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Reduced Bank of Kalman Filters ⋆

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Abstract: A reduction of the computational complexity of bank of Kalman filters is proposed. The algorithm is focused in fault detection and isolation problems. It is shown that the orders of individual filters in the bank can be lower than the respective filtered process model order. The original model state variables are not estimated. Linear functions of noise samples are the newly estimated variables.

Keywords: Fault detection and diagnosis; Filtering and smoothing.

1. INTRODUCTION

1.1 Bank of Kalman filters

The bank of Kalman filters is the standard solution to the hybrid linear system state estimation problem, see bar-Shalom et al. [2001]. Such system state consists of a vector of continuous real state variables $\mathbf{x}(t)$, which are governed by a system of differential equations linear in $\mathbf{x}(t)$. In addition, the other system state variable includes an integer-valued parameter $\mu(t)$, which can be understood as the system mode. Some applications do not require the system state variables $\mathbf{x}(t)$ to be estimated. Occasionally, only the mode estimate $\hat{\mu}(t|t)$ may be of interest. Our approach focuses in these particular problems.

A linear process subject to failures can often be described by a quadruple of matrices $\mathbf{A}_\mu$, $\mathbf{B}_\mu$, $\mathbf{C}_\mu$, $\mathbf{D}_\mu$ for each mode, see Venkatasubramanian [2003] or Basseville et al. [2003] or Simon and Kobayashi [2006]. The model has both manipulated and random Gaussian inputs. We will refer to the latter as unknown inputs. Then $\mu(t)$ is often modelled as Markov process changing its value randomly based on its current value. The Bayesian solution to this filtering problem has been developed in Bar-Shalom et al. [1988].

The algorithm presented in this text is confined to situations that only those columns of $\mathbf{B}_\mu$ and $\mathbf{D}_\mu$ which correspond to the unknown inputs are functions of μ. The remaining state space model parameters are the same irrespective μ. The sensor and actuator faults can often be modelled in this way, see Basseville et al. [2003].

1.2 Notation

Real column vectors will be distinguished by bold roman lowercase font, e.g. $\mathbf{v}$. The length of that vector $\mathbf{v}$ will be denoted as $n(\mathbf{v})$. A real matrix $\mathbf{Z}$ will be written in uppercase using roman bold face. The matrix $\mathbf{Z}'$ is $\mathbf{Z}$ transposed, the vector $\mathbf{v}'$ is the row vector. The discrete vector time series $\mathbf{v}(t)$ sampled on an interval of time $0, 1, \ldots, t$ will be denoted as $\mathbf{v}\{0, \ldots, t\}$. The same time series understood as the column vector will be denoted as $\mathbf{v}_t^0$. Vector of all zeros will be denoted as $\mathbf{o}$, the $n \times n$ identity matrix will be denoted as $\mathbf{I}_n$. The $m \times n$ matrix of all zeros will be denoted $\mathbf{O}_{m\times n}$. Let $p(\mathbf{x}|\mathbf{y})$ be the probability density function of $\mathbf{x}$ conditioned on $\mathbf{y}$. $\hat{\mathbf{x}}(t|k)$ is the estimate of $\mathbf{x}(t)$ conditioned on all data available at time k.

2. MOTIVATION PROBLEM

2.1 Formulation

Let us suppose the following problem: A random vector $\mathbf{v} \in \mathbb{R}^{n(\mathbf{v})}$ has normal probability distribution with zero mean and unit variance, i.e. $\mathbf{v} \sim \mathcal{N}(\mathbf{o}, \mathbf{I})$. The vector $\mathbf{v}$ is hidden. Only the value of certain linear function $\mathbf{y} = \mathbf{Z}_\mu \mathbf{u}$, $\mathbf{y} \in \mathbb{R}^{n(\mathbf{y})}$ can be observed.

We will suppose the matrices $\mathbf{Z}_\mu$ have full row rank. Therefore, it holds $n(\mathbf{y}) \leq n(\mathbf{v})$, the observed vector is a projection of $\mathbf{v}$ to a subspace. The statistical decision problem concerns hypotheses about the matrix $\mathbf{Z}_\mu$. Suppose there is a number of competing values for $\mathbf{Z}_\mu$; $\mu \in \{0,\ 1, \ldots, M\}$; $M \geq 1$.

To decide about the hypotheses it is necessary to evaluate $p(\mathbf{y}|\mu)$ for all μ. Then the likelihood of the hypotheses together with their priors $p(\mu)$ define the posterior probability distribution according to the Bayes rule:

$$p(\mu|\mathbf{y}) = p(\mathbf{y}|\mu)p(\mu) / \sum_{\mu=1}^{M} (p(\mathbf{y}|\mu)p(\mu)) \qquad (1)$$

The posterior probability distribution (1) may be considered to be the solution to the problem in the Bayes sense. Because the observed vector $\mathbf{y}$ has normal distribution of probability with zero mean and the variance matrix $(\mathbf{Z}_\mu \mathbf{Z}_\mu')^{-1}$, the quadratic forms ρ_μ defined by (2)

$$\rho_\mu = \mathbf{y}'(\mathbf{Z}_\mu \mathbf{Z}_\mu')^{-1}\mathbf{y} \qquad (2)$$

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are sufficient statistics. The information in the original data $\mathbf{y}$ and the information in ρ_μ for all μ are equivalent as long as the decision problem is concerned. Using (3) and the fact C is independent on $\mathbf{y}$, one can verify the sufficiency condition is satisfied.

$$p(\mu|\mathbf{y},\rho_\mu) = C(\mu)\exp\left(-\frac{1}{2}\rho_\mu\right) = p(\mu|\rho_\mu) \qquad (3)$$

Thus, to solve this decision problem, one can evaluate the ρ_μ statistics, which are sufficient information to evaluate likelihood of all the particular hypotheses. A way alternative to (2), how ρ_μ can be evaluated is based on the maximum a posteriori (MAP) $\hat{\mathbf{v}}_\mu$ estimates. Because the vector $\mathbf{v}$ has the $\mathcal{N}(\mathbf{o},\mathbf{I})$ distribution of probability, its max. a posteriori estimate is defined by (4).

$$\hat{\mathbf{v}}_\mu = \arg\min_{\mathbf{v}} \mathbf{v}'\mathbf{v}, \text{ subject to } \mathbf{y} = \mathbf{Z}_\mu\mathbf{v} \qquad (4)$$

We will show that $\rho_\mu = \hat{\mathbf{v}}'_\mu\hat{\mathbf{v}}_\mu$. To show this we will use two orthonormal matrices: $\mathbf{N}_\mu = \text{null}\{\mathbf{Z}_\mu\}$ and $\mathbf{R}_\mu = \text{null}\{\text{null}\{\mathbf{Z}_\mu\}'\}$. The columns of $\mathbf{N}_\mu$ span the null space of $\mathbf{Z}_\mu$, the columns of $\mathbf{R}_\mu$ span a complementary perpendicular linear subspace; it is an orthonormal basis of the row space of $\mathbf{Z}_\mu$. Note that the matrix $\mathbf{R}_\mu$ is $n(\mathbf{y}) \times n(\mathbf{y})$ because we supposed $n(\mathbf{y})$ is the rank of $\mathbf{Z}_\mu$.

Now the vector $\mathbf{v}$ can be decomposed into the two perpendicular components: $\mathbf{v} = \mathbf{v}^{\mathbf{N}}_\mu + \mathbf{v}^{\mathbf{R}}_\mu$. The component $\mathbf{v}^{\mathbf{N}}_\mu$ is in the column space of $\mathbf{N}_\mu$ and the component $\mathbf{v}^{\mathbf{R}}_\mu$ is in the column space of $\mathbf{R}_\mu$. Both components have zero mean and unit variance matrix and they are independent. The MAP criterion can be rewritten to (5) to become a minimization with respect to both components.

$$\min_{\mathbf{v}^{\mathbf{R}}_\mu,\mathbf{v}^{\mathbf{N}}_\mu} \mathbf{v}'^{\mathbf{R}}_\mu\mathbf{v}^{\mathbf{R}}_\mu + \mathbf{v}'^{\mathbf{N}}_\mu\mathbf{v}^{\mathbf{N}}_\mu, \text{ subject to } \mathbf{y} = \mathbf{Z}_\mu\mathbf{v}^{\mathbf{R}}_\mu \qquad (5)$$

Looking at the minimization (5) it is clear the minimizing $\mathbf{v}^{\mathbf{N}}_\mu = \mathbf{o}$. In other words, the MAP estimate $\hat{\mathbf{v}}_\mu$ is within the column space of the matrix $\mathbf{R}_\mu$. Thus, it is correct to write $\hat{\mathbf{v}}_\mu = \mathbf{R}_\mu\hat{\mathbf{w}}_\mu$, where $\hat{\mathbf{w}}_\mu \in \mathbb{R}^{n(\mathbf{y})}$. Hence (6) follows (index μ omitted in this equation for brevity).

$$\begin{aligned}\rho &= \hat{\mathbf{v}}'\mathbf{Z}'(\mathbf{Z}\mathbf{Z}')^{-1}\mathbf{Z}\hat{\mathbf{v}} \\ &= \hat{\mathbf{w}}'\mathbf{R}'\mathbf{Z}'(\mathbf{Z}\mathbf{R}\mathbf{R}'\mathbf{Z}')^{-1}\mathbf{Z}\mathbf{R}\hat{\mathbf{w}} \\ &= \hat{\mathbf{w}}'\mathbf{R}'\mathbf{Z}'(\mathbf{R}'\mathbf{Z}')^{-1}(\mathbf{Z}\mathbf{R})^{-1}\mathbf{Z}\mathbf{R}\hat{\mathbf{w}} \\ &= \hat{\mathbf{w}}'\hat{\mathbf{w}} \\ &= \hat{\mathbf{w}}'\mathbf{R}'\mathbf{R}\hat{\mathbf{w}} \\ &= \hat{\mathbf{v}}'\hat{\mathbf{v}} \\ &\text{for all } \mu,\ \text{index } \mu \text{ omitted}\end{aligned} \qquad (6)$$

Note that the matrix $\mathbf{Z}_\mu\mathbf{R}_\mu$ has full rank because the column space of $\mathbf{R}_\mu$ is perpendicular to the null space of $\mathbf{Z}_\mu$ by definition. The inverse $(\mathbf{Z}_\mu\mathbf{R}_\mu)^{-1}$ exists therefore.

The conclusion drawn from this example is the following: To evaluate the likelihood of the hypotheses on μ, one can either (a) substitute the observed $\mathbf{y}$ to the probability density functions $p(\mathbf{y}|\mu)$ or (b) to find the minimum quadratic norm of the unobserved random vector $\mathbf{v}$ satisfying the equality $\mathbf{y} = \mathbf{Z}_\mu\mathbf{v}$.

2.2 Dynamic problem

The two solutions (a) and (b) developed in the previous section for the static case shall be examined considering the linear dynamic hybrid system mode estimation. Here, the dynamic hybrid system will be represented as a set of linear competing state space models differing only in the unknown input related matrices $\mathbf{F}_\mu$, $\mathbf{G}_\mu$ as in the state space model (7).

$$\begin{aligned}\mathbf{x}(t{+}1) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) + \mathbf{F}_\mu\mathbf{v}(t) \\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t) + \mathbf{G}_\mu\mathbf{v}(t) + \mathbf{e}(t)\end{aligned} \qquad (7)$$

Each mode is characterized by particular values of the two matrices. This special problem structure is often encountered when solving fault detection problems.

Again, the likelihood $p(\mathbf{y}^0_t|\mu,\mathbf{u}^0_t)$ is sought. The vector $\mathbf{u}(t) \in \mathbb{R}^{n(\mathbf{u})}$ represents the manipulated inputs, the vector $\mathbf{v}(t) \in \mathbb{R}^{n(\mathbf{v})}$ represents the random phenomena. Without loosing generality $\mathbf{v}(t)$ can be considered to be a white Gaussian noise process with unit variance matrix. An appropriate model augmentation can add correlation and autocorrelation phenomena.

The structure of this decision problem resembles the linear problem solved in the previous section. The observed data represent a projection of the unknown random series $\mathbf{v}(t)$. Unless $n(\mathbf{v}) \geq n(\mathbf{y})$, i.e. there are more or the same number of unknown inputs than the number of measured outputs, the model may happen not to be able to match the observed data. To avoid such singularity we add a measurement errors to the measured output: $\mathbf{e}(t)$. In the following development, we will consider $\mathbf{I}_{n(\mathbf{e})} \gg \text{var}\{\mathbf{e}(t)\}$. To simplify the situation, we will also suppose μ is time invariant.

For this problem the substitution of the observed time series $\mathbf{y}^0_t$ to its probability density function alike in (2) is not possible algorithmically. As the time goes on and more data from the process are collected, the dimension of the series $\mathbf{y}^0_t$ grows and the ρ_μ evaluation would require a manipulation with vectors and matrices of growing dimension. That is why Kalman filter is normally used to evaluate the μ likelihood. Kalman filter calculates its prediction errors which are mutually independent. This makes the likelihood a product of simpler functions of those errors. Each factor in the product is than characterized by a prediction covariance matrix of fixed dimension. Thus, no dimension growth occurs. One can see the Kalman filter as a mechanism removing serial correlation from the data to make the likelihood evaluation easy.

In the following, we will use the quadratic norm of the MAP estimate of $\mathbf{v}\{0,\ \ldots,\ t\}$ as the sufficient statistics. This approach differs from that of the Kalman filter approach. To evaluate the likelihood of μ, we propose to calculate the minimum quadratic norm of the time series $\mathbf{v}\{0,\ \ldots,\ t\}$ directly using the dynamic programming mechanism. Though it is very closely related to Kalman filtering, it may be simpler as the process state $\mathbf{x}(t)$ estimation is not a part of this problem formulation.

The sum of squares defined below has the χ^2 distribution of probability with $n(\mathbf{v})(t+1)$ degrees of freedom.

$$\sum_{k=0}^{t} \mathbf{v}'(k)\mathbf{v}(k) \sim \chi^2_{n(\mathbf{v})\,(t+1)} \tag{8}$$

We will define the statistics $\rho(t,\mu)$ as the minimum value of the sum of squares (8). The minimization has to be done respecting the data observed and for each system mode (9). The values of $\mathbf{v}(k)$ which result from the minimization must satisfy the input output relationship for the respective mode μ and the $\mathbf{u}_t^0$, $\mathbf{y}_t^0$ data measured.

$$\rho(t,\mu) = \min_{\substack{\mathbf{v}(k) \\ \text{w.r. } \mu,\mathbf{y}_t^0,\mathbf{u}_t^0}} \sum_{k=0}^{t} \mathbf{v}'(k)\mathbf{v}(k) \sim \chi^2_{n(\mathbf{y})\,(t+1)} \tag{9}$$

This statistics (9) also has the χ^2 distribution of probability, but with fewer degrees of freedom. This probability is the likelihood of the mode.

To evaluate the marginal probability distribution of the system mode μ for the model (7) it is sufficient to evaluate the minimum value of the sum of squares (8) for each mode. The optimization is subject to a set of linear constraints. This set contains all data samples observed satisfying (7). Note that we have neglected $\mathbf{e}(t)$ at this point.

3. ALGORITHM

3.1 Minimization

To calculate $\rho(t,\mu)$, it is necessary to minimize the sum of squares subject to different sets of linear constraints, one set per system mode. The standard bank of Kalman filters actually minimizes the sum of squares M times as it solves M Riccati equations in parallel. The Ricatti equation appears in many minimization problems where a quadratic form is minimized recursively.

We propose to solve this optimization problem recursively via the dynamic programming technique not directly but after a convenient linear transform. The set of linear constraints is (10).

$$\begin{aligned}
\begin{bmatrix} \mathbf{y}(0) \\ \mathbf{y}(1) \\ \vdots \\ \mathbf{y}(t) \end{bmatrix} &= \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \vdots \\ \mathbf{CA}^t \end{bmatrix} \mathbf{x}(0) + \\
&+ \begin{bmatrix} \mathbf{G}_\mu & \mathbf{O} & \dots & \mathbf{O} \\ \mathbf{CF}_\mu & \mathbf{G}_\mu & \dots & \mathbf{O} \\ \vdots & \vdots & & \vdots \\ \mathbf{CA}^{t-1}\mathbf{F}_\mu & \mathbf{CA}^t\mathbf{F}_\mu & \dots & \mathbf{G}_\mu \end{bmatrix} \begin{bmatrix} \mathbf{v}(0) \\ \mathbf{v}(1) \\ \vdots \\ \mathbf{v}(t) \end{bmatrix} + \\
&+ \begin{bmatrix} \mathbf{D} & \mathbf{O} & \dots & \mathbf{O} \\ \mathbf{CB} & \mathbf{D} & \dots & \mathbf{O} \\ \vdots & \vdots & & \vdots \\ \mathbf{CA}^{t-1}\mathbf{B} & \mathbf{CA}^t\mathbf{B} & \dots & \mathbf{D} \end{bmatrix} \begin{bmatrix} \mathbf{u}(0) \\ \mathbf{u}(1) \\ \vdots \\ \mathbf{u}(t) \end{bmatrix}
\end{aligned} \tag{10}$$

We propose to transform the linear system of constraints (10) to a simpler system by means of a deadbeat observer $\mathcal{O}$ used as a $\mathbf{u}(t)$, $\mathbf{y}(t)$ data prefilter. The deadbeat state observer processes the $\mathbf{u}(t)$, $\mathbf{y}(t)$ data to produce residuals $\mathbf{r}(t)$ on its output. The residuals will be a moving average of certain number of the previous $\mathbf{v}(t)$ values as in (11).

$$\mathbf{r}(t) = \sum_{k=0}^{m} \mathbf{H}_k \mathbf{v}(t-k) \tag{11}$$

The dead-beat observer $\mathcal{O}$ is designed for the original state space model (7) not considering the inputs $\mathbf{v}(t)$, $\mathbf{e}(t)$. Thus, the observer $\mathcal{O}$ is the same for all modes, it is independent on μ. Neglecting the unknown initial condition (its effect on $\mathbf{r}(t)$ vanishes after a number of samples) the linear system of constraints for the filtered data will have the band matrix form (12).

$$\begin{aligned}
\begin{bmatrix} \mathbf{r}(m) \\ \mathbf{r}(m+1) \\ \vdots \\ \mathbf{r}(t) \end{bmatrix} = &\begin{bmatrix} \mathbf{H}_m & \mathbf{H}_{m-1} & \dots & \mathbf{H}_0 \\ \mathbf{O} & \mathbf{H}_m & \dots & \mathbf{O} \\ \vdots & \vdots & & \vdots \\ \mathbf{O} & \mathbf{O} & \dots & \mathbf{O} \end{bmatrix} \\
&\quad \begin{matrix} \dots & \mathbf{O} & \mathbf{O} \\ \dots & \mathbf{O} & \mathbf{O} \\ & \vdots & \vdots \\ \dots & \mathbf{H}_1 & \mathbf{H}_0 \end{matrix} \Bigg] \begin{bmatrix} \mathbf{v}(0) \\ \mathbf{v}(1) \\ \vdots \\ \mathbf{v}(t) \end{bmatrix}
\end{aligned} \tag{12}$$

We will suppose the two systems of constraints (10), (12) are equivalent in the sense that for any vector $\mathbf{v}_t^0$ satisfying any of the two also the other is satisfied.

The deadbeat observer $\mathcal{O}$ impulse response can be evaluated as (13).

$$\begin{aligned} \mathbf{H}_0 &= \mathbf{G}, \\ \mathbf{H}_k &= \mathbf{C}(\mathbf{A}-\mathbf{LC})^{k-1}(\mathbf{F}-\mathbf{LG}), k > 0 \end{aligned} \tag{13}$$

In (13) the matrix $\mathbf{L}$ is a state injection matrix so that $\mathbf{A}-\mathbf{LC}$ has zero eigenvalues. Technically the eigenvalues need not to be exactly zero, it is sufficient the observer $\mathcal{O}$ had a fast impulse response which can be approximated by a finite impulse response $\mathbf{H}_k$.

Using the form (12) we will minimize the sum of squares by the following recursion based on the dynamic programming idea. First, the sum of squares will be decomposed to two sums (14) breaking the summation at the mth sample.

$$\begin{aligned} \sum_{k=0}^{t} \mathbf{v}'(k)\mathbf{v}(k) &= \sum_{k=0}^{m} \mathbf{v}'(k)\mathbf{v}(k) + \sum_{k=m+1}^{t} \mathbf{v}'(k)\mathbf{v}(k) \\ &= \Sigma_1 + \Sigma_2 \end{aligned} \tag{14}$$

The two sums $\Sigma_{1,2}$ can be related to the diagonal form of the constraints (12) as follows. The first sum Σ_1 is a function of those $\mathbf{v}(k)$ which affect the fist residual value $\mathbf{r}(m)$. Then, the first sum in (14) has to be minimized with respect to the $\mathbf{v}(0)$ regarding the linear constraint given by $\mathbf{r}(m)$. The minimizing $\mathbf{v}(0)$ value denoted as $\hat{\mathbf{v}}(0|m)$

will be an affine function of $\mathbf{v}(m)\ \dots\ \mathbf{v}(1)$. Substituting this affine function to Σ_1 one changes its analytic form to:

$$\rho(m,\mu)+\sum_{k=1}^{m}(\mathbf{v}(k)-\hat{\mathbf{v}}(k|m))'\mathbf{P}(m)(\mathbf{v}(k)-\hat{\mathbf{v}}(k|m)) \quad (15)$$

Here, $\mathbf{P}(m)$ is a symmetric matrix and $\hat{\mathbf{v}}(k|m)$ vectors. Their numeric values will be known at this point as they will be defined by the minimization with respect to $\mathbf{v}(0)$.

Absorbing the first square from the second sum adding it to the first sum and repeating this process recursively we can proceed up to the point were all terms from the second sum have already been absorbed by the first sum.

Then the minimum value of the sum of squares is $\rho(t,\mu)$ substituting zeros for $\mathbf{v}(k)-\hat{\mathbf{v}}(k|t)$, $t-m+1 \geq k \leq t$.

An interesting question is how many variables are the arguments to Σ_1, because this define the order of a filter which is equivalent to this recursive minimization. The conjecture is that a Kalman filter with order $n(\mathbf{v})m$ should be sufficient.

It is interesting this filter order can be lower than the original system order $n(\mathbf{x})$. In particular for $n(\mathbf{v})=1$ the order is roughly $n(\mathbf{y})$ times lower. This filter state is a vector of lagged $\mathbf{v}(k)$.

3.2 Equivalent System

Let the system for which the filter performing the recursive minimization is designed be called the equivalent system. The equivalent system as described in the previous section always has shift-register dynamics, i.e. the $\mathbf{A}_s$ matrix of its state representation has no non-zero eigenvalues. This $\mathbf{A}_s$ matrix is a zero matrix $(1{+}m)n(\mathbf{v})\times(1{+}m)n(\mathbf{v})$ with ones on its $n(\mathbf{v})$th upper diagonal – thus implementing shifting by $n(\mathbf{v})$. We will examine the minimum realization of the equivalent system in this section.

Suppose making a prediction of future $\mathbf{r}(k)$ residuals at time t, i.e. predicting $\mathbf{r}(t{+}1)$, $\mathbf{r}(t{+}2)$ and further on. Using the equivalent system we get (16).

$$\begin{aligned}\begin{bmatrix}\mathbf{r}(t{+}1)\\ \mathbf{r}(t{+}2)\\ \vdots\\ \mathbf{r}(t{+}m)\end{bmatrix} &= \dots\\ \dots &= \underbrace{\begin{bmatrix}\mathbf{H}_m & \mathbf{H}_{m-1} & \dots & \mathbf{H}_1\\ \mathbf{O} & \mathbf{H}_m & \dots & \mathbf{H}_2\\ \vdots & \vdots & & \vdots\\ \mathbf{O} & \mathbf{O} & \dots & \mathbf{H}_m\end{bmatrix}}_{\mathbf{W}}\begin{bmatrix}\mathbf{v}(t-m+1)\\ \mathbf{v}(t-m+2)\\ \vdots\\ \mathbf{v}(t)\end{bmatrix}\\ \dots &+ \begin{bmatrix}\mathbf{H}_0 & \mathbf{O} & \dots & \mathbf{O}\\ \mathbf{H}_1 & \mathbf{H}_0 & \dots & \mathbf{O}\\ \vdots & \vdots & & \vdots\\ \mathbf{H}_{m-1} & \mathbf{H}_{m-2} & \dots & \mathbf{H}_0\end{bmatrix}\begin{bmatrix}\mathbf{v}(t{+}1)\\ \mathbf{v}(t{+}2)\\ \vdots\\ \mathbf{v}(t{+}m)\end{bmatrix}\end{aligned} \quad (16)$$

Here $\mathbf{W}$ denotes a matrix defining the linear relationship between future residuals $\mathbf{r}_{t+m}^{t+1}$ and the lagged unknown inputs $\mathbf{v}_t^{t-m+1}$. We can think $\mathbf{W}$ is a matrix projecting the past unknown inputs to the future residuals. Thus, the minimal representation of the equivalent system state must be a set of linear functions of $\mathbf{v}_t^{t-m+1}$ which form a basis of the rows in $\mathbf{W}$. These linear functions allow the future residuals to be predicted. Thus, they can represent the system state. It is convenient to use the singular value decomposition of $\mathbf{W}$ as defined by (17) to find a suitable basis.

$$\mathbf{W}=[\,\mathbf{U}\ \mathbf{U}_0\,]\begin{bmatrix}\mathbf{\Sigma} & \mathbf{O}\\ \mathbf{O} & \mathbf{O}\end{bmatrix}[\,\mathbf{V}\ \mathbf{V}_0\,]' \quad (17)$$

Provided the significantly non-zero singular values in (17) are in $\mathbf{\Sigma}$, the equivalent system states are the following linear functions of $\mathbf{v}_t^{t-m+1}$: $\mathbf{\Sigma V}'$. For these state variables the following equivalent state space system can be defined (18).

$$\begin{aligned}\mathbf{A}_e &= \mathbf{\Sigma V}'\mathbf{A}_s\mathbf{V\Sigma}^{-1}\\ \mathbf{B}_e &= \mathbf{\Sigma V}'\begin{bmatrix}\mathbf{O}_{m\,n(\mathbf{v})\times n(\mathbf{v})}\\ \mathbf{I}_{n(\mathbf{v})}\end{bmatrix}\\ \mathbf{C}_e &= [\,\mathbf{H}_m\ \mathbf{H}_{m-1}\ \dots\ \mathbf{H}_1\,]\,\mathbf{V\Sigma}^{-1}\\ \mathbf{D}_e &= \mathbf{H}_0\end{aligned} \quad (18)$$

Matrix $\mathbf{A}_s$ is the shifting matrix with ones on its $n(\mathbf{v})$th upper diagonal. The minimal equivalent system representation is (19).

$$\begin{aligned}\mathbf{x}_e(t{+}1) &= \mathbf{A}_e\mathbf{x}_e(t)+\mathbf{B}_e\mathbf{v}(t)\\ \mathbf{r}(t) &= \mathbf{C}_e\mathbf{x}(t)+\mathbf{D}_e\mathbf{v}(t)+\mathbf{e}_e(t)\end{aligned} \quad (19)$$

Note that this equivalent system has the unknown input $\mathbf{v}(t)$ as its only input, $\mathbf{u}(t)$ has vanished. It has the dead-beat state observer residuals on its output. The matrix $\mathbf{A}_e$ will have all eigenvalues equal to zero.

For the case the signal $\mathbf{v}(t)$ has not enough degrees of freedom to satisfy the input output relationships (7), we have added an extra random term $\mathbf{e}_e(t)$ in (19). This term should be understood as a white noise measurement error with a variance much smaller than $\mathbf{v}(t)$.

We have changed the original white noise measurement errors $\mathbf{e}(t)$ considered in (7) to the white noise measurement errors $\mathbf{e}_e(t)$ considered in (19) as errors of the equivalent system outputs. Though this is inconsistent, we recall the term $\mathbf{e}(t)$ was added to the model equation to ensure the model can be consistent with any input/output data. This approximation is possible provided the variance of $\mathbf{e}(t)$ is negligible. Though the errors $\mathbf{e}(t)$ were not considered when the equivalent system was derived, we now re-define statistics $\rho(t,\mu)$ to (20).

$$\rho(t,\mu)=\sum_{k=0}^{t}\left(\hat{\mathbf{v}}'(k|t)\hat{\mathbf{v}}(k|t)+\hat{\mathbf{e}}_e'(k|t)\mathrm{var}^{-1}\{\mathbf{e}\}\,\hat{\mathbf{e}}_e(k|t)\right) \quad (20)$$

As already noted, this approximation is justified for $\hat{\mathbf{e}}(k|t)\to\mathbf{o}$.

3.3 The Main Result

Claim 1. The minimum order of the linear filter calculating system mode likelihood cannot be greater than $n(\mathbf{v})m$. Here, $m+1$ is the number of non-zero values in the deadbeat impulse response from the unknown inputs to the $\mathcal{O}$ residuals. The $n(\mathbf{v})$ is the number of the unknown inputs.

Proof: The order of the linear filter calculating the sufficient statistics equals the number of non-zero singular values in the SVD decomposition (17). This number cannot be greater than $n(\mathbf{v})m$, which the number of $\mathbf{W}$ columns.

The reduced bank consists of Kalman filters designed for the equivalent system, not for the original model of the mode. All Kalman filters accept the dead-beat observer residuals on their inputs. Note that we have used the dynamic programming idea to find the reduced Kalman filter, but the manipulation (14) related to the dynamic programming minimization is actually not a specific part of the reduced bank as the minimization can be done by the Kalman filters.

4. NUMERICAL EXAMPLE

4.1 System

Let us consider the state-space model (21). The system has two possible faults a and b modelled as two unknown inputs. The total number of fault scenarios is four: $\mathbf{F}_1 =$ empty matrix (no fault active), $\mathbf{F}_2 = \mathbf{F}_a$ (a active), $\mathbf{F}_3 = \mathbf{F}_b$ (b active), $\mathbf{F}_4 = [\mathbf{F}_a,\ \mathbf{F}_b]$ (both active). The matrices $\mathbf{G}_{1,2,3,4}$ analogously.

$$\mathbf{A} = \frac{1}{7}\begin{bmatrix} 1&1&0&0&0&1\\ 0&2&1&0&0&0\\ 0&0&3&1&0&0\\ 0&0&0&4&1&0\\ 0&0&0&0&5&1\\ 1&0&0&0&0&6 \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} 1&0\\0&1\\0&0\\0&0\\0&0\\0&0 \end{bmatrix} \mathbf{F}_a = \begin{bmatrix}1\\0\\0\\0\\0\\0\end{bmatrix} \mathbf{F}_b = \begin{bmatrix}0\\1\\0\\0\\0\\0\end{bmatrix} \qquad (21)$$

$$\mathbf{C} = \begin{bmatrix} 1&1&0&0&0&0\\ 0&0&1&1&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1 \end{bmatrix}$$

$$\mathbf{D} = \begin{bmatrix}0&0\\0&0\\0&0\\0&0\end{bmatrix} \mathbf{G}_a = \begin{bmatrix}1\\0\\0\\0\end{bmatrix} \mathbf{G}_b = \begin{bmatrix}0\\-1\\1\\0\end{bmatrix}$$

The deadbeat observer can be designed using the state injection matrix (22). It can be verified the matrix $\mathbf{A}-\mathbf{LC}$ has zero eigenvalues; up to rounding errors.

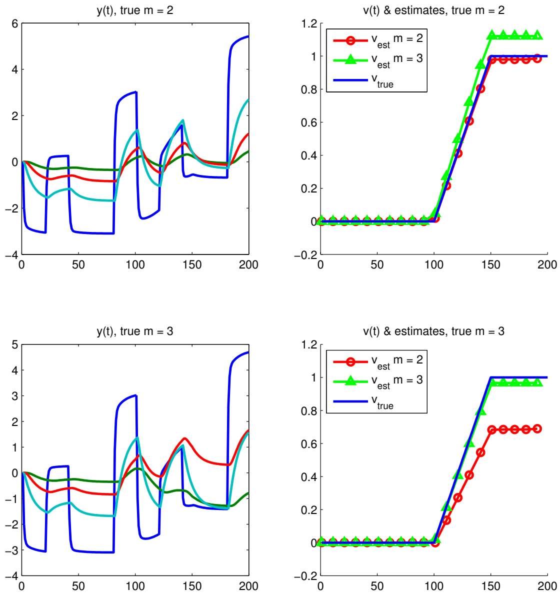

Fig. 1. System outputs in the left, true $\mathbf{v}(t)$ and $\hat{\mathbf{v}}(t|t,\mu)$ estimated by two filters $\mu = 2, 3$.

$$\mathbf{L} = \begin{bmatrix} 0.1429 & -0.0000 & 0 & 0.1429\\ 0.2531 & -0.1591 & 0 & -0.0651\\ 0.0432 & -0.1213 & 0 & 0.0863\\ -0.0863 & 1.0997 & 0.1429 & -0.1727\\ 0 & 0 & 0.7143 & 0.1429\\ 0.0271 & 0.0135 & 0 & 0.9113 \end{bmatrix} \qquad (22)$$

The deadbeat observer residuals can be represented as a moving average of the current unknown input value $\mathbf{v}(t)$ and two lagged values, i.e. $m = 2$. Now the state space representation of the reduced equivalent system for each system mode can be calculated using (18). For example for the mode $\mu = 2$ equivalent system is (23).

$$\mathbf{A}_2 = \begin{bmatrix} -0.3010 & -0.5354\\ 0.1692 & 0.3010 \end{bmatrix} \mathbf{B}_2 = \begin{bmatrix} -0.6550\\ -0.0413 \end{bmatrix}$$

$$\mathbf{C}_2 = \begin{bmatrix} -0.9333 & 0.1758\\ -0.0479 & -0.2854\\ 0 & 0\\ 0.0863 & -0.7144 \end{bmatrix} \mathbf{D}_2 = \begin{bmatrix}1\\0\\0\\0\end{bmatrix} \qquad (23)$$

For the two single fault scenarios $\mu = 2$ and $\mu = 3$ the equivalent systems for which the respective Kalman filters in the bank should be designed have order two. This is a considerable simplification as the original system order is six.

4.2 Simulation results

The left half of Fig. 1 shows 200 samples output data generated by the system (21) excited by a PRBS (random binary switched) signal on the two $\mathbf{u}(t)$ inputs. At the right the true fault signal $\mathbf{v}(t)$ is shown. It was a ramp signal starting at the sample 100 and increasing up to level one during 50 samples. From sample 150 up to 200 the fault signal was 1. This Figure also shows the smoothed

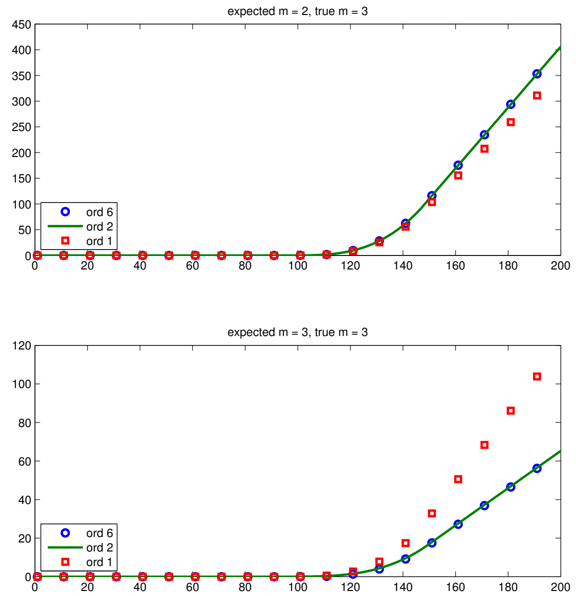

Fig. 2. $\rho(t, \mu)$ evaluated for the sixth order system (21) using either the full order Kalman filter or the reduced Kalman filter.

unknown input estimate estimated by two filters: for the modes $\mu = 2$ and $\mu = 3$. The filters were designed taking $\text{var}\{\mathbf{v}(t)\} = \mathbf{I}_{n(\mathbf{v})}$ and $\text{var}\{\mathbf{e}_e(t)\} = 1/25\mathbf{I}_{n(\mathbf{y})}$.

Fig. 2 shows how the quadratic norm $\rho(t, \mu)$ had been calculated when processing these data and the system was in the mode $\mu = 3$. Especially note the $\rho(t, \mu)$ values calculated by the full order Kalman filter of the sixth order and the reduced second order Kalman filter are the same, up to rounding errors ($\mathbf{e}(t)$ were not simulated). The line marked by squares shows the $\rho(t, \mu)$ values produced by a first order filter obtained taking only the largest singular value in (17). These results are no more equivalent to the full order filter results though the true modes could still be detected correctly.

5. CONCLUSION

This text answers a simple question related to the bank of Kalman filters: "Is it necessary to estimate the state variables as many times as there are process modes to evaluate the marginal probabilities of those modes?" The answer given here is negative (conditionally). We have proven it is sufficient to use the Kalman filter whose order is related to certain impulse response length (from unknown input to the auxiliary deadbeat observer residuals). Though we have assumed the process mode μ is constant, the algorithm can be generalized to time varying $\mu(t)$. The bank of reduced filters must be supplemented by a common linear time invariant prefilter. Thus, a part of the dynamics is factored out from the Kalman filters in the bank to be included in the prefilter for all of them only once.

Note that the result presented in this text is of practical importance. For a fault detection system designed for large scale linearized model, the order reduction can lead to substantial computational performance improvement; especially if $n(\mathbf{y}) \gg n(\mathbf{v})$.

Though our approach uses the dead beat observer, it is not a deterministic approach. The dead beat observer is not used to estimate the process state. Its use should not exaggerate the sensitivity to the noise therefore. Our method is a bridge between the stochastic and deterministic approaches. The more the number of unknown inputs $n(\mathbf{v})$ is reduced, the simpler are the Kalman filters. If $n(\mathbf{v}) \geq n(\mathbf{x})$, no reduction is possible.

REFERENCES

D. L. Simon, T. Kobayashi, *Hybrid Kalman Filter Approach for Aircraft Engine In-Flight Diagnostics: Sensor Fault Detection Case*, NASA/TM2006-214418 technical report, 2006

S. Krzemiñski and T. Kaczorek, Perfect reduced-order unknown-input observer for standard systems, *Bulletin Of The Polish Academy Of Sciences: Technical Sciences* Vol. 52, No. 2, 2004

V. Venkatasubramanian, R. Rengaswamy, K. Yin, S. N. Kavuri, *A review of process fault detection and diagnosis,* Computers and Chemical Engineering 27 (2003) 293-311

M. Basseville I.V. Nikiforov, *Detection of Abrupt Changes, Theory and Application*, Prentice-Hall, 2003, ISBN:0-13-126780-9

I. Jikuya, M. Verhaegen, *Deadbeat Observer Based Detection And Estimation Of A Jump In LTI Systems* In the 15th Triennial World Congress of IFAC, Barcelona, Spain, 2002

Y. Bar-Shalom, X.-R. Li, T. Kirubarajan, *Estimation with Applications to Tracking and Navigation,* Wiley Interscience, 2001.

H.A.P. Blom and Y. Bar-Shalom, The interacting multiple model algorithm for system with Markovian switching coefficients, *IEEE Transactions on Automatial Control*, 33, pp. 780-783, 1988.

H. E. Rauch, F. Tung, C. T. Striebel, *Maximum likelihood estimates of linear dynamic systems,* AIAA Journal 3(8) (1965) 1445-1450.