \lambda_i(t) A_{kl}^i, \\ B_2^t &= B_{20} + \sum{}_{i=1}^{N} \lambda_i(t) B_2^i \end{aligned} \tag{3}$$

where $A_{11}^0 \in R^{(n-m)\times(n-m)}$, $B_{20} \in R^{m\times m}$ is non singular and $T \in R^{p\times p}$ is an orthogonal matrix. The matrices have appropriate dimensions. It is assumed that, for all $i \in \{1,..,N\}$, the pair of matrices $(A_{kl}^0 + A_{kl}^i, B_{20})$ is controllable. The scalar functions λ_i are such that:

$$\forall i = 1,..,N,\ \lambda_i(t) \in [0,\ \ 1],\ \sum_{i=1}^{N} \lambda_i(t) = 1. \tag{4}$$

As it is possible to remove some uncertainties, the system is rewritten as:

$$\begin{aligned} \dot{x}(t) &= \begin{bmatrix} A_{11}^t & A_{12}^t \\ A_{21}^t & A_{22}^0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ B_{20} \end{bmatrix} (u(t) + \psi_0(t,y,u)) \\ y(t) &= \begin{bmatrix} 0 & T \end{bmatrix} x(t) \end{aligned} \tag{5}$$

where the matched uncertainties are represented by:

$$\psi_0(t,y,u) = B_{20}^{-1} \left(\sum_{i=1}^{N} \lambda_i(t)(A_{22}^i x_2(t) + B_2^i u(t)) \right) + \psi(t,y)$$

In Edwards and Spurgeon [1995] a sliding surface $S = \{x \in R^n : FCx(t) = 0\}$ is proposed, where $F = F_2[K\ I_m]T^T$, $K \in R^{m\times(p-m)}$ and $F_2 \in R^{m\times m}$ is a non singular matrix. The sliding motion is governed by the choice of K. If a further coordinate change is introduce based on the nonsingular transformation $z = \hat{T}x$ with $\hat{T}$ defined by

$$\hat{T} = \begin{bmatrix} I_{n-m} & 0 \\ KC_1 & I_m \end{bmatrix}$$

with

$$C_1 = [0_{(p-m)\times(n-p)}\ \ I_{(p-m)}]$$

Then as argued in Edwards and Spurgeon [1995], the dynamics of the reduced order sliding motion is governed by:

$$\dot{x}_1(t) = (A_{11}^t - A_{12}^t KC_1)x_1(t) \tag{6}$$

To show the motivation of such a study, consider initially constant parameters, ie. $\lambda_i(t) = 0$ for all $i = 1,..,N$ and $t \geq t_0$. The fictitious system $(A_{11}^0, A_{12}^0, C_1)$ is assumed to be output stabilizable i.e., there exists a matrix K such that the matrix $A_{11}^0 - A_{12}^0 KC_1$ is Hurwitz. It is shown in Edwards and Spurgeon [1995] that a necessary condition for $(A_{11}^0, A_{12}^0, C_1)$ to be stabilizable is that the invariant zeros of (A, B, C) lie in the open left half-plane. However the design of a gain K such that the matrix $A_{11}^0 - A_{12}^0 KC_1$ is Hurwitz is not always straightforward or even possible. Consider for instance the system (6) with

$$A_{11}^0 = \begin{bmatrix} 0 & -2 \\ 1 & 0.1 \end{bmatrix}, \quad A_{12}^0 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \quad C_1 = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

The output feedback stabilization problem in this case becomes one of finding a scalar k such that the matrix $\begin{bmatrix} 0 & -2+k \\ 1 & 0.1 \end{bmatrix}$ has strictly negative eigenvalues, which is clearly not possible. In this situation some authors Bag et al. [1997], Edwards and Spurgeon [1998b], Edwards et al. [2003] have employed a compensator in order to stabilize the system. However these methods increase the order of the controller and have an associated computational overhead both in terms of design and implementation. The proposed method seeks to introduce an artificial delay in the system such that the system can be stabilized by static output feedback without the need to introduce a compensator.

3. DESIGN OF A NEW SLIDING MODE SURFACE WITH DELAY

In this section, the design of a new type of sliding surface will be discussed. The objective is to define a sliding surface of the form of S but which introduces a delay in the reduced order dynamics. Consider

$$S' = \{x : FCx(t) + F_\tau Cx(t-\tau) = 0\} \tag{7}$$

where as before the matrix $F = F_2[K\ I_m]T^T$ and where $F_\tau = F_2[K_\tau\ 0_m]T^T$. Here without loss of generality, the matrix F_2 and T are chosen as identity. In this definition τ is an artificial fixed and known delay which has to be chosen to stabilize the reduced order system and represents a design parameter. The existence of such a delay and constructive methods to choose it will be discussed in a latter section. Instead of $\hat{T}$, consider the coordinate change $x \mapsto T_\tau x$:

$$\begin{aligned} \tilde{x}_1(t) &= x_1(t) \\ \tilde{x}_2(t) &= x_2(t) + KC_1 x_1(t) + K_\tau C_1 x_1(t-\tau) \end{aligned}$$

By construction the sliding function associated with S' is $s(t) = \tilde{x}_2(t)$. This leads to

$$\begin{aligned} \dot{\tilde{x}}_1(t) &= (A_{11}^t - A_{12}^t KC_1)\tilde{x}_1(t) + A_{12}^t \tilde{x}_2(t) \\ &\quad - A_{12}^t K_\tau C_1 \tilde{x}_1(t-\tau) \\ \dot{\tilde{x}}_2(t) &= (A_{21}^t + KC_1 A_{11}^t)\tilde{x}_1(t) \\ &\quad + K_\tau C_1 A_{11}^{t-\tau} \tilde{x}_1(t-\tau) + (A_{22}^0 + KC_1 A_{12}^t)\tilde{x}_2(t) \\ &\quad + K_\tau C_1 A_{12}^{t-\tau} \tilde{x}_2(t-\tau) + B_{20}(u(t) + \psi_0(t,y,u)) \\ &\quad - (A_{22}^0 + KC_1 A_{12}^t) KC_1 \tilde{x}_1(t) \\ &\quad - K_\tau C_1 A_{12}^{t-\tau} K_\tau C_1 \tilde{x}_1(t-2\tau) - (KC_1 A_{12}^t K_\tau \\ &\quad + A_{22}^0 K_\tau + K_\tau C_1 A_{12}^{t-\tau} K) C_1 \tilde{x}_1(t-\tau) \end{aligned} \tag{8}$$

Remark 1. Note that the system (8) is a particular delay system. Since the delay τ is artificially introduced, it is known and can be chosen to improve the stability of the closed-loop system.

Remark 2. Note that the delayed sliding motion produces a more complicated system since the resulting equation contains several delayed terms and two different delays, τ and 2τ.

Note that the last four lines of the previous equation only depend on the output information and thus the following output feedback control law can be defined:

$$\begin{aligned} u(t) = -(B_{20})^{-1}\{&(A_{22}^0 + KC_1A_{12}^0)\tilde{x}_2(t) \\ &+K_\tau C_1A_{12}^0\tilde{x}_2(t-\tau) + \nu - (A_{22}^0 \\ &+KC_1A_{12}^0)KC_1\tilde{x}_1(t) - (KC_1A_{12}^0K_\tau \\ &+A_{22}^0K_\tau + K_\tau C_1A_{12}^0K)C_1\tilde{x}_1(t-\tau) \\ &-K_\tau C_1A_{12}^0K_\tau C_1\tilde{x}_1(t-2\tau) - G_l\tilde{x}_2(t)\} \end{aligned} \tag{9}$$

where G_l is a Hurwitz matrix. The closed loop system satisfies the following equations:

$$\begin{aligned} \dot{\tilde{x}}_1(t) &= (A_{11}^t - A_{12}^tKC_1)\tilde{x}_1(t) \\ &\quad -A_{12}^tK_\tau C_1\tilde{x}_1(t-\tau) + A_{12}^t\tilde{x}_2(t) \\ \dot{\tilde{x}}_2(t) &= G_l\tilde{x}_2(t) + (A_{21}^t + KC_1A_{11}^t)\tilde{x}_1(t) \\ &\quad +K_\tau C_1A_{11}^t\tilde{x}_1(t-\tau) - \nu + \psi_1(t,y,u) \end{aligned} \tag{10}$$

where

$$\begin{aligned} &\psi_1(t,y,u) = \\ &\quad \sum{}_{i=1}^N\lambda_i(t)[KC_1A_{12}^i\tilde{x}_2(t) + KC_1A_{12}^iKC_1\tilde{x}_1(t) \\ &\quad -KC_1A_{12}^iK_\tau C_1\tilde{x}(t-\tau)] + \sum{}_{i=1}^N\lambda_i(t-\tau) \\ &\quad [K_\tau C_1A_{12}^i\tilde{x}_2(t-\tau) + K_\tau C_1A_{12}^iKC_1\tilde{x}_1(t-\tau) \\ &\quad -K\tau C_1A_{12}^iK_\tau C_1\tilde{x}(t-2\tau)] + B_{20}\psi_0(t,y,u) \end{aligned}$$

As ψ_1 only depends on t, y and u, there exists a positive function Ψ_{02} such that:

$$\|\psi_1(t,y,u)\| \leq \|B_{20}\|\Psi_2(t,y,u) + \Psi_{21}(t,y,u)$$

Now define the discontinuous sliding function ν by:

$$\nu = \begin{cases} \rho(t,y,u)\dfrac{Q_2\tilde{x}_2(t)}{\|Q_2\tilde{x}_2(t)\|} & \text{if } \tilde{x}_2(t) \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

where Q_2 is a symmetric positive definite matrix in $R^{m\times m}$ and

$$\rho(t,y,u) = \|B_{20}\|\Psi_2(t,y,u) + \Psi_{21}(t,y,u) + \delta \tag{11}$$

where δ is a positive scalar gain.

Remark 3. Note that the control law (9) does not have a heavy computational overhead.

4. STABILIZATION OF THE CLOSED LOOP SYSTEM

This section focusses on the stability of the whole system (10). In particular, it needs to be established that $\tilde{x}_2 = 0$ in finite time, i.e. a sliding motion is achieved.

4.1 Exponential Stability Condition

Theorem 1. System (10) is exponentially stable for given output feedback gains K and K_τ with decay rate α if there exist $P_1 > 0$, P_2, P_3, $S_p = S_p^T$, Q_p, $R_{pq} = R_{qp}^T$, $p,q = 0,...,\bar{N}$ in $R^{(n-m)\times(n-m)}$ and $Q_2 > 0 \in R^{m\times m}$ which satisfy LMI's (12) and (13) with $h = \tau/\bar{N}$ and for all $i = 1,..,N$:

$$\begin{bmatrix} \Xi_\alpha^i & D^s & D^a & \begin{bmatrix} (\Lambda_{21}^i + KC_1\Lambda_{11}^i)^TQ_2 + P_1\Lambda_{12}^i \\ 0 \\ e^{\alpha\tau}(K_\tau C_1\Lambda_{11}^i)^TQ_2 \end{bmatrix} \\ * & -R_d - S_d & 0 & 0 \\ * & * & -3S_d & 0 \\ * & * & * & Q_2G_l + G_l^TQ_2 + 2\alpha Q_2 \end{bmatrix} < 0 \tag{12}$$

$$\begin{bmatrix} P_1 & \tilde{Q} \\ * & \tilde{R} + \tilde{S} \end{bmatrix} > 0 \tag{13}$$

where the matrix Ξ_α^i is given by:

$$\begin{bmatrix} \Psi^i & P^T\begin{bmatrix} 0_n \\ e^{\alpha\tau}\Lambda_1^i \end{bmatrix} - \begin{bmatrix} Q_N \\ 0_n \end{bmatrix} \\ * & -S_N \end{bmatrix}, \tag{14}$$

where $\forall i = 1,..,N$ and $(k,l) \in [1,2]^2$:

$$\begin{aligned} \Psi^i &= \begin{bmatrix} Q_0 + Q_0^T + S_0 & 0 \\ 0 & 0 \end{bmatrix} + P^T\begin{bmatrix} 0 & I_n \\ \Lambda_0^i + \alpha I_n & -I_n \end{bmatrix} \\ &\quad + \begin{bmatrix} 0 & I_n \\ \Lambda_0^i + \alpha I_n & -I_n \end{bmatrix}^T P, \\ \Lambda_0^i &= \Lambda_{11}^i + \Lambda_{12}^iKC_1,\ \Lambda_1^i = -\Lambda_{12}^iKC_1, \\ \Lambda_{kl}^i &= A_{kl}^0 + A_{kl}^i, \end{aligned}$$

and for $p,q = 1,..,\bar{N}$

$$\begin{aligned} P &= \begin{bmatrix} P_1 & 0 \\ P_2 & P_3 \end{bmatrix},\ \tilde{Q} = [Q_0\ Q_1\ \ldots\ Q_{\bar{N}}], \\ \tilde{S} &= diag\{1/hS_0, 1/hS_1, \ldots, 1/hS_{\bar{N}}\}, \\ S_d &= diag\{S_0 - S_1, ..., S_{\bar{N}-1} - S_{\bar{N}}\}, \\ \tilde{R} &= \begin{bmatrix} R_{00} & R_{01} & \ldots & R_{0\bar{N}} \\ R_{01} & R_{11} & \ldots & R_{1\bar{N}} \\ \vdots & \vdots & \ddots & \vdots \\ R_{\bar{N}0} & R_{\bar{N}1} & \ldots & R_{\bar{N}\bar{N}} \end{bmatrix}, \\ R_d &= \begin{bmatrix} R_{d11} & R_{d12} & \ldots & R_{d1\bar{N}} \\ R_{d21} & R_{d22} & \ldots & R_{d2\bar{N}} \\ \vdots & \vdots & \ddots & \vdots \\ R_{d\bar{N}1} & R_{d\bar{N}2} & \ldots & R_{d\bar{N}\bar{N}} \end{bmatrix}, \\ D^s &= [D_1^s\ D_2^s\ \ldots\ D_{\bar{N}}^s], \\ D^a &= [D_1^a\ D_2^a\ \ldots\ D_{\bar{N}}^a], \\ R_{dpq} &= h(R_{(p-1)(q-1)} - R_{pq}), \\ D_p^s &= \begin{bmatrix} (R_{0(p-1)} + R_{0p}) - (Q_{p-1} - Q_p) \\ h/2(Q_{p-1} + Q_p) \\ -h/2(R_{\bar{N}(p-1)} + R_{\bar{N}p}) \end{bmatrix}, \\ D_p^a &= \begin{bmatrix} -h/2(R_{0(p-1)} - R_{0p}) \\ -h/2(Q_{p-1} - Q_p) \\ h/2(R_{\bar{N}(p-1)} - R_{\bar{N}p}) \end{bmatrix}. \end{aligned}$$

Proof. As in Seuret et al. [2004], consider new variables $\tilde{x}_{1\alpha}(t) = \tilde{x}_1(t)e^{\alpha t}$ and $\tilde{x}_{2\alpha}(t) = \tilde{x}_2(t)e^{\alpha t}$. The new closed-loop system satisfies the following equations:

$$\begin{aligned} \dot{\tilde{x}}_{1\alpha}(t) &= (A_{11}^t - A_{12}^tKC_1 + \alpha I_{n-m})\tilde{x}_{1\alpha}(t) \\ &\quad -e^{\alpha\tau}A_{12}^tK_\tau C_1\tilde{x}_{1\alpha}(t-\tau) + A_{12}^t\tilde{x}_{2\alpha}(t) \\ \dot{\tilde{x}}_{2\alpha}(t) &= (G_l + \alpha I_m)\tilde{x}_{2\alpha}(t) + (A_{21}^t \\ &\quad +KC_1A_{11}^t)\tilde{x}_{1\alpha}(t) - e^{\alpha t}(\nu - \psi_1(t,y,u)) \\ &\quad +e^{\alpha\tau}K_\tau C_1A_{11}^{t-\tau}\tilde{x}_{1\alpha}(t-\tau) \end{aligned} \tag{15}$$

Consider the Lyapunov-Krasovskii functional:

$$V_\alpha(t) = V_{1\alpha}(t) + V_{2\alpha}(t),$$

where $V_{2\alpha}(t) = \tilde{x}_{2\alpha}^T(t)Q_2\tilde{x}_{2\alpha}(t)$ and:

$$\begin{aligned} V_{1\alpha}(t) &= \tilde{x}_{1\alpha}^T(t)P_1\tilde{x}_{1\alpha}(t) + 2\tilde{x}_{1\alpha}^T(t)\int_{-\tau}^0 Q(\zeta)\tilde{x}_{1\alpha}(t+\zeta)d\zeta \\ &\quad + \int_{-\tau}^0 \tilde{x}_{1\alpha}^T(t+\zeta)S(\xi)\tilde{x}_{1\alpha}(t+\zeta)d\zeta \\ &\quad + \int_{-\tau}^0\int_{-\tau}^0 \tilde{x}_{1\alpha}^T(t+s)R(s,\zeta)ds\tilde{x}_{1\alpha}(t+\zeta)d\zeta \end{aligned} \tag{16}$$

where $P_1 > 0, Q(\zeta) \in R^{(n-m)\times(n-m)}$, $R(s,\xi) = R^T(\zeta,s) \in R^{(n-m)\times(n-m)}$, $S(\zeta) \in R^{(n-m)\times(n-m)}$, and Q, R, S are continuous matrix functions. From Gu et al. [2003] (p. 185) $V_{1\alpha}$ is positive definite if the LMI (13) holds. Then the proof follows along the lines of Fridman [2006] using a descriptor representation, Fridman and Shaked [2002], and discretization Gu [2001]. Differentiating the Lyapunov functional $V_{1\alpha}$ along the trajectories of (15) and using the convexity of functions λ_i leads to:

$$\begin{aligned}\dot V_{1\alpha}(t) &= 2\dot{\tilde x}_{1\alpha}^T(t)\left[P_1\tilde x_{1\alpha}(t)+\int_{-\tau}^{0}Q(\zeta)\tilde x_{1\alpha}(t+\zeta)d\zeta\right]\\ &+2\tilde x_{1\alpha}^T(t)\int_{-\tau}^{0}Q(\zeta)\dot{\tilde x}_{1\alpha}(t+\zeta)d\zeta\\ &+2\int_{-\tau}^{0}\int_{-\tau}^{0}\dot{\tilde x}_{1\alpha}^T(t+s)R(s,\zeta)ds\tilde x_{1\alpha}(t+\zeta)d\zeta\\ &+2\int_{-\tau}^{0}\dot{\tilde x}_{1\alpha}^T(t+\zeta)S(\zeta)\tilde x_{1\alpha}(t+\zeta)d\zeta\end{aligned} \tag{17}$$

Integrating the expression above by parts, the following equality can be established as in Fridman [2006]:

$$\begin{aligned}\dot V_{1\alpha}(t) &= \sum_{i=1}^{N}\lambda_i(t)\left\{\xi^T(t)\Xi_\alpha^i\xi(t)\right.\\ &+2\dot{\tilde x}_{1\alpha}^T(t)\int_{-\tau}^{0}Q(\zeta)\tilde x_{1\alpha}(t+\zeta)d\zeta\\ &-\int_{-\tau}^{0}x_\alpha^T(t+\zeta)\dot S(\zeta)\tilde x_{1\alpha}(t+\zeta)d\zeta\\ &-\int_{-\tau}^{0}\int_{-\tau}^{0}\tilde x_{1\alpha}^T(t+s)\left(\frac{\partial}{\partial s}R(s,\zeta)\right.\\ &\left.+\frac{\partial}{\partial\zeta}R(s,\zeta)\right)ds\tilde x_{1\alpha}(t+\zeta)d\zeta\\ &+2\tilde x_{1\alpha}^T\int_{-\tau}^{0}\left[-\dot Q(\zeta)+R(0,\zeta)\right]\tilde x_{1\alpha}(t+\zeta)d\zeta\\ &\left.-2\tilde x_{1\alpha}^T(t)\int_{-\tau}^{0}R(-\tau,\zeta)\tilde x_{1\alpha}(t+\zeta)d\zeta\right\}\end{aligned} \tag{18}$$

where $\xi(t)=col\{\bar{\tilde x}_{1\alpha}(t),\tilde x_{1\alpha}(t-\tau)\}$ and Ξ_α^i has the form in (14) with $Q(0)$, $Q(-\tau)$, $S(0)$ and $S(-\tau)$ instead of Q_0, Q_N, S_0 and S_N respectively. The Lyapunov functional is now expressed in an appropriate representation to apply the discretization. The continuous matrix functions $Q(\zeta)$ and $S(\zeta)$ are chosen to be linear within each interval and the continuous matrix function $R(s,\zeta)$ is chosen to be linear within each triangle. The proposed matrix functions are:

$$\begin{aligned}&Q(\theta_p+\beta h)=(1-\beta)Q_p+\beta Q_{p-1}\\ &S(\theta_p+\beta h)=(1-\beta)S_p+\beta S_{p-1}\\ &R(\theta_p+\beta h,\theta_q+\gamma h)=\\ &\begin{cases}(1-\beta)R_{pq}+\gamma R_{(p-1)(q-1)}+(\beta-\gamma)R_{(p-1)q}, & \beta\ge\gamma\\ (1-\gamma)R_{pq}+\beta R_{(p-1)(q-1)}+(\gamma-\beta)R_{(p-1)q}, & \beta\le\gamma\end{cases}\end{aligned}$$

for $0\le\beta\le1$ and $0\le\gamma\le1$. Simple definitions of the derivative of the matrix functions can be obtained which are, for appropriate values of p and q:

$$\begin{aligned}&\dot S(\xi)=1/h(S_{p-1}-S_p),\ \dot Q(\xi)=1/h(Q_{p-1}-Q_p)\\ &\frac{\partial}{\partial s}R(s,\xi)+\frac{\partial}{\partial\xi}R(s,\xi)=1/h(R_{(p-1)(q-1)}-R_{pq})\end{aligned} \tag{19}$$

From Fridman [2006], differentiating $V_{1\alpha}$ along the trajectory of (15a) leads to the following inequality:

$$\begin{aligned}\dot V_{1\alpha} &\le \sum\nolimits_{i=1}^{N}\lambda_i(t)\\ &\left\{\xi^T(t)\Xi_\alpha^i\xi(t)-\int_0^1\phi^T(\beta)S_d\phi(\beta)d\beta\right.\\ &+2\xi^T(t)\int_0^1\left[D^s+(1-2\beta)D^a\right]\phi(\beta)d\beta\\ &\left.-\int_0^1\int_0^1\phi^T(\beta)R_d\phi(\gamma)d\beta d\gamma+\tilde x_{1\alpha}^T(t)P_1\Lambda_{12}^i\tilde x_{2\alpha}(t)\right\}\end{aligned} \tag{20}$$

where Ξ_α^i is defined in (14) and the functions $\phi(\beta)=col\{\tilde x_{1\alpha}(t-h+\beta h),..,\tilde x_{1\alpha}(t-Nh+\beta h)\}$. Differentiating $V_{2\alpha}$ along the trajectory of (15b) leads to:

$$\begin{aligned}&\dot V_{2\alpha}(t)\le-\tilde x_{2\alpha}^T(t)Q_2e^{\alpha t}(\nu+\psi_1(t,y,u))\\ &\quad\sum\nolimits_{i=1}^{N}\lambda_i(t)\left\{\tilde x_{2\alpha}^T(t)Q_2[(A_{21}^t+KC_1A_{11}^t)\tilde x_{1\alpha}(t)\right.\\ &\quad+e^{\alpha\tau}K_\tau C_1A_{11}^{t-\tau}\tilde x_{1\alpha}(t-\tau)]\\ &\quad\left.\tilde x_{2\alpha}^T(t)(G_l^TQ_2+Q_2G_l+2\alpha I_m)\tilde x_{2\alpha}(t)\right\}\end{aligned} \tag{21}$$

Then by combining (20) and (21) and by defining $\xi'(t)=col\{\tilde x_{1\alpha}(t),\dot{\tilde x}_{1\alpha}(t),\tilde x_{1\alpha}(t-\tau),\tilde x_{2\alpha}(t)\}$, the following inequality holds:

$$\begin{aligned}&\dot V_\alpha(t)\le\sum\nolimits_{i=1}^{N}\lambda_i(t)\{\\ &\xi'^T(t)\begin{bmatrix}\Xi_\alpha^i & \begin{bmatrix}(\Lambda_{21}^i+KC_1\Lambda_{11}^i)^TQ_2+P_1\Lambda_{12}^i\\ 0_{(n-m)\times m}\\ e^{\alpha\tau}(K_\tau C_1\Lambda_{11}^i)^TQ_2\end{bmatrix}\\ * & G_l^TQ_2+Q_2G_l\end{bmatrix}\xi'(t)\\ &-\int_0^1\phi^T(\beta)S_d\phi(\beta)d\beta\\ &-\int_0^1\int_0^1\phi^T(\beta)R_d\phi(\gamma)d\beta d\gamma\\ &\left.+2\xi^T(t)\int_0^1\left[D^s+(1-2\beta)D^a\right]\phi(\beta)d\beta\right\}\\ &-x_{2\alpha}^T(t)Q_2e^{\alpha t}(\nu-\psi_1(t,y,u))\end{aligned} \tag{22}$$

Note that from (11) $-x_{2\alpha}^T(t)Q_2e^{\alpha t}(\nu-\psi_1(t,y,u))\le-\delta e^{\alpha t}\|Q_2x_{2\alpha}(t)\|$. The last term is thus negative. Applying Proposition 5.21 from Gu et al. [2003] to (22) and using the convexity of λ_i, it can be concluded that $\dot V(t)<0$ if LMI's (12) hold for all $i=1,..,N$. ■

4.2 Reachability to the sliding manifold in finite time

Corollary 1. An ideal sliding motion takes place on the surface $\tilde x_2(t)=0$ in the domain

$$\begin{aligned}&\Omega=\{(\tilde x_1,\tilde x_2)\in[t-\tau,t]\mapsto R^{n-m}\times R^m\ :\\ &\max_{i,j=1,..,N}(\|(\Lambda_{21}^i+KC_1\Lambda_{11}^i)\|+\|K_\tau C_1\Lambda_{11}^j\|)|\tilde x_1|_\tau<\delta-\eta\}\end{aligned}$$

where η is a small scalar satisfying $0<\eta<\delta$.

Proof. Consider the following Lyapunov function $V_s(t)=\tilde x_2^T(t)Q_2\tilde x_2(t)$. By differentiating V_s along the trajectories of (10b), it follows that:

$$\begin{aligned}\dot V_s(t)&=\tilde x_2^T(t)(Q_2G_l+G_l^TQ_2)\tilde x_2(t)\\ &+2\tilde x_2^T(t)Q_2\left[(A_{21}^t-KC_1A_{11}^t)\tilde x_1(t)-\nu\right.\\ &\left.+\psi_1(t,y,u)-K_\tau C_1A_{11}^{t-\tau}\tilde x_1(t-\tau)\right]\end{aligned}$$

As the matrix G_l is Hurwitz, Q_2 is chosen such that $Q_2G_l+G_l^TQ_2<0$. If the system satisfies the conditions from Theorem 1, the state $\tilde x_1$ converges to the solution $\tilde x_1=0$ with an exponential decay rate. Then the domain Ω is reached in finite time. Since the gain ρ of the sliding function is defined as $\rho(t,y,u)=\|B_{20}\|\Psi_2(t,y,u)+\Psi_{21}(t,y,u)+\delta$, the following inequality holds:

$$\dot V_s(t)\le-\eta\sqrt{V_s}(t)$$

This conclude the proof. ■

Remark 4. As usual, the problem of designing the output feedback gain is not straightforward. Moreover LMI (12) is not an appropriate form for synthesis purposes because the gains K and K_τ appear in different ways in Ξ_α than in $(KC_1\Lambda_{11}^i)^TQ_2$ and $(K_\tau C_1\Lambda_{11}^i)^TQ_2$. Congruence and other classical LMI transformations will probably not facilitate constructive conditions. A constructive method at this time is to test the stability of the closed-loop system for a given set of values of K and K_τ as shown in the following example.

5. EXAMPLE

Consider the non-delayed system (5) with:

$$A_{11}^t = \begin{bmatrix} 0 & -2 \\ 1 & 0.1 \end{bmatrix}, \quad A_{12}^t = \begin{bmatrix} -1 \\ 0.1(1-cos(x_1)) \end{bmatrix},$$
$$A_{21}^t = \begin{bmatrix} -0.1 & -1 \end{bmatrix},\ A_{22}^t = \begin{bmatrix} 1 \end{bmatrix},$$
$$B^{tT} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T, \quad C = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

As in Edwards and Spurgeon [1995], this system is not output stabilizable using traditional static (ie. non delayed output feedback). The objective here is to design the controller (9) with appropriate gains K, $K_\tau \in R$ and an artificial delay τ such that the closed-loop system is exponentially stable with decay rate α.

5.1 Design of the output feedback

This section proposes a method to obtain the optimal controller (K, K_τ, τ). The idea is to test if, for a set of values of K and K_τ, the LMIs from Theorem 1 have a solution and if it is possible to find the delay which ensures the greatest exponential decay rate.

Theorem 1 can only be satisfied when K lies in the interval $[-4;\ 2]$ and K_τ in $[0;\ 4]$. For each value of the gains K and K_τ, an optimization process is used to obtain the best value of α by tuning τ upwards from zero until the LMIs are not satisfied.

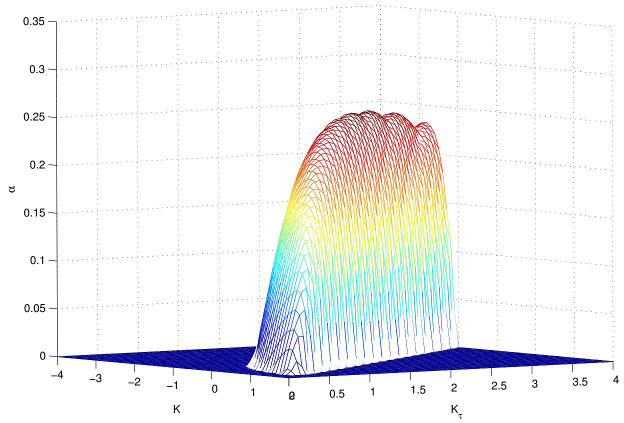

Fig. 1. Maximum decay rate α with respect to K and K_τ for $N = 1$

Figure 1 shows the relationship between the output feedback gains and the decay rate α using Theorem 1 with $N = 1$. It appears that the solution design a direction of the plane defined by (K, K_τ). The length of the set increase when the discretization number N also increase. Figure 1 also shows that the graph has a maximum at $K = -1.7$ and $K_\tau = 2.8$. This selection of gains K and K_τ ensures the system is exponentially stable with a decay rate $\alpha = 0.254$. The corresponding delay is $\tau = 0.25$.

For $N = 3$ the same gains are $K = -1.7$ and $K_\tau = 2.8$. The corresponding delay is $\tau = 0.425$. For such parameters the decay rate is $\alpha = 0.445$. Theorem 1 also ensures for $N = 6$ that the same gains exponentially stabilize the system (5) with a decay rate $\alpha = 0.538$ with $\tau = 0.425$.

Remark 5. Starting from $N = 2$, the computation of the conditions from Theorem 1 become very heavy. The optimization problem has not been tested for $N \geq 3$.

5.2 Simulation results

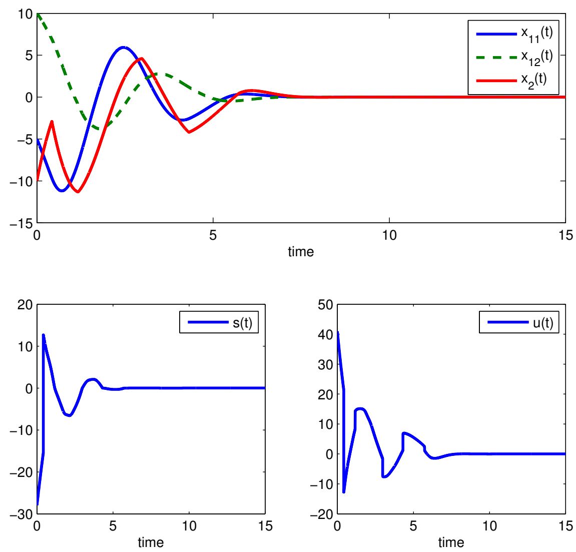

Fig. 2. Simulation results for $K = -1.7$, $K_\tau = 2.8$ and $\tau = 0.425s$

In the results which follow system (5) is controlled using (9) with $K = -1.7$, $K_\tau = 2.8$ and $\tau = 0.425$.

Figure 2 shows the state, the input and the sliding function. The state converges exponentially to $x(t) = 0$ with an exponential decay rate $\alpha = 0.538$. The sliding function converges to $s(t) = 0$ in finite time. The evolution of the control signal is also shown in Figure 2.

The robustness with respect to the delay is shown in Figure 3. Assuming that the gains K and K_τ are still $K = -1.7$ and $K_\tau = 2.8$, Figure 3 presents several simulations for different delays. It can be seen that the delay has a influence on the stability. For small delays ($\tau = 0.01$ or $\tau = 0.9$), system (5) is unstable. However, when the delay is sufficiently close to the optimal delay ($\tau = 0.3$ or $\tau = 0.6$), system (5) becomes stable. It can also be seen in figures 3 that the decay rates for $\tau = 0.3$ or $\tau = 0.6$ are less than in the optimal delay case shown in figure 2. This proves the efficiency the proposed method.

6. CONCLUSION

A new sliding mode controller has been suggested for systems for which finding a traditional static output feedback sliding mode controller is not possible. The controller introduces a *stabilizing delay* in the closed loop system. The controller is simple and does not require heavy computation and it ensures robust exponential stability of the closed-loop system. An example has been used to

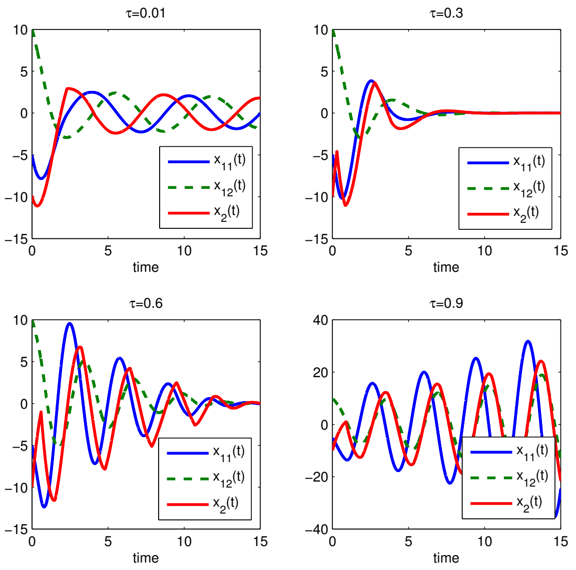

Fig. 3. Simulation results for $K = -1.7$, $K_\tau = 2.8$ and different values of τ

demonstrate a method to design the gains and the delay of the controller. It also shows that this controller still stabilizes the system even when system has vertices which have invariant zeros on the right side of the complex plane.

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