Proceedings of the 17th World Congress
The International Federation of Automatic Control
Seoul, Korea, July 6-11, 2008

Extracting Worst Case Perturbations for Robustness Analysis of Parameter-Dependent LTI Systems

Yusuke Onishi Yoshio Ebihara Tomomichi Hagiwara

Department of Electrical Engineering, Kyoto University, Kyotodaigaku-Katsura, Nishikyo-ku, Kyoto 615-8510, Japan (Tel: +81-75-383-{2253,2252,2250}; E-mail: onishi@jaguar.kuee.kyoto-u.ac.jp, {ebihara, hagiwara}@kuee.kyoto-u.ac.jp).

Abstract: In this paper, we deal with robust performance analysis problems of LTI systems depending on uncertain parameters. By following existing scaling-based approaches, we firstly derive computationally tractable parameter-independent LMI conditions to assess the robust performance, which are conservative in general. What makes the present approach novel is to take the dual of those LMIs so that we can conclude the exactness of the analysis results. More precisely, we clarify that if the computed dual solution satisfies a certain rank condition, then we can ensure that the robust performance is never attained. In particular, we can extract the worst case perturbation that violates the underlying performance. Thus we provide viable tests for the exactness verification of LMI-based robust performance analysis.

Keywords: Robust performance, uncertain systems, linear matrix inequalities, duality theory, linear fractional transformation.

1. INTRODUCTION

This paper is concerned with the robustness analysis problems of linear time-invariant (LTI) systems depending on uncertain parameters (Barmish [1994]). These problems are naturally formulated as feasibility problems of linear matrix inequalities (LMIs) whose coefficient matrices are affected by the uncertain parameters. These LMIs, so called robust LMIs, arise when we deal with whole variety of robustness analysis and synthesis problems (see, ex., Scherer [2005, 2006]). Unfortunately, however, robust LMI problems are essentially intractable NP-hard problems. In view of these facts, main focus has been laid upon deriving sufficient LMI conditions that are less conservative and efficiently solved via LMI solvers.

Recently, stimulated by the theoretical advances on polynomial optimization via sum-of-squares decompositions (Lasserre [2001], Parrilo [2003]), novel contributions have been made to deal with robust LMIs in an asymptotically exact fashion (see, ex., Bliman [2004a], Henrion et al. [2004], Scherer [2005, 2006], Scherer and Hol [2006]). Among them, Scherer [2005, 2006] and Scherer and Hol [2006] showed a unified way for LMI relaxation, which enables us to obtain a hierarchy of LMIs with theoretical guarantee of asymptotic exactness. In addition, by taking the dual of these LMIs, viable tests for the exactness verification have been provided (Scherer [2005, 2006]).

In Ebihara et al. [2007], the authors pursued the direction related to but yet distinct from Scherer [2005, 2006], focusing on robustness analysis problems of continuous-time uncertain LTI systems. More precisely, the authors provided sound rank conditions for the exactness verification based on the particular block-moment matrix structure of the dual solution. This result is closely related to the LMI relaxation for polynomial matrix inequality (PMI) problems suggested in Henrion and Lasserre [2006], which is a genuine matrix counterpart of those in Lasserre [2001], Henrion and Lasserre [2005]. In comparison with the direct formulation as PMIs, one of the salient feature of the approach in Ebihara et al. [2007] is that it exploits the block-moment matrix structure of the dual solution so that the associated computational burden can keep moderate.

Our primary concern in this paper is to extend the results in Ebihara et al. [2007] so that we can deal with discrete-time system analysis in a unified fashion. To this end, we first analyze the generalized Lyapunov inequality (Scherer [2005]) for matrices depending on uncertain parameters. By following Ebihara et al. [2007], we convert this robust LMI into a numerically verifiable LMI via (D, G)-scaling (Meinsma et al. [1997]) and take its dual for the exactness verification. Based on these preliminary results, we next clarify that, if the computed dual solution satisfies a certain rank condition, then the original robust LMI never holds. In particular, we can extract the worst case parameter perturbation that violates the robust LMI. We also show that these results can readily be extended to robust dissipative performance analysis by using the idea of Hamiltonian eigenvalue tests (Hagiwara [2005], Zhou and Doyle [1998]). Thus we can obtain consistent results to our preceding results for continuous-time system analysis (Ebihara et al. [2007]).

We use the following notations in this paper. The symbol $\mathbf{S}_n$ denotes the set of real symmetric matrices of the size n. For a matrix $A \in \mathbf{R}^{n\times n}$, we denote by $\lambda(A)$ the set of its eigenvalues. For matrices $A, B \in \mathbf{R}^{n\times n}$, the symbol $\lambda(A, B)$ denotes the set of their generalized eigenvalues, i.e, the set of $\lambda \in \mathbf{C}$ satisfying $Ax = \lambda Bx$ for $x \in \mathbf{C}^n \setminus 0$. For $A \in \mathbf{S}_n$, we denote by $\lambda_k(A)$ $(k = 1, \cdots, n)$

its k-th eigenvalue. In addition, we use the notation $\mathrm{In}(A) = (p, \nu, \zeta)$ to indicate that the number of the positive, negative and zero eigenvalues of A are p, ν and ζ, respectively. For a matrix $A \in \mathbf{R}^{n\times m}$, we denote its Moore-Penrose generalized inverse by $A^\dagger$. For a matrix A with partition

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad A_{11} \in \mathbf{R}^{n_1\times m_1}, \quad A_{22} \in \mathbf{R}^{n_2\times m_2},$$

we define $\lceil A \rfloor^{n_1} := [\ A_{11}\ A_{12}\ ]$ and $\lceil A \rfloor_{n_2} := [\ A_{21}\ A_{22}\ ]$. In particular, if A is square and $n_i = m_i$ $(i = 1, 2)$, we define $\langle A \rangle^{n_1} := A_{11}$ and $\langle A \rangle_{n_2} := A_{22}$. Finally, for given $q, r, s \in \mathbf{R}$ satisfying $qr - s^2 < 0$, we define $\mathcal{D}(q, r, s)$ and $\partial\mathcal{D}(q, r, s)$ as follows:

$$\begin{aligned} \mathcal{D}(q, r, s) &:= \left\{ \lambda \in \mathbf{C} : \begin{bmatrix} 1 \\ \lambda \end{bmatrix}^* \begin{bmatrix} r & s \\ s & q \end{bmatrix} \begin{bmatrix} 1 \\ \lambda \end{bmatrix} < 0 \right\}, \\ \partial\mathcal{D}(q, r, s) &:= \left\{ \lambda \in \mathbf{C} : \begin{bmatrix} 1 \\ \lambda \end{bmatrix}^* \begin{bmatrix} r & s \\ s & q \end{bmatrix} \begin{bmatrix} 1 \\ \lambda \end{bmatrix} = 0 \right\}. \end{aligned}$$

2. PROBLEM FORMULATION AND GENERALIZED LYAPUNOV INEQUALITY

Let us consider the rational functions $M(\theta) : \mathbf{R}^L \to \mathbf{R}^{n\times n}$ and $J(\theta) : \mathbf{R}^L \to \mathbf{R}^{n\times n}$ represented by

$$\begin{aligned} M(\theta) &= M_{11} + M_{12}(I_l - \Delta_l(\theta)M_{22})^{-1}\Delta_l(\theta)M_{21}, \\ J(\theta) &= J_{11} + J_{12}(I_l - \Delta_l(\theta)J_{22})^{-1}\Delta_l(\theta)J_{21}, \\ \Delta_l(\theta) &= \sum_{i=1}^{L} \theta_i E_{l,i}. \end{aligned} \tag{1}$$

Here, the uncertain parameter θ varies over $\Theta_\delta := [-\delta, \delta]^L$. The matrices $E_{l,i} \in \mathbf{R}^{l\times l}$ $(i = 1, \cdots, L)$ are nonzero diagonal matrices whose diagonal entries are zero or one and satisfy $\sum_{i=1}^{L} E_{l,i} = I_l$. We assume that the LFTs in (1) are *well-posed*, i.e.,

$$\det(I_l - \Delta_l(\theta)M_{22}) \neq 0,\ \det(I_l - \Delta_l(\theta)J_{22}) \neq 0,\ \forall\theta \in \Theta_\delta^L.$$

Under these preparations, in this paper, we consider the next problem which is basically motivated from the robust $\mathcal{D}$-stability analysis problem discussed in Barmish [1994].

Problem 1. For given $q, r, s \in \mathbf{R}$ with $qr - s^2 < 0$, determine whether the following condition holds or not:

$$\{\lambda(M(\theta), J(\theta)) \cap \partial\mathcal{D}(q, r, s)\} = \emptyset \quad \forall\theta \in \Theta_\delta^L. \tag{2}$$

The next Lemma shows that Problem 1 can be reduced into a feasibility test of a robust LMI. This result readily follows from the generalized Lyapunov inequality for uncertainty-free matrices (Scherer [2005]).

Lemma 1. For given $q, r, s \in \mathbf{R}$ satisfying $qr - s^2 < 0$, the condition (2) holds if and only if there exists $P(\theta) : \mathbf{R}^L \to \mathbf{S}_n$ such that

$$\begin{bmatrix} rP(\theta) & sP(\theta) \\ sP(\theta) & qP(\theta) \end{bmatrix} - \begin{bmatrix} M(\theta)^T \\ -J(\theta)^T \end{bmatrix} \begin{bmatrix} M(\theta)^T \\ -J(\theta)^T \end{bmatrix}^T \prec 0 \quad \forall\theta \in \Theta_\delta^L. \tag{3}$$

The matrix-valued function $P(\theta)$ is often referred to as a parameter-dependent multiplier (Chen and Sugie [1996]). From the facts that (i) $M(\theta)$ and $J(\theta)$ are continuous with respect to $\theta \in \Theta_\delta^L$, (ii) the set Θ_δ^L is compact, (iii) the inequality (3) is strict, the multiplier $P(\theta)$ can be restricted to be a polynomial without loss of generality (Bliman [2004b]). Nevertheless, the robust LMI (3) is still intractable mainly due to the following reasons:

1. The explicit degree of the polynomial $P(\theta)$ that enables us to achieve exact analysis is not known a priori, even though several results were obtained for robust $\mathcal{D}$-stability analysis cases (Henrion et al. [2004], Zhang et al. [2003]).
2. Suppose we fix the multiplier $P(\theta)$ to be a polynomial of finite-degree. Then, in the single parameter case, we can reduce the robust LMI (3) into a parameter-independent LMI in an exact fashion via (D, G)-scaling (Meinsma et al. [1997]). However, this is not straightforwardly achieved in the multiple uncertain parameter cases, and those LMIs resulting from LMI relaxations are expected to be conservative in general.

In view of these current state of the art, all we can do amounts to resorting to LMI conditions that are conservative in general. To obtain exactness certificates even under these difficult situation, in this paper, we follow the idea in Ebihara et al. [2007] and consider the dual LMI problem. This enables us to derive viable tests for the exactness verification as we see in the next two sections. We note that the discussion in Ebihara et al. [2007] is restricted to the case where $J(\theta) = I$ and $(q, r, s) = (0, 0, 1)$ in Problem 1.

3. ROBUST $\partial\mathcal{D}$-NONSINGULARITY ANALYSIS

3.1 Single Uncertain Parameter Case

Let us first consider the single parameter case. In this case, we can state the next result.

Theorem 1. For given odd number N and $\delta > 0$, let us consider the following LMI-LME conditions with respect to $\mathcal{H} \in \mathbf{S}_{(N+3)n_b}$ $(n_b := n + l)$:
Find $\mathcal{H} \in \mathbf{S}_{(N+3)n_b} \setminus \{0\}$ such that

$$\begin{aligned} &\mathcal{H} = \begin{bmatrix} \mathcal{H}_0 & \mathcal{H}_1 & \cdots & \mathcal{H}_{(N+1)/2} \\ \mathcal{H}_1 & \iddots & \iddots & \vdots \\ \vdots & \iddots & \iddots & \mathcal{H}_N \\ \mathcal{H}_{(N+1)/2} & \cdots & \mathcal{H}_N & \mathcal{H}_{N+1} \end{bmatrix} \succeq 0, \\ &\delta^2 \langle \mathcal{H} \rangle^{(N+1)n_b} - \langle \mathcal{H} \rangle_{(N+1)n_b} \succeq 0, \\ &\begin{bmatrix} 0_{2n,2l} \\ -I_{2l} \\ \hdashline \mathcal{V} \end{bmatrix}^T \begin{bmatrix} \mathcal{H}_0 & \mathcal{H}_1 \\ \mathcal{H}_1 & \mathcal{H}_2 \end{bmatrix} \begin{bmatrix} 0_{2n,2l} \\ -I_{2l} \\ \hdashline \mathcal{V} \end{bmatrix} = 0, \quad \mathcal{W}^T \mathcal{H}_0 \mathcal{W} = 0, \\ &\langle \begin{bmatrix} rI_n & sI_n \\ sI_n & qI_n \end{bmatrix} \langle \mathcal{H}_j \rangle^{2n} \rangle^n + \langle \begin{bmatrix} rI_n & sI_n \\ sI_n & qI_n \end{bmatrix} \langle \mathcal{H}_j \rangle^{2n} \rangle_n = 0 \\ &\qquad (j = 0, \cdots, N) \end{aligned} \tag{4}$$

where

$$\mathcal{V} := \begin{bmatrix} M_{21} & 0 & M_{22} & 0 \\ 0 & J_{21} & 0 & J_{22} \end{bmatrix}^T, \ \mathcal{W} := [\, M_{11} \ -J_{11} \ M_{12} \ -J_{12} \,]^T.$$

Then, the following two assertions hold:
(i) If (4) is infeasible, then the condition (2) holds.
(ii) Suppose (4) is feasible and has a solution $\mathcal{H}$. Let us denote the full-rank factorization of $\mathcal{H}$ by

$$\mathcal{H} = \begin{bmatrix} H_0 \\ \vdots \\ H_{(N+1)/2} \end{bmatrix} \begin{bmatrix} H_0 \\ \vdots \\ H_{(N+1)/2} \end{bmatrix}^T \quad (H_j \in \mathbf{R}^{2n_b\times m}) \tag{5}$$

and define

$$\begin{aligned} \overline{H} &:= \left[(\lceil H_0 \rfloor^n)^T \ \cdots \ (\lceil H_{(N-1)/2} \rfloor^n)^T \right]^T \in \mathbf{R}^{(N+1)n/2\times m}, \\ \underline{H} &:= \left[(\lceil H_1 \rfloor^n)^T \ \cdots \ (\lceil H_{(N+1)/2} \rfloor^n)^T \right]^T \in \mathbf{R}^{(N+1)n/2\times m}. \end{aligned}$$

Then, if

$$\mathrm{rank}(\overline{H}) = \mathrm{rank}(\mathcal{H}), \tag{6}$$

the condition (2) never holds. More precisely, if we define $\Omega := \overline{H}^{\dagger}\underline{H}$, then this matrix Ω satisfies $\Omega \in \mathbf{S}_m$ and $\lambda(\Omega) \subset \Theta_\delta$. In addition, we have $\{\lambda(M(\lambda_k(\Omega)), J(\lambda_k(\Omega))) \cap \partial\mathcal{D}(q,r,s)\} \neq \emptyset$ for all $k = 1, \cdots, m$.

Due to limited space, we omit the proof of this theorem. It should be noted that, in the case where we have only one uncertain parameter, we can apply those results in (Ebihara and Hagiwara [2005], Meinsma et al. [1997]) so that we can obtain an LMI that ensures the existence of N-th degree polynomial $P_N(\theta)$ satisfying (3) in an exact fashion. The LMI (4) corresponds to the dual of this LMI, which follows immediately from the convex duality theory (Balakrishnan and Vandenberghe [2003]). From this procedure, we see that the LMI (4) is infeasible if and only if (3) is feasible via $P(\theta) = P_N(\theta)$. Thus, the assertion (i) in Theorem 1 readily follows.

The importance of the theorem lies in the assertion (ii), which provides a viable tests for the exactness verification. Namely, if the dual LMI (4) is feasible and if the computed dual solution satisfies the rank condition (6), then we can conclude that (2) never holds. In addition, the worst case parameter perturbation θ_{w} such that $\{\lambda(M(\theta_{\mathrm{w}}), J(\theta_{\mathrm{w}})) \cap \partial\mathcal{D}(q,r,s)\} \neq \emptyset$ can be obtained as eigenvalues of the matrix $\Omega \in \mathbf{S}_m$, which can be readily computed by constructing Ω from the dual solution $\mathcal{H}$. This result surely goes beyond the standard primal LMI approach that allows us to conclude the assertion (i) only. We note that the key to derive the result (ii) lies on the particular block-Hankel matrix structure of the dual solution $\mathcal{H}$.

Remark 1. The size of the LMI (3) and the rank condition for the exactness verification (6) of course depend on N, the degree of the employed multiplier. By increasing N, we can show that the condition (6) becomes more likely to be satisfied in the sense that if there exists a dual solution satisfying (3) and (6) for $N = N_1$, then there always exists a dual solution satisfying (3) and (6) for $N \geq N_1$.

Remark 2. The exactness test (6) should be compared with those reported in the literature (Scherer [2005, 2006]). To this end, for simplicity, let us consider the robust $\mathcal{D}$-stability analysis case where $J(\theta) = I$. In this case, we can confirm that the existence of the worst case parameter is also ensured if the following condition holds:

$$\exists\theta \in \Theta_\delta \text{ such that } \mathcal{H}_k = \theta^k\mathcal{H}_0 \ (k = 1, \cdots, N+1). \tag{7}$$

It is also true that if $\mathrm{rank}(\mathcal{H}) = 1$, the above condition is automatically satisfied. Namely, the exactness verification test (7) goes beyond the common rank-one exactness principle $\mathrm{rank}(\mathcal{H}) = 1$.

On the other hand, from the form of the equality constraints in (4), we can readily prove that if (7) holds, then (6) holds. Here, we note that, even though the discussion in Scherer [2005, 2006] does not clearly mention the block-Hankel matrix structure of the dual solution, we can confirm that the exactness verification test to (4), in the spirit of Scherer [2005, 2006], can be given as (7). It follows that, at least in the case where we deal with robust $\mathcal{D}$-stability analysis problems, the suggested exactness test (6) is more general than those in Scherer [2005, 2006].

3.2 Multiple Uncertain Parameter Case

Let us next consider the case where we have L-multiple uncertain parameters in Problem 1. To tackle this problem, we consider an affine multiplier of the form $P(\theta) = P_0 + \sum_{i=1}^{L}\theta_i P_i$ in (3). Then, by following a close argument to Ebihara et al. [2007], we are led to the next result.

Theorem 2. Let us consider the following LMI-LME conditions with respect to $\mathcal{H} \in \mathbf{S}_{2(L+1)n_b}$:
Find $\mathcal{H} \in \mathbf{S}_{2(L+1)n_b} \setminus \{0\}$ such that

$$\begin{aligned}
&\mathcal{H} = \begin{bmatrix} \mathcal{H}_{00} & \mathcal{H}_{01} & \cdots & \mathcal{H}_{0L} \\ \mathcal{H}_{01} & \mathcal{H}_{11} & & \vdots \\ \vdots & & \ddots & \mathcal{H}_{L-1,L} \\ \mathcal{H}_{0L} & \cdots & \mathcal{H}_{L-1,L} & \mathcal{H}_{L,L} \end{bmatrix} \succeq 0, \\
&\delta^2\mathcal{H}_{00} - \mathcal{H}_{ii} \succeq 0 \ (i = 1, \cdots, L), \\
&\begin{bmatrix} 0_{2n,2l} \\ -I_{2l} \\ \hline \mathcal{V}_1 \\ \vdots \\ \mathcal{V}_L \end{bmatrix}^T \mathcal{H} \begin{bmatrix} 0_{2n,2l} \\ -I_{2l} \\ \hline \mathcal{V}_1 \\ \vdots \\ \mathcal{V}_L \end{bmatrix} = 0, \quad \mathcal{W}^T\mathcal{H}_{00}\mathcal{W} = 0, \\
&\langle \begin{bmatrix} rI_n & sI_n \\ sI_n & qI_n \end{bmatrix} \langle\mathcal{H}_{0j}\rangle^{2n}\rangle^n + \langle \begin{bmatrix} rI_n & sI_n \\ sI_n & qI_n \end{bmatrix} \langle\mathcal{H}_{0j}\rangle^{2n}\rangle_n = 0 \\
&\qquad (j = 0, \cdots, L)
\end{aligned} \tag{8}$$

where

$$\mathcal{V}_i := \begin{bmatrix} E_{l,i}M_{21} & 0 & E_{l,i}M_{22} & 0 \\ 0 & E_{l,i}J_{21} & 0 & E_{l,i}J_{22} \end{bmatrix}^T \ (i = 1, \cdots, L).$$

Then, the following two assertions hold:
(i) If (8) is infeasible, then condition (2) holds.
(ii) Suppose (8) is feasible and has a solution $\mathcal{H}$. Then, if

$$\mathrm{rank}(\langle\mathcal{H}_{00}\rangle^n) = \mathrm{rank}(\mathcal{H}), \tag{9}$$

the condition (2) never holds. More precisely, if we denote the full-rank factorization of $\mathcal{H}$ by

$$\mathcal{H} = \begin{bmatrix} H_0 \\ \vdots \\ H_L \end{bmatrix} \begin{bmatrix} H_0 \\ \vdots \\ H_L \end{bmatrix}^T, \ H_j \in \mathbf{R}^{2n_b \times m} \ (j = 0, \cdots, L) \tag{10}$$

and define $\Omega_i := (\lceil H_0 \rfloor^n)^{\dagger} \lceil H_i \rfloor^n \ (i = 1, \cdots, L)$, then these matrices satisfy $\Omega_i \in \mathbf{S}_m$, $\lambda(\Omega_i) \subset \Theta_\delta^L \ (i = 1, \cdots, L)$ and share all eigenvectors $u_k \in \mathbf{C}^m (k = 1, \cdots, m)$ in common. In addition, if we denote by $\lambda_k(\Omega_i) \ (i = 1, \cdots, L)$ the eigenvalue of Ω_i corresponding to the common eigenvector u_k, we have $\{\lambda(M(\theta_{\mathrm{w},k}), J(\theta_{\mathrm{w},k})) \cap \partial\mathcal{D}(q,r,s)\} \neq \emptyset$ for all $k = 1, \cdots, m$ where $\theta_{\mathrm{w},k} = [\ \lambda_k(\Omega_1) \ \cdots \lambda_k(\Omega_L)\ ]^T$.

We also omit the proof for this theorem. Again, the assertion (ii) is important in this theorem, which indicates that if the dual LMI (8) is feasible and the computed dual solution satisfies the rank condition (9), then we can conclude that (2) never holds. In addition, the worst case parameter perturbations can readily be extracted as eigenvalues of $\Omega_i \in \mathbf{S}_m \ (i = 1, \cdots, L)$ corresponding to the common eigenvector $u_k \ (k = 1, \cdots, m)$.

Remark 3. Even though we have restricted our attention to the affine multiplier in Theorem 2, it is possible to employ higher-degree polynomial multipliers and derive corresponding rank conditions for the exactness verification. However, extensive numerical experiments indicate that, for most of problem instances, we can obtain exact results via the affine multiplier. Thus, we do not pursue the direction of higher-degree polynomial multipliers in this paper.

3.3 Numerical Example

Let us consider the discrete-time system described by

$$x(t+1) = A(\theta)x(t),\ A(\theta) = A_0 + \theta_1 E_1 + \theta_2 E_2. \tag{11}$$

Here, the matrices A_0, E_1 and E_2 are given in Example 3 of Ramos [2001] and $\theta_1 \in [\,-0.0615, 0.8822\,]$, $\theta_2 \in [\,-0.0793, 0.7977\,]$. The problem we posed here is to analyze the robust stability of this discrete-time system.

Since the parameter variation is asymmetrical with respect to the origin, our method cannot be applied directly to this problem. To get around this difficulty, we first determine the nominal parameter $\theta_{\rm c} = [\,0.41035\ 0.35920\,]^T$ by taking the center of each parameter's variation. This allows us to represent the parameter variation as $\theta_i \in [\,\theta_{{\rm c}i} - \theta_{{\rm r}i}\,, \theta_{{\rm c}i} + \theta_{{\rm r}i}\,]$ $(i = 1, 2)$ where $\theta_{\rm r} = [\,0.47185\ 0.43850\,]^T$. It follows that we can describe $A(\theta)$ as an LFT form (1) where

$$M_{11} = A_0 + \theta_{\rm c1}E_1 + \theta_{\rm c2}E_2,\ M_{12} = [\,I_4\ I_4\,],\ M_{22} = 0,$$
$$M_{21} = \begin{bmatrix} \theta_{\rm r1}E_1 \\ \theta_{\rm r2}E_2 \end{bmatrix},\ \Delta_8(\hat\theta) = \begin{bmatrix} \hat\theta_1 I_4 & 0 \\ 0 & \hat\theta_2 I_4 \end{bmatrix}\ (\hat\theta \in \widehat\Theta_1^2 = [-1,1]^2).$$

Through this equivalent problem reformulation, we solved the dual LMI (8) in Theorem 2. It turns out that (8) is infeasible and thus we can readily conclude this system is robustly stable.

We next seek for the robust stability margin $\delta_{\max}$, which is defined by the maximal value such that the system remains stable for all $\hat\theta \in \widehat\Theta_\delta^2 = [-\delta, \delta]^2$. To this end, we carried out a bisection search over δ by regarding the parameter range as $\widehat\Theta_\delta^2$. At the minimal value of δ such that (8) is feasible, we examined whether the rank condition (9) holds. It turns out that $\mathrm{rank}(\langle\mathcal{H}_{00}\rangle^4) = \mathrm{rank}(\mathcal{H}) = 2$ and thus the suggested rank condition is satisfied. The worst case parameter that destabilizes this system was obtained as $\theta_{\rm w} = [\,-0.0616\ -0.0794\,]^T$. We can confirm that

$$\lambda(A(\theta_{\rm w})) = \{0.4066 \pm 0.9136i,\ -0.1473 \pm 0.3345i\},$$

whose absolute values are 1.0000, 1.0000, 0.3655 and 0.3655, respectively.

4. ROBUST DISSIPATION PERFORMANCE ANALYSIS

Based on the preceding detailed analysis on the generalized Lyapunov inequality, we next move on to the robust dissipation performance analysis of uncertain LTI systems. For simplicity, we focus our attention on the discrete-time robust H_∞ performance analysis problem described below.

Problem 2. Let us given rational functions $A(\theta) : \mathbf{R}^L \to \mathbf{R}^{n\times n}$, $B(\theta) : \mathbf{R}^L \to \mathbf{R}^{n\times p_1}$, $C(\theta) : \mathbf{R}^L \to \mathbf{R}^{p_2\times n}$, $D(\theta) : \mathbf{R}^L \to \mathbf{R}^{p_2\times p_1}$ with no poles over Θ_δ^L. We assume that $A(\theta)$ is Schur stable for all $\theta \in \Theta_\delta^L$. With these matrices, let us consider the discrete-time LTI system described by

$$P(z,\theta) = \left[\begin{array}{c|c} A(\theta) & B(\theta) \\ \hline C(\theta) & D(\theta) \end{array}\right]. \tag{12}$$

Then, for given $\gamma > 0$, determine whether

$$\|P(z,\theta)\|_\infty < \gamma \quad \forall\theta \in \Theta_\delta^L \tag{13}$$

holds or not.

In Ebihara et al. [2007], the continuous-time counterpart of this problem was investigated. In particular, by using the idea of the Hamiltonian eigenvalue tests (Hagiwara [2005], Zhou and Doyle [1998]), the problem is first reduced into a $\partial\mathcal{D}$-nonsingularity analysis problem. Then, based on the similar results to Theorems 1 and 2, effective analysis methods with exactness verification have been proposed.

We follow this strategy to deal with the discrete-time robust H_∞ performance analysis problem. To this end, we introduce the next lemma.

Lemma 2. (Hagiwara [2005]) Let us consider the discrete-time LTI system $P(z) = \{A, B, C, D\}$ with A being Schur stable. Then, $\|P(z)\|_\infty < \gamma$ holds if and only if the following three conditions hold:

(i) The matrix D satisfies
$$R_\gamma := D^TD - \gamma^2 I \prec 0. \tag{14}$$
(ii) For one z_0 taken from $\partial\mathcal{D}(1,-1,0)$ at one's discretion,
$$\mathrm{In}\,(F_\gamma - F_{z_0}) = (n, n, 0) \tag{15}$$
holds where
$$F_\gamma := \begin{bmatrix} -BR_\gamma^{-1}B^T & A - BR_\gamma^{-1}D^TC \\ A^T - C^TDR_\gamma^{-1}B^T & C^TC - C^TDR_\gamma^{-1}D^TC \end{bmatrix},$$
$$F_{z_0} := \begin{bmatrix} 0 & z_0 I \\ z_0^* I & 0 \end{bmatrix}.$$
(iii) The generalized eigenvalue condition
$$\{\lambda(M_\gamma, J_\gamma) \cap \partial\mathcal{D}(1,-1,0)\} = \emptyset \tag{16}$$
holds where
$$J_\gamma := \begin{bmatrix} 0 & I \\ A^T - C^TDR_\gamma^{-1}B^T & C^TC - C^TDR_\gamma^{-1}D^TC \end{bmatrix},$$
$$M_\gamma := \begin{bmatrix} -BR_\gamma^{-1}B^T & A - BR_\gamma^{-1}D^TC \\ I & 0 \end{bmatrix}.$$

When dealing with uncertainty-free systems, it is straightforward to verify the conditions (i), (ii) and (iii). However, if the system matrices are affected by the uncertain parameters as in Problem 2, those matrices R_γ, F_γ, J_γ and M_γ depend on the parameter θ as in $R_\gamma(\theta)$, $F_\gamma(\theta)$, $J_\gamma(\theta)$ and $M_\gamma(\theta)$ and thus it is far from obvious to check the corresponding conditions (i), (ii) and (iii). In addition, the conditions (i) and (ii) particularly appear for discrete-time system analysis and this fact makes the problem more complicated in comparison with the continuous-time case.

To deal with Problem 2 by means of Lemma 2, we first note that $D(\theta)$, $F_\gamma(\theta)$, $J_\gamma(\theta)$ and $M_\gamma(\theta)$ admit LFT representation of the form

$$\begin{aligned} D(\theta) &= D_{11} + D_{12}(I_q - \Delta_q(\theta)D_{22})^{-1}\Delta_q(\theta)D_{21}, \\ F_\gamma(\theta) &= F_{11} + F_{12}(I_r - \Delta_r(\theta)F_{22})^{-1}\Delta_r(\theta)F_{21}, \\ J_\gamma(\theta) &= J_{11} + J_{12}(I_r - \Delta_r(\theta)J_{22})^{-1}\Delta_r(\theta)J_{21}, \\ M_\gamma(\theta) &= M_{11} + M_{12}(I_r - \Delta_r(\theta)M_{22})^{-1}\Delta_r(\theta)M_{21} \end{aligned} \tag{17}$$

where $\Delta_q(\theta) = \sum_{i=1}^L \theta_i E_{q,i}$, $\Delta_r(\theta) = \sum_{i=1}^L \theta_i E_{r,i}$ and $\sum_{i=1}^L E_{q,i} = I_q$, $\sum_{i=1}^L E_{r,i} = I_r$. These LFT representations are always possible since $A(\theta)$, $B(\theta)$, $C(\theta)$ and $D(\theta)$ are rational. Then, it is apparent from Lemma 2 that $\|P(z,\theta)\|_\infty < \gamma$ $(\forall\theta \in \Theta_\delta^L)$ holds if and only if the following three conditions hold:

$$D(\theta)^TD(\theta) - \gamma^2 I \prec 0 \quad \forall\theta \in \Theta_\delta^L, \tag{18}$$

$$\mathrm{In}\,(F_\gamma(\theta) - F_{z_0}) = (n, n, 0) \quad \forall\theta \in \Theta_\delta^L, \tag{19}$$

$$\{\lambda(M_\gamma(\theta), J_\gamma(\theta)) \cap \partial\mathcal{D}(1,-1,0)\} = \emptyset \quad \forall\theta \in \Theta_\delta^L. \tag{20}$$

Since we can apply Theorems 1 and 2 for the analysis of (20), it remains to show how we deal with (18) and (19).

In the sequel, we naturally assume that the nominal performance condition $\|P(z,0)\|_\infty < \gamma$ holds. It should be emphasized that this in particular implies

$$D_{11}^T D_{11} - \gamma^2 I \prec 0, \tag{21}$$

$$\mathrm{In}\,(F_{11} - F_{z_0}) = (n, n, 0)\,. \tag{22}$$

Then, from continuity arguments, we can rewrite (18) and (19) equivalently as follows:

$$\det(D(\theta)^T D(\theta) - \gamma^2 I) \neq 0 \quad \forall \theta \in \Theta_\delta^L, \tag{23}$$

$$\det(F_\gamma(\theta) - F_{z_0}) \neq 0 \quad \forall \theta \in \Theta_\delta^L. \tag{24}$$

It is also true that the above condition is equivalent to

$$(F_\gamma(\theta) - F_{z_0})^*(F_\gamma(\theta) - F_{z_0}) \succ 0 \quad \forall \theta \in \Theta_\delta^L. \tag{25}$$

With these equivalent reformulations, we can deal with (18) and (19) very easily in the single parameter case.

4.1 Single Uncertain Parameter Case

Let us consider the single parameter case. In this case, we can verify (18) and (19) exactly by computing eigenvalues of a fixed matrix as shown in the next Lemmas.

Lemma 3. The condition (23) holds if and only if

$$\left\{\lambda\left(\begin{bmatrix} -\gamma^2 I & D_{11}^T & 0 & 0 \\ D_{11} & -I & 0 & D_{12} \\ 0 & 0 & 0 & -I \\ 0 & D_{12}^T & -I & 0 \end{bmatrix}^{-1}\begin{bmatrix} 0 & 0 & D_{21}^T & 0 \\ 0 & 0 & 0 & 0 \\ D_{21} & 0 & 0 & D_{22} \\ 0 & 0 & D_{22}^T & 0 \end{bmatrix}\right) \cap \{(-\infty, -\frac{1}{\delta}] \cup [\frac{1}{\delta}, \infty)\}\right\} = \emptyset. \tag{26}$$

Lemma 4. The condition (24) holds if and only if

$$\left\{\lambda\left(\begin{bmatrix} F_{11} - F_{z_0} & F_{12} \\ 0 & I \end{bmatrix}^{-1}\begin{bmatrix} 0 & 0 \\ -F_{21} & -F_{22} \end{bmatrix}\right) \cap \{(-\infty, -\frac{1}{\delta}] \cup [\frac{1}{\delta}, \infty)\}\right\} = \emptyset. \tag{27}$$

With these lemmas and Theorem 1, we can obtain the next result.

Theorem 3. Let us consider Problem 2 with $L = 1$.

(i) If either of the conditions (26) and (27) fails, we can conclude that (13) never holds.
(ii) If both of the conditions (26) and (27) are satisfied and the dual LMI (4) corresponding to (20) is infeasible, then we can conclude that (13) holds.
(iii) If the dual LMI (4) corresponding to (20) is feasible and the computed dual solution satisfies the rank condition (6), we can conclude that (13) never holds.

4.2 Multiple Uncertain Parameter Case

Problem 2 becomes much more intractable in the case of multiple uncertain parameters. Obviously, we can apply Theorem 2 to assess the condition (20). Thus the rest of this subsection is devoted to the technical details how we deal with the conditions (18) and (19).

As before, it is hard to deal with (18) and (25) exactly since they depend on multiple uncertain parameters. To get around this difficulty, as in Subsection 3.2, we apply the modified (D, G)-scaling (Ebihara et al. [2007]) to these parameter-dependent inequalities and obtain numerically tractable but conservative in general LMI conditions. By considering corresponding dual LMIs, it turns out that we can obtain consistent results to Theorem 2. The results are summarized in the next two lemmas.

Lemma 5. Let us consider the following LMI with respect to $\mathcal{G} \in \mathbf{S}_{(L+1)(p_1+q)} \setminus \{0\}$:

$$\mathcal{G} = \begin{bmatrix} \mathcal{G}_{00} & \mathcal{G}_{01} & \cdots & \mathcal{G}_{0L} \\ \mathcal{G}_{01} & \mathcal{G}_{11} & & \vdots \\ \vdots & & \ddots & \mathcal{G}_{L-1,L} \\ \mathcal{G}_{0L} & \cdots & \mathcal{G}_{L-1,L} & \mathcal{G}_{L,L} \end{bmatrix} \succeq 0, \quad \begin{bmatrix} 0_{p_1,q} \\ -I_q \\ \hline \mathcal{S}_1 \\ \vdots \\ \mathcal{S}_L \end{bmatrix}^T \mathcal{G} \begin{bmatrix} 0_{p_1,q} \\ -I_q \\ \hline \mathcal{S}_1 \\ \vdots \\ \mathcal{S}_L \end{bmatrix} = 0, \tag{28}$$

$$\delta^2 \mathcal{G}_{00} - \mathcal{G}_{ii} \succeq 0 \ \ (i = 1, \cdots, L)\,,$$
$$\mathrm{trace}\left(\begin{bmatrix} D_{11}^T D_{11} - \gamma^2 I_{p_1} & D_{11}^T D_{12} \\ D_{12}^T D_{11} & D_{12}^T D_{12} \end{bmatrix} \mathcal{G}_{00}\right) \geq 0.$$

Here, we defined $\mathcal{S}_i := [\, E_{q,i} D_{21} \ E_{q,i} D_{22} \,]^T \ \ (i = 1, \cdots, L)$.
(i) If (28) is infeasible, then the condition (18) holds.
(ii) Suppose (28) is feasible and has a solution $\mathcal{G}$. Then, if

$$\mathrm{rank}(\mathcal{G}_{00}) = \mathrm{rank}(\mathcal{G}), \tag{29}$$

the condition (18) never holds. More precisely, if we denote the full-rank factorization of $\mathcal{G}$ by

$$\mathcal{G} = \begin{bmatrix} G_0 \\ \vdots \\ G_L \end{bmatrix}\begin{bmatrix} G_0 \\ \vdots \\ G_L \end{bmatrix}^T, G_j \in \mathbf{R}^{(p_1+q)\times m} \ (j = 0, \cdots, L),$$

and define $\Omega_i := G_0^\dagger G_i$ $(i = 1, \cdots, L)$, then these matrices satisfy $\Omega_i \in \mathbf{S}_m$, $\lambda(\Omega_i) \subset \Theta_\delta^L$ $(i = 1, \cdots, L)$ and share all eigenvectors $u_k \in \mathbf{C}^m (k = 1, \cdots, m)$ in common. In addition, if we denote by $\lambda_k(\Omega_i)$ $(i = 1, \cdots, L)$ the eigenvalue of Ω_i corresponding to the common eigenvector u_k, we have $D(\theta_{\mathrm{w},k})^T D(\theta_{\mathrm{w},k}) - \gamma^2 I \not\prec 0$ for all $k = 1, \cdots, m$ where $\theta_{\mathrm{w},k} = [\ \lambda_k(\Omega_1) \ \cdots \lambda_k(\Omega_L)\ ]^T$.

Lemma 6. Let us consider the following LMI with respect to $\mathcal{G} \in \mathbf{S}_{(L+1)(2n+r)} \setminus \{0\}$:

$$\mathcal{G} = \begin{bmatrix} \mathcal{G}_{00} & \mathcal{G}_{01} & \cdots & \mathcal{G}_{0L} \\ \mathcal{G}_{01} & \mathcal{G}_{11} & & \vdots \\ \vdots & & \ddots & \mathcal{G}_{L-1,L} \\ \mathcal{G}_{0L} & \cdots & \mathcal{G}_{L-1,L} & \mathcal{G}_{L,L} \end{bmatrix} \succeq 0, \quad \begin{bmatrix} 0_{2n,r} \\ -I_r \\ \hline \mathcal{T}_1 \\ \vdots \\ \mathcal{T}_L \end{bmatrix}^T \mathcal{G} \begin{bmatrix} 0_{2n,r} \\ -I_r \\ \hline \mathcal{T}_1 \\ \vdots \\ \mathcal{T}_L \end{bmatrix} = 0, \tag{30}$$

$$\delta^2 \mathcal{G}_{00} - \mathcal{G}_{ii} \succeq 0 \ \ (i = 1, \cdots, L)\,,$$
$$\begin{bmatrix} F_{11}^T - F_{z_0}^* \\ F_{12}^T \end{bmatrix}^* \mathcal{G}_{00} \begin{bmatrix} F_{11}^T - F_{z_0}^* \\ F_{12}^T \end{bmatrix} = 0.$$

Here, we defined $\mathcal{T}_i := [\, E_{r,i} F_{21} \ E_{r,i} F_{22} \,]^T \ \ (i = 1, \cdots, L)$.
(i) If (30) is infeasible, then the condition (25) holds.
(ii) Suppose (30) is feasible and has a solution $\mathcal{G}$. Then, if

$$\mathrm{rank}(\mathcal{G}_{00}) = \mathrm{rank}(\mathcal{G}), \tag{31}$$

the condition (25) never holds. More precisely, if we denote the full-rank factorization of $\mathcal{G}$ by

$$\mathcal{G} = \begin{bmatrix} G_0 \\ \vdots \\ G_L \end{bmatrix}\begin{bmatrix} G_0 \\ \vdots \\ G_L \end{bmatrix}^T, \ G_j \in \mathbf{R}^{(2n+r)\times m} \ (j = 0, \cdots, L),$$

and define $\Omega_i := G_0^\dagger G_i$ $(i = 1, \cdots, L)$, then these matrices satisfy $\Omega_i \in \mathbf{S}_m$, $\lambda(\Omega_i) \subset \Theta_\delta^L$ $(i = 1, \cdots, L)$ and share all eigenvectors $u_k \in \mathbf{C}^m (k = 1, \cdots, m)$ in common. In addition, if we denote by $\lambda_k(\Omega_i)$ $(i = 1, \cdots, L)$ the eigenvalue of Ω_i corresponding to the common eigenvector u_k, we have $\mathrm{In}\,(F_\gamma(\theta_{\mathrm{w},k}) - F_{z_0}) \neq (n,n,0)$ for all $k = 1,\cdots,m$ where $\theta_{\mathrm{w},k} = [\ \lambda_k(\Omega_1)\ \cdots \lambda_k(\Omega_L)\ ]^T$.

We see that these results are surely consistent with Theorem 2. With these results and Theorem 2, we are led to the next result.

Theorem 4. Let us consider Problem 2.

(i) If all of the dual LMIs (28), (30) and (8) corresponding respectively to (18), (19) and (20) are infeasible, then (13) holds.
(ii) If the dual LMI (28) corresponding to (18) is feasible and the computed dual solution satisfies the rank condition (29), then (13) never holds.
(iii) If the dual LMI (30) corresponding to (19) is feasible and the computed dual solution satisfies the rank condition (31), then (13) never holds.
(iv) If the dual LMI (8) corresponding to (20) is feasible and the computed dual solution satisfies the rank condition (9), then (13) never holds.

We note that, if we resort to the standard primal-LMI-based approaches, all we can conclude is the assertion (i) in Theorem 4 (this corresponds to the case where all of the primal LMIs of (28), (30) and (8) are feasible). By investigating the dual LMIs and considering the structure of the dual solution, we have succeeded in deriving exactness verification tests as in (ii), (iii) and (iv). More precisely, if one of the rank conditions (29), (31) and (9) is satisfied, we can readily extract the worst case parameter perturbations.

5. CONCLUSION

In this paper, we considered robust performance analysis problems of LTI systems depending on uncertain parameters. We extended our dual LMI approach in Ebihara et al. [2007] so that we can deal with discrete-time system analysis in a unified fashion. This has been achieved by the detailed analysis on the generalized Lyapunov inequalities depending rationally upon the uncertain parameters. In stark contrast with the standard primal-LMI-based approaches, the suggested dual LMI approach would be effective to extract the worst case parameter perturbations and to conclude the exactness of the computed results. From numerical experiments, we confirmed that the suggested method is surely effective to achieve exact analysis.

At the same time, this paper showed that the robust dissipation performance analysis based on the Hamiltonian eigenvalue tests becomes rather complicated particularly for discrete-time systems. Alternative approach by means of KYP-lemma should be promising and this topic is currently under investigation.

REFERENCES

V. Balakrishnan and L. Vandenberghe, Semidefinite Programming Duality and Linear Time-Invariant Systems. *IEEE Transactions on Automatic Control*, volume 48, no. 1, pages 30–41, 2003.

B. R. Barmish, *New Tools for Robustness of Linear Systems*, Macmillan Publishing Company, 1994.

P. A. Bliman, A Convex Approach to Robust Stability for Linear Systems with Uncertain Scalar Parameters. *SIAM Journal on Control and Optimization*, volume 42, no. 6, pages 2016–2042, 2004a.

P. A. Bliman, An Existence Result for Polynomial Solutions of Parameter-Dependent LMIs. *Systems and Control Letters*, volume 51, pages 165–169, 2004b.

G. Chen and T. Sugie, New Upper Bound of the Real μ Based on the Parameter Dependent Multiplier. Proc. Conference on Decision and Control, pages 1293–1294, 1996.

Y. Ebihara and T. Hagiwara, A Dilated LMI Approach to Robust Performance Analysis of Linear Time-Invariant Uncertain Systems. *Automatica*, volume 41, no. 11, pages 1933–1941, 2005.

Y. Ebihara, Y. Onishi and T. Hagiwara, Robust Performance Analysis of Uncertain LTI Systems: Dual LMI Approach and Verifications for Exactness. Proc. Conference on Decision and Control, pages 6304–6309, 2007.

T. Hagiwara, Inertia Approach to the Hamiltonian Eigenvalue Tests for Frequency-Domain Inequality Conditions. *Dynamics of Continuous, Discrete and Impulsive Systems, Series B: Applications & Algorithms*, Special Issue, pages 279–283, 2005.

D. Henrion, D. Arzelier, D. Peaucelle and J. B. Lasserre, On Parameter-Dependent Lyapunov Functions for Robust Stability of Linear Systems. Proc. Conference on Decision and Control, pages 887–892, 2004.

D. Henrion and J. B. Lasserre, Detecting Global Optimality and Extracting Solutions in GloptiPoly. *Lecture Notes in Control and Information Sciences*, volume 312, Springer Verlag, 2005.

D. Henrion and J. B. Lasserre, Convergent Relaxations of Polynomial Matrix Inequalities and Static Output Feedback. *IEEE Transactions on Automatic Control*, volume 51, no. 2, pages 192–202, 2006.

J. B. Lasserre, Global Optimization with Polynomials and the Problem of Moments. *SIAM Journal on Optimization*, volume 11, no. 3, pages 796–817, 2001.

G. Meinsma, Y. Shrivastava and M. Fu, A Dual Formulation of Mixed μ and on the Losslessness of (D,G) Scaling. *IEEE Transactions on Automatic Control*, volume 42, no. 7, pages 1032–1036, 1997.

P. A. Parrilo, Semidefinite Programming Relaxations for Semialgebraic Problems. *Mathematical Programming*, volume 96, pages 293–320, 2003.

D. C. W. Ramos and P. L. D. Peres, A Less Conservative LMI Condition for the Robust Stability of Discrete-Time Uncertain Systems. *Systems and Control Letters*, volume 43, pages 371–378, 2001.

C. W. Scherer, Relaxations for Robust Linear Matrix Inequality Problems with Verifications for Exactness. *SIAM Journal on Matrix Analysis and Applications*, volume 27, no. 2, pages 365–395, 2005.

C. W. Scherer, LMI Relaxations in Robust Control. *European Journal of Control*, volume 12, no. 1, 2006.

C. W. Scherer and C. W. J. Hol, Matrix Sum-of-Squares Relaxations for Robust Semi-Definite Programs. *Mathematical Programming*, volume 107, pages 189–211, 2006.

X. Zhang, P. Tsiotras and T. Iwasaki, Parameter-Dependent Lyapunov Functions for Exact Stability Analysis of Single-Parameter Dependent LTI Systems. Proc. Conference on Decision and Control, pages 5168–5173, 2003.

K. Zhou and J. C. Doyle, *Essentials of Robust Control*, Prentice Hall, 1998.