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Reduced complexity existence conditions for robust $\mathcal{L}_2$-gain feedforward controllers for uncertain systems using dynamic IQCs

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Abstract: For uncertain systems described in the standard LFT form, we consider the problem of designing robust $\mathcal{L}_2$-gain disturbance feedforward controllers if the uncertain blocks are described by integral quadratic constraints (IQCs). For technical reasons related to the feedforward problem we work with the duals of the constraints involved in robustness analysis using IQCs. Based on an elimination of the controller parameters, we develop in this paper reduced complexity LMI conditions for the existence of a stable feedforward controller that guarantees a given $\mathcal{L}_2$-gain for the closed-loop system.

1. INTRODUCTION

The robust disturbance feedforward control problem involves the design of a stable controller, C, for the nominally stable plant G which is perturbed by the uncertainties captured in Δ as shown in Figure 1. In addition to recent research efforts in the literature [Devasia, 2002, Ferreres and Roos, 2005, de Gelder et al., 2006, Giusto and Paganini, 1999, Scorletti and Fromion, 2006], we have obtained a general solution to the robust $\mathcal{L}_2$-gain feedforward problem by controller parameter transformation in Köse and Scherer [2007a]. In this paper, we propose an alternative solution based on eliminating the controller matrices from the matrix inequality conditions related to the closed-loop robust performance characterization, and we arrive at reduced complexity solvability conditions. The extension of the elimination step, as well-known for static multipliers, is nontrivial if characterizing the uncertainties by dynamic multipliers. In order to avoid substantial overlap with Köse and Scherer [2007a], we confine the present discussion to the technical details involved in controller elimination. A more comprehensive motivation of the problem as well as a numerical example can be found both in Köse and Scherer [2007a] and in the journal version of the paper [Köse and Scherer, 2007b] which has been recently submitted for publication.

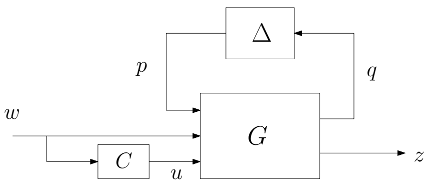


Fig. 1. The robust feedforward control problem.

By following the dual versions of the steps taken in the present paper, the results apply in a similar manner to the robust $\mathcal{L}_2$-gain estimation problem which is omitted due to space limitations. Let us also stress that the developed techniques are essential for obtaining solvability conditions for gain-scheduled controllers using IQCs, with some recent findings in this direction being reported in Scherer and Köse [2007a] and Scherer and Köse [2007b].

We begin in Section 2 with a dual of the stability characterization in Scherer and Köse [2008], since the nature of the feedforward control problem necessitates the use of dual forms of stability conditions. In Section 3, we give necessary and sufficient LMI conditions for the existence of robust disturbance feedforward controllers using the matrix elimination theorem in the Appendix. Following some remarks on the numerical aspects of the solvability conditions, we give a proof of the main theorem of the paper. Application of the results to the example in Köse and Scherer [2007a] give the same numerical values in that paper and therefore is omitted. We conclude with a summary and some final remarks in Section 4. The main technical tools for obtaining dual constraints are given in the appendix.

Notation. $\mathcal{L}_{2+}$ and $\mathcal{L}_{2-}$ denote the spaces of vector-valued square integrable functions defined on $[0, \infty)$ and $(-\infty, 0]$, respectively, with the usual inner product given by $\langle \cdot, \cdot \rangle$. The space of matrix-valued functions with entries that are essentially bounded on the imaginary axis is denoted by $\mathcal{L}_\infty$. The symbol $\mathbb{C}^0$ is used for the extended imaginary axis $i\mathbb{R} \cup \{\infty\}$. The inertia of a Hermitian matrix M is $\mathbf{in}(M) = (n_+, n_-, n_0)$, where n_+, n_-, n_0 denote the number of positive, negative and zero eigenvalues of M. For any matrix A, we denote by $A_\perp$ a basis matrix of the orthogonal complement of the image of A. We also use $\mathcal{M}(X, M) := \begin{pmatrix} 0 & X & 0 \\ X & 0 & 0 \\ 0 & 0 & M \end{pmatrix}$.

2. PRELIMINARIES

We begin with the dual form of the stability characterization with dynamic IQCs, which is necessary for the feedforward problem. The dual forms in frequency-domain and state-space are given in Sections 2.1 and 2.2. The main results of this section are Lemma 2 and Theorem 3 at the end of the section.

The main result on robustness analysis of the configuration in Figure 2(a) using IQCs is as follows.

Theorem 1. [Megretski and Rantzer, 1997] Suppose G_{qp} is stable and

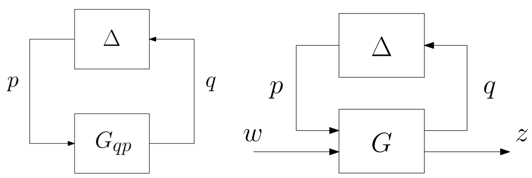
(i) the feedback interconnection of $\tau\Delta$ and G_{qp} is well-posed for all $\tau \in [0,1]$,
(ii) $\tau\Delta$ satisfies the IQC defined by Π for all $\tau \in [0,1]$. I.e.,

$$\left\langle \begin{pmatrix} q \\ \tau\Delta q \end{pmatrix}, \Pi \begin{pmatrix} q \\ \tau\Delta q \end{pmatrix} \right\rangle \geq 0 \quad \forall \tau \in [0,1], \quad \forall q \in \mathcal{L}_{2+}. \quad (1)$$

(iii) G_{qp} satisfies $\begin{pmatrix} G_{qp} \\ I \end{pmatrix}^* \Pi \begin{pmatrix} G_{qp} \\ I \end{pmatrix} \prec 0 \quad \text{on} \quad \mathbb{C}^0.$

Then, the feedback interconnection of G_{qp} and Δ is stable.

In this paper, we are interested in robust performance rather than robust stability. Therefore, we consider the system in Figure 2(b). If conditions (i) and (ii) in Theorem 1 hold, and if



(a) Robust stability (b) Robust performance

Fig. 2. The robust stability and $\mathcal{L}_2$-gain performance analysis problems.

$$\begin{pmatrix} G \\ I \end{pmatrix}^* \left(\begin{array}{cc|cc} \Pi_{11} & 0 & \Pi_{12} & 0 \\ 0 & \gamma^{-1}I & 0 & 0 \\ \hline \Pi_{12}^* & 0 & \Pi_{22} & 0 \\ 0 & 0 & 0 & -\gamma I \end{array}\right) \begin{pmatrix} G \\ I \end{pmatrix} \prec 0 \quad \text{on } \mathbb{C}^0, \quad (2)$$

where Π is partitioned as $\Pi = \begin{pmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{12}^* & \Pi_{22} \end{pmatrix}$, then we can conclude that the system is robustly stable and the $\mathcal{L}_2$-gain from w to z is less than γ [Scherer and Weiland, 1999].

As mentioned in the introduction, the feedforward problem necessitates the use of dual forms of the conditions involving the multiplier Π and G. Therefore, we now discuss dual IQCs and the resulting conditions in state-space.

2.1 Dual IQCs

Suppose the conditions in Theorem 1 hold. It is easily shown that we can assume $\Pi_{11} \succ 0$ on $\mathbb{C}^0$ without loss of generality. Due to Theorem 1 (iii), Π has as many negative eigenvalues as the number of inputs of G_{qp}, and since $\mathbf{in}(\Pi) = \mathbf{in}(\Pi_{11}) + \mathbf{in}(\Pi_{22} - \Pi_{12}^*\Pi_{11}^{-1}\Pi_{12})$, we obtain $\Pi_{11} \succ 0$ and $\Pi_{22} - \Pi_{12}^*\Pi_{11}^{-1}\Pi_{12} \prec 0$ on $\mathbb{C}^0$. We can now apply Lemma 5 to condition (iii) in Theorem 1 and obtain the equivalent condition

$$\begin{pmatrix} I \\ -G_{qp}^* \end{pmatrix}^* \Pi^{-1} \begin{pmatrix} I \\ -G_{qp}^* \end{pmatrix} \succ 0 \quad \text{on} \quad \mathbb{C}^0. \quad (3)$$

Let us define $\Pi^{-1} =: \Theta = \begin{pmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{21}^* & \Theta_{22} \end{pmatrix}$ and deduce that

$$\Theta_{22} \prec 0 \quad \text{and} \quad \Theta_{11} - \Theta_{12}\Theta_{22}^{-1}\Theta_{12}^* \succ 0 \quad \text{on} \quad \mathbb{C}^0. \quad (4)$$

Due to the nature of the feedforward problem, instead of condition (iii) in Theorem 1, we will be using (3), where Θ satisfies (4). For some subtleties involving the choice of Θ so that (1) is satisfied, we refer the reader to Köse and Scherer [2007a].

2.2 Dual IQCs in State-space

Let us now assume that Θ is factorized as $\Theta = \phi N \phi^* = (-\phi)N(-\phi^*)$ with a stable ϕ that is in general wide. After partitioning the rows of ϕ compatibly with the columns of $(I \ \ -G_{qp})$, let us introduce the state-space realization

$$\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \left[\begin{array}{cc|c} A_1 & A_3 & B_1 \\ 0 & A_2 & B_2 \\ \hline C_1 & C_3 & D_1 \\ 0 & C_2 & D_2 \end{array}\right],$$

where we can assume, without loss of generality, that (C_2, A_2) is observable and that A_1, A_2 are stable. For any realization $G_{qp} = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$, we arrive at the state-space description

$$-\phi^* \begin{pmatrix} I \\ -G_{qp}^* \end{pmatrix} = \left[\begin{array}{ccc|c} -A_1^T & 0 & 0 & -C_1^T \\ -A_3^T & -A_2^T & C_2^T B^T & -C_3^T + C_2^T D^T \\ 0 & 0 & -A^T & -C^T \\ \hline -B_1^T & -B_2^T & D_2^T B^T & -D_1^T + D_2^T D^T \end{array}\right] =: \left[\begin{array}{c|c} -\mathcal{A}^T & -\mathcal{C}^T \\ \hline -\mathcal{B}^T & -\mathcal{D}^T \end{array}\right].$$

Condition (3) then reads as

$$\begin{pmatrix} I \\ -G_{qp}^* \end{pmatrix}^* (-\phi)N(-\phi^*) \begin{pmatrix} I \\ -G_{qp}^* \end{pmatrix} \succ 0 \quad \text{on} \quad \mathbb{C}^0. \quad (5)$$

Due to the KYP lemma, this is equivalent to the existence of $Y = Y^T$ such that

$$(\star)^T \mathcal{M}(Y,N) \begin{pmatrix} I & 0 \\ -\mathcal{A}^T & -\mathcal{C}^T \\ -\mathcal{B}^T & -\mathcal{D}^T \end{pmatrix} \succ 0. \quad (6)$$

In the sequel we partition Y compatibly with $\mathcal{A}$ as

$$Y = \begin{pmatrix} Y_{11} & Y_{12} & Y_{1G} \\ Y_{21} & Y_{22} & Y_{2G} \\ Y_{G1} & Y_{G2} & Y_{GG} \end{pmatrix}.$$

Moreover, for future reference, let us also note that the condition $(-\phi_2)N(-\phi_2)^* = \Theta_{22} \prec 0$ on $\mathbb{C}^0$ is equivalent to the existence of an $X = X^T$ such that

$$(\star)^T \mathcal{M}(X,N) \begin{pmatrix} I & 0 \\ -A_2^T & -C_2^T \\ -B_2^T & -D_2^T \end{pmatrix} \prec 0. \quad (7)$$

At this point, we are ready to give the dual of the stability characterization in Lemma 5 of Scherer and Köse [2008]. Full details can be found in the journal version of this paper [Köse and Scherer, 2007b].

Lemma 2. Suppose the LMIs (6) and (7) are feasible.

(i) There exist stable transfer functions $\tilde{\phi}_{11}$, $\tilde{\phi}_{12}$ and $\tilde{\phi}_{22}$ such that $\tilde{\phi}_{11}$ and $\tilde{\phi}_{22}$ are bi-proper and stable, while $\tilde{\phi}_{11}$ has a stable inverse and an $\tilde{N} := \operatorname{diag}(\tilde{N}_1, N_2)$ with $\tilde{N}_1 \succ 0$ and $N_2 \prec 0$ such that
$$\Theta = (\star)N\begin{pmatrix} -\phi_1 \\ -\phi_2 \end{pmatrix}^* = (\star)\tilde{N}\begin{pmatrix} -\tilde{\phi}_{11} & -\tilde{\phi}_{12} \\ 0 & -\tilde{\phi}_{22} \end{pmatrix}^*. \quad (8)$$

(ii) If W is the anti-stabilizing solution of the ARE corresponding to (7), any solution X of (7) satisfies $W \prec X$ and X can be taken arbitrarily close to W.

(iii) Let $\tilde{K}$ satisfy $\tilde{A}\tilde{K} + \tilde{K}\tilde{A}^T \prec 0$, with $\tilde{A}$ defined in Köse and Scherer [2007b]. Define $J := \begin{pmatrix} I & 0 \end{pmatrix}^T$ and $K := J\tilde{K}J^T$. Then, Y is a solution of (6) iff for all small $\delta > 0$, the matrix $\tilde{Y}_e \succ 0$, given by
$$\begin{pmatrix} J^TY_{11}J - \tilde{W}_{11} & J^TY_{12} - \tilde{W}_{12} & J^TY_{12} & J^TY_{1G} \\ \star & Y_{22} - \tilde{W}_{22} & Y_{22} & Y_{2G} \\ \star & \star & Y_{22} - W + \delta K & Y_{2G} \\ \star & \star & \star & Y_{GG} \end{pmatrix}$$
satisfies
$$(\star)^T\mathcal{M}(\tilde{Y}_e, \tilde{N})\begin{pmatrix} I & 0 \\ -\tilde{\mathcal{A}}^T & -\tilde{\mathcal{C}}^T \\ -\tilde{\mathcal{B}}^T & -\tilde{\mathcal{D}}^T \end{pmatrix} \succ 0, \quad (9)$$
where $\left[\begin{array}{c|c} -\tilde{\mathcal{A}}^T & -\tilde{\mathcal{C}}^T \\ \hline -\tilde{\mathcal{B}}^T & -\tilde{\mathcal{D}}^T \end{array}\right] := \begin{pmatrix} -\tilde{\phi}_{11}^* \\ -\tilde{\phi}_{12}^* + \tilde{\phi}_{12}^*G^* \end{pmatrix}$.

(iv) A is Hurwitz iff there exists a positive definite solution of (9) iff all solutions of (9) are positive definite.

Based on this stability characterization, we obtain the following dual of Theorem 4 in Scherer and Köse [2008].

Theorem 3. Matrix A is Hurwitz and conditions (3) and (4) hold if and only if there exist solutions $X = X^T$ and $Y = Y^T$ of LMIs (7) and (6) that are coupled as
$$\begin{pmatrix} Y_{22} - X & Y_{2G} \\ Y_{2G}^T & Y_{GG} \end{pmatrix} \succ 0. \quad (10)$$

3. ROBUST DISTURBANCE FEEDFORWARD CONTROL

3.1 Preliminaries

Consider the robust disturbance feedforward problem as described in Figure 1, where G is stable. The objective is to design a stable controller C such that the closed-loop system has a prescribed $\mathcal{L}_2$-gain. The state-space representations for the plant and the controller are
$$G = \left[\begin{array}{c|ccc} A & B_p & B_w & B_u \\ \hline C_q & D_{qp} & D_{qw} & D_{qu} \\ C_z & D_{zp} & D_{zw} & D_{zw} \\ 0 & 0 & I & 0 \end{array}\right] \text{ and } C = \left[\begin{array}{c|c} \mathbf{A}_C & \mathbf{B}_C \\ \hline \mathbf{C}_C & \mathbf{D}_C \end{array}\right] \quad (11)$$
where A is Hurwitz. The closed-loop system becomes
$$\left[\begin{array}{c|cc} A^{cl} & B_p^{cl} & B_w^{cl} \\ \hline C_q^{cl} & D_{qp}^{cl} & D_{qw}^{cl} \\ C_z^{cl} & D_{zp}^{cl} & D_{zw}^{cl} \end{array}\right]$$
$$:= \left[\begin{array}{c|cc} A^a + B_u^a\mathbf{K}C_y^a & B_p^a + B_u^a\mathbf{K}D_{yp}^a & B_w^a + B_u^a\mathbf{K}D_{yw}^a \\ \hline C_q^a + D_{qu}^a\mathbf{K}C_y^a & D_{qp}^a + D_{qu}^a\mathbf{K}D_{yp}^a & D_{qw}^a + D_{qu}^a\mathbf{K}D_{yw}^a \\ C_z^a + D_{zu}^a\mathbf{K}C_y^a & D_{zp}^a + D_{zu}^a\mathbf{K}D_{yp}^a & D_{zw}^a + D_{zu}^a\mathbf{K}D_{yw}^a \end{array}\right]$$
with obvious definitions for the augmented system matrices with superscript "a" and $\mathbf{K} := \begin{pmatrix} \mathbf{D}_C & \mathbf{C}_C \\ \mathbf{B}_C & \mathbf{A}_C \end{pmatrix}$. Using (2), the condition for robust closed-loop stability with $\mathcal{L}_2$-gain less than γ becomes
$$[\star]^*\begin{pmatrix} \phi_1N\phi_1^* & 0 & \phi_2N\phi_2^* & 0 \\ 0 & \gamma I & 0 & 0 \\ \phi_2N\phi_1^* & 0 & \phi_2N\phi_2^* & 0 \\ 0 & 0 & 0 & -\gamma^{-1}I \end{pmatrix}$$
$$\left[\begin{array}{c|cc} -A^{cl^T} & -C_q^{cl^T} & -C_z^{cl^T} \\ \hline 0 & I & 0 \\ 0 & 0 & I \\ -B_p^{cl^T} & -D_{qp}^{cl^T} & -D_{zp}^{cl^T} \\ -B_w^{cl^T} & -D_{qw}^{cl^T} & -D_{zw}^{cl^T} \end{array}\right] \succ 0. \quad (12)$$
We can factorize the middle matrix on the left-hand-side of the inequality above as
$$(\star)^*\begin{pmatrix} N & 0 & 0 \\ 0 & \gamma I & 0 \\ 0 & 0 & -\gamma^{-1}I \end{pmatrix}\begin{pmatrix} -\phi_1^* & 0 & -\phi_2^* & 0 \\ 0 & -I & 0 & 0 \\ 0 & 0 & 0 & -I \end{pmatrix}$$
and define
$$\begin{pmatrix} -\phi_1^* & 0 & -\phi_2^* & 0 \\ 0 & -I & 0 & 0 \\ 0 & 0 & 0 & -I \end{pmatrix} = \left[\begin{array}{c|cccc} -A_\phi^T & -C_{\phi_1}^T & 0 & -C_{\phi_2}^T & 0 \\ \hline -B_\phi^T & -D_{\phi_1}^T & 0 & -D_{\phi_2}^T & 0 \\ 0 & 0 & -I & 0 & 0 \\ 0 & 0 & 0 & 0 & -I \end{array}\right]$$
$$=: \left[\begin{array}{c|c} -A_\Phi^T & -C_\Phi^T \\ \hline -B_\Phi^T & -D_\Phi^T \end{array}\right]. \quad (13)$$
For later use, we also introduce
$$\begin{pmatrix} -\phi_1^* & 0 & -\phi_2^* \\ 0 & -I & 0 \end{pmatrix} = \left[\begin{array}{c|cccc} -A_\phi^T & -C_{\phi_1}^T & 0 & -C_{\phi_2}^T \\ \hline -B_\phi^T & -D_{\phi_1}^T & 0 & -D_{\phi_2}^T \\ 0 & 0 & -I & 0 \end{array}\right]$$
$$=: \left[\begin{array}{c|c} -\mathcal{A}_\Phi^T & -\mathcal{C}_\Phi^T \\ \hline -\mathcal{B}_\Phi^T & -\mathcal{D}_\Phi^T \end{array}\right]. \quad (14)$$

3.2 Main result

The following theorem gives necessary and sufficient conditions for the existence of a stable controller that satisfies (12).

Theorem 4. There exists a stable feedforward controller such that the closed-loop system satisfies (12) if and only if there exist $Y = Y^T$, and $Z = Z^T$ such that
$$(\star)^T(\star)^T\mathcal{M}(Y, \operatorname{diag}(N, \gamma I, -\gamma^{-1}I))$$
$$\left(\begin{array}{ccc} I & 0 & \begin{pmatrix} 0 & 0 \end{pmatrix} \\ 0 & I & \begin{pmatrix} 0 & 0 \end{pmatrix} \\ \hdashline -A_\Phi^T & -C_\Phi^T\begin{pmatrix} 0 \\ 0 \\ -B_p^T \\ -B_w^T \end{pmatrix} & -C_\Phi^T\begin{pmatrix} I & 0 \\ 0 & I \\ -D_{qp}^T & -D_{zp}^T \\ -D_{qw}^T & -D_{zw}^T \end{pmatrix} \\ 0 & -A^T & \begin{pmatrix} -C_q^T & -C_z^T \end{pmatrix} \\ \hdashline -B_\Phi^T & -D_\Phi^T\begin{pmatrix} 0 \\ 0 \\ -B_p^T \\ -B_w^T \end{pmatrix} & -D_\Phi^T\begin{pmatrix} I & 0 \\ 0 & I \\ -D_{qp}^T & -D_{zp}^T \\ -D_{qw}^T & -D_{zw}^T \end{pmatrix} \end{array}\right)$$
$$\begin{pmatrix} I & 0 \\ 0 & \\ 0 & \begin{pmatrix} B_u \\ D_{qu} \\ D_{zu} \end{pmatrix}_\perp \\ 0 & \end{pmatrix} \succ 0, \quad (15)$$

$$
(\star)^T \mathcal{M}(Z, \mathrm{diag}(N, \gamma I))
\left(\begin{array}{ccc}
I & 0 & (\,0\ 0\,) \\
0 & I & (\,0\ 0\,) \\ \hdashline
-\mathcal{A}_\Phi^T & -\mathcal{C}_\Phi^T \begin{pmatrix} 0 \\ 0 \\ -B_p^T \end{pmatrix} & -\mathcal{C}_\Phi^T \begin{pmatrix} I & 0 \\ 0 & I \\ -D_{qp}^T & -D_{zp}^T \end{pmatrix} \\
0 & -A^T & (\,-C_q^T\ -C_z^T\,) \\ \hdashline
-\mathcal{B}_\Phi^T & -\mathcal{D}_\Phi^T \begin{pmatrix} 0 \\ 0 \\ -B_p^T \end{pmatrix} & -\mathcal{D}_\Phi^T \begin{pmatrix} I & 0 \\ 0 & I \\ -D_{qp}^T & -D_{zp}^T \end{pmatrix}
\end{array}\right) \succ 0, \tag{16}
$$

coupled with the solution $X = X^T$ of (7) as

$$
Y - Z \succ 0 \quad \text{and} \quad T^T Z T - \begin{pmatrix} X & 0 \\ 0 & 0 \end{pmatrix} \succ 0, \tag{17}
$$

where $Y := \begin{pmatrix} Y_{11} & Y_{12} & Y_{1G} \\ \star & Y_{22} & Y_{2G} \\ \star & \star & Y_{GG} \end{pmatrix}$, $Z := \begin{pmatrix} Z_{11} & Z_{12} & Z_{1G} \\ \star & Z_{22} & Z_{2G} \\ \star & \star & Z_{GG} \end{pmatrix}$ and $T := \begin{pmatrix} 0 & 0 \\ I & 0 \\ 0 & I \end{pmatrix}$.

Before the proof of Theorem 4, let us provide some clarifying remarks:

- When the solvability conditions above are satisfied, a suitable controller can be obtained through a straightforward calculation. The details of the controller construction can be found in the proof of the sufficiency direction. The resulting controller has an order equal to the sum of the orders of the plant and the outer factor of the multiplier. That is, $\dim(\mathbf{A}_C) := \dim(A_\Phi) + \dim(A) = \dim(A_1) + \dim(A_2) + \dim(A)$.
- For a fixed multiplier, conditions (15)-(17), combined with (7), constitute convex conditions in the variables X, Y, Z and γ. Additionally, the multiplier Θ can be parametrized by fixing ϕ appropriately and N can be taken as another matrix variable. Finally, the overall problem can be turned into an LMI problem by applying the Schur complement formula to inequality (15) with respect to the block $-\gamma^{-1} I$. Hence, the guaranteed $\mathcal{L}_2$-gain γ can be minimized over the feasible set of the solvability conditions using commercially available semi-definite programming packages.
- If compared to the approach based on parameter transformations [Köse and Scherer, 2007a], we have reduced the number of decision variables by the number of elements in $\mathbf{K}$. The reduction is by exactly $(\dim(\mathbf{A}_C) + m_u) \times (\dim(\mathbf{A}_C) + p_y)$ variables, where m_u and p_y represent dimensions of the control input and measured output of G. This turns into a great practical advantage, in particular for sophisticated multiplier descriptions that require outer factors of relatively high McMillan degree.
- For robust $\mathcal{H}_\infty$-estimation in Scherer and Köse [2008], a similar elimination of the estimator parameters can be performed in order to reduce the size of the synthesis LMIs, by making full use of the dual version of Lemma 3. Along similar lines, partial parameter elimination is possible for robust $\mathcal{H}_2$-estimation.
- When the multiplier is restricted to be frequency-independent, we recover the results of Giusto and Paganini [1999] exactly. When the multiplier is in the D/G-structure, we recover the results of Scorletti and Fromion [2006].

Proof of necessity: Applying the KYP lemma to (12), we obtain

$$
\Gamma_{cl}^T \Omega_{cl} \Gamma_{cl} := (\star)^T \mathcal{M}(Y_{cl}, \mathrm{diag}(N, \gamma I, -\gamma^{-1} I))
\left(\begin{array}{ccc}
I & 0 & 0 \\
0 & I & 0 \\ \hdashline
-A_\Phi^T & -C_\Phi^T \begin{pmatrix} 0 \\ -B_{cl}^T \end{pmatrix} & -C_\Phi^T \begin{pmatrix} I \\ -D_{cl}^T \end{pmatrix} \\
0 & -A_{cl}^T & -C_{cl}^T \\ \hdashline
-B_\Phi^T & -D_\Phi^T \begin{pmatrix} 0 \\ -B_{cl}^T \end{pmatrix} & -D_\Phi^T \begin{pmatrix} I \\ -D_{cl}^T \end{pmatrix}
\end{array}\right) \succ 0 \tag{18}
$$

where $Y_{cl} := \begin{pmatrix} Y_{11} & Y_{12} & Y_{1G} & Y_{1C} \\ \star & Y_{22} & Y_{2G} & Y_{2C} \\ \star & \star & Y_{GG} & Y_{GC} \\ \star & \star & \star & Y_{CC} \end{pmatrix}$. Due to Lemma 2, this inequality implies

$$
\tilde{\Gamma}_{cl}^T \tilde{\Omega}_{cl_e} \tilde{\Gamma}_{cl} := (\star)^T \mathcal{M}(\tilde{Y}_{cl_e}, \mathrm{diag}(N, \gamma I, -\gamma^{-1} I))
\left(\begin{array}{ccc}
I & 0 & 0 \\
0 & I & 0 \\ \hdashline
-\tilde{A}_\Phi^T & -\tilde{C}_\Phi^T \begin{pmatrix} 0 \\ -B_{cl}^T \end{pmatrix} & -\tilde{C}_\Phi^T \begin{pmatrix} I \\ -D_{cl}^T \end{pmatrix} \\
0 & -A_{cl}^T & -C_{cl}^T \\ \hdashline
-\tilde{B}_\Phi^T & -\tilde{D}_\Phi^T \begin{pmatrix} 0 \\ -B_{cl}^T \end{pmatrix} & -\tilde{D}_\Phi^T \begin{pmatrix} I \\ -D_{cl}^T \end{pmatrix}
\end{array}\right) \succ 0 \tag{19}
$$

for small $\delta > 0$, where $\tilde{N}$ and $\left[\begin{array}{c|c} \tilde{A}_\Phi & \tilde{B}_\Phi \\ \hline \tilde{C}_\Phi & \tilde{D}_\Phi \end{array}\right] := \begin{pmatrix} \tilde{\phi}_{11} & \tilde{\phi}_{12} \\ 0 & \tilde{\phi}_{22} \end{pmatrix}$ are as in Lemma 2 and $\tilde{Y}_{cl_e}$ is defined as

$$
\left(\begin{array}{cccc:c}
J^T Y_{11} J - \tilde{W} & J^T Y_{12} - \tilde{W}_{12} & J^T Y_{12} & J^T Y_{1G} & J^T Y_{1C} \\
\star & Y_{22} - \tilde{W}_{22} & Y_{22} & Y_{2G} & Y_{2C} \\
\star & \star & Y_{22} - W + \delta K & Y_{2G} & Y_{2C} \\
\star & \star & \star & Y_{GG} & Y_{GC} \\ \hdashline
\star & \star & \star & \star & Y_{CC}
\end{array}\right)
$$

$$
=: \left(\begin{array}{c:c} \tilde{Y}_e & Y_{\star C} \\ \hdashline \star & Y_{CC} \end{array}\right) =: \left(\begin{array}{c:c} \tilde{Z}_e^{-1} & \star \\ \hdashline \star & \star \end{array}\right)^{-1} \succ 0. \tag{20}
$$

Since $\tilde{Z}_e = \tilde{Y}_e - Y_{\star C} Y_{CC}^{-1} Y_{\star C}^T$, it clearly has the structure

$$
\begin{pmatrix}
J^T Z_{11} J - \tilde{W}_{11} & J^T Z_{12} - \tilde{W}_{12} & J^T Z_{12} & J^T Z_{1G} \\
\star & Z_{22} - \tilde{W}_{22} & Z_{22} & Z_{2G} \\
\star & \star & Z_{22} - W + \delta K & Z_{2G} \\
\star & \star & \star & Z_{GG}
\end{pmatrix}
$$

Now express $\tilde{\Gamma}_{cl}$ as

$$
\left(\begin{array}{ccc}
I & 0 & 0 \\
0 & I & 0 \\ \hdashline
-\tilde{A}_\Phi^T & -\tilde{C}_\Phi^T \begin{pmatrix} 0 \\ 0 \\ -B_p^{a\,T} \\ -B_w^{a\,T} \end{pmatrix} & -\tilde{C}_\Phi^T \begin{pmatrix} I & 0 \\ 0 & I \\ -D_{qp}^{a\,T} & -D_{zp}^{a\,T} \\ -D_{qw}^{a\,T} & -D_{zw}^{a\,T} \end{pmatrix} \\
0 & -A^{a\,T} & (\,-C_q^{a\,T}\ -C_z^{a\,T}\,) \\ \hdashline
-\tilde{B}_\Phi^T & -\tilde{D}_\Phi^T \begin{pmatrix} 0 \\ 0 \\ -B_p^{a\,T} \\ -B_w^{a\,T} \end{pmatrix} & -\tilde{D}_\Phi^T \begin{pmatrix} I & 0 \\ 0 & I \\ -D_{qp}^{a\,T} & -D_{zp}^{a\,T} \\ -D_{qw}^{a\,T} & -D_{zw}^{a\,T} \end{pmatrix}
\end{array}\right)
$$

$$+\begin{pmatrix} 0 \\ 0 \\ -\tilde{C}_\Phi^T \begin{pmatrix} 0 \\ 0 \\ -{D_{yp}^a}^T \\ -{D_{yw}^a}^T \end{pmatrix} \\ -{C_y^a}^T \\ -\tilde{D}_\Phi^T \begin{pmatrix} 0 \\ 0 \\ -{D_{yp}^a}^T \\ -{D_{yw}^a}^T \end{pmatrix} \end{pmatrix} \mathbf{K} \begin{pmatrix} 0 \\ B_u^a \\ \begin{pmatrix} D_{qu}^a \\ D_{zu}^a \end{pmatrix} \end{pmatrix}^T$$

which is abbreviated as $\tilde{\Gamma}_{cl} =: \tilde{\Gamma}_A + \tilde{\Gamma}_B \mathbf{K} \tilde{\Gamma}_C$. We are now in a position to apply Lemma 7. The solvability condition

$$\left(\tilde{\Gamma}_C^T\right)_\perp^T \tilde{\Gamma}_A^T \tilde{\Omega}_{cl_e} \tilde{\Gamma}_A \left(\tilde{\Gamma}_C^T\right)_\perp \succ 0 \tag{21}$$

yields

$$(\star)^T (\star)^T \mathcal{M}(\tilde{Y}_e, \mathrm{diag}(\tilde{N}, \gamma I, -\gamma^{-1} I))$$
$$\begin{pmatrix} I & 0 & (\,0\ 0\,) \\ 0 & I & (\,0\ 0\,) \\ -\tilde{A}_\Phi^T & -\tilde{C}_\Phi^T \begin{pmatrix} 0 \\ 0 \\ -B_p^T \\ -B_w^T \end{pmatrix} & -\tilde{C}_\Phi^T \begin{pmatrix} I & 0 \\ 0 & I \\ -D_{qp}^T & -D_{zp}^T \\ -D_{qw}^T & -D_{zw}^T \end{pmatrix} \\ 0 & -A^T & (\,-C_q^T\ -C_z^T\,) \\ -\tilde{B}_\Phi^T & -\tilde{D}_\Phi^T \begin{pmatrix} 0 \\ 0 \\ -B_p^T \\ -B_w^T \end{pmatrix} & -\tilde{D}_\Phi^T \begin{pmatrix} I & 0 \\ 0 & I \\ -D_{qp}^T & -D_{zp}^T \\ -D_{qw}^T & -D_{zw}^T \end{pmatrix} \end{pmatrix}$$
$$\begin{pmatrix} I & 0 \\ 0 & \begin{pmatrix} B_u \\ D_{qu} \\ D_{zu} \end{pmatrix} \\ 0 & \\ 0 & \end{pmatrix}_\perp \succ 0. \tag{22}$$

Since this holds for all $\delta > 0$, we can apply Lemma 2 to obtain (15). The second solvability condition, namely

$$\left(\,\tilde{\Gamma}_A\ \tilde{\Gamma}_B\,\right)_\perp^T \tilde{\Omega}_{cl_e} \left(\,\tilde{\Gamma}_A\ \tilde{\Gamma}_B\,\right)_\perp \prec 0, \tag{23}$$

gives

$$L_\perp^T \mathcal{M}(\tilde{Z}_e^{-1}, \tilde{N}^{-1}, \gamma^{-1} I) L_\perp \prec 0,$$

where

$$L := \begin{pmatrix} I & 0 & (\,0\ 0\,) \\ 0 & I & (\,0\ 0\,) \\ -\tilde{\mathcal{A}}_\Phi^T & -\tilde{\mathcal{C}}_\Phi^T \begin{pmatrix} 0 \\ 0 \\ -B_p^T \end{pmatrix} & -\tilde{\mathcal{C}}_\Phi^T \begin{pmatrix} I & 0 \\ 0 & I \\ -D_{qp}^T & -D_{zp}^T \end{pmatrix} \\ 0 & -A^T & (\,-C_q^T\ -C_z^T\,) \\ -\tilde{\mathcal{B}}_\Phi^T & -\tilde{\mathcal{D}}_\Phi^T \begin{pmatrix} 0 \\ 0 \\ -B_p^T \end{pmatrix} & -\tilde{\mathcal{D}}_\Phi^T \begin{pmatrix} I & 0 \\ 0 & I \\ -D_{qp}^T & -D_{zp}^T \end{pmatrix} \end{pmatrix}.$$

By Appendix B, this inequality is equivalent to

$$L^T \mathcal{M}(\tilde{Z}_e^{-1}, \tilde{N}, \gamma I) L \succ 0. \tag{24}$$

Hence, by Lemma 2 again, (24) implies (16).

By Theorem 3, stability of $\mathbf{A}_C$ is equivalent to the condition

$$\begin{pmatrix} Y_{22} - X & Y_{2G} & Y_{2C} \\ \star & Y_{GG} & Y_{GC} \\ \star & \star & Y_{CC} \end{pmatrix} \succ 0.$$

We infer that $Y - Z \succ 0$ since $Y_{CC} \succ 0$ (and using a perturbation argument if necessary). Finally, we have

$$\begin{pmatrix} Z_{22} - X & Z_{2G} \\ \star & Z_{GG} \end{pmatrix} = \begin{pmatrix} Y_{22} - X & Y_{2G} \\ \star & Y_{GG} \end{pmatrix} - (\star) Y_{CC}^{-1} \begin{pmatrix} Y_{2C} \\ Y_{GC} \end{pmatrix}^T \succ 0$$

due to the Schur complement formula, which proves (17).

Proof of sufficiency: Suppose conditions (15)-(17) are satisfied. Construct Y_{cl} as $Y_{cl} = \begin{pmatrix} Y & I \\ I & (Y - Z)^{-1} \end{pmatrix}$. Due to Lemma 2, we can hence infer that (15), (16) imply (22), (24) for all small $\delta > 0$ (without the need to use different parameters). As also exploited above, conditions (22) and (24) are seen to be equivalent to (21) and (23) respectively. It is hence guaranteed that there exists a $\mathbf{K}$ such that

$$(\tilde{\Gamma}_A + \tilde{\Gamma}_B \mathbf{K} \tilde{\Gamma}_C)^T \tilde{\Omega}_{cl_e} (\tilde{\Gamma}_A + \tilde{\Gamma}_B \mathbf{K} \tilde{\Gamma}_C) \succ 0. \tag{25}$$

For clarity, let $\eta \times \rho$ and $\mu \times \nu$ be the dimensions of $\tilde{\Gamma}_A$ and $\mathbf{K}$, respectively. Since

$$\mathbf{in}(\tilde{\Omega}_{cl_e}) = \mathbf{in}\begin{pmatrix} 0 & \tilde{Y}_{cl_e} \\ \tilde{Y}_{cl_e} & 0 \end{pmatrix} + \mathbf{in}(\tilde{N}) + \mathbf{in}\begin{pmatrix} \gamma I & 0 \\ 0 & -\gamma^{-1} I \end{pmatrix},$$

it is easily verified that $\mathbf{in}(\tilde{\Omega}_{cl_e}) = (\rho, \eta, 0)$. A $\mathbf{K}$ that satisfies (25) can be obtained as follows. Let

$$\Lambda := \left(\,\tilde{\Gamma}_A\ \tilde{\Gamma}_B\,\right)^T \tilde{\Omega}_{cl_e} \left(\,\tilde{\Gamma}_A\ \tilde{\Gamma}_B\,\right),$$

and rewrite (25) as

$$\begin{pmatrix} I_\rho \\ \mathbf{K}\tilde{\Gamma}_C \end{pmatrix}^T \Lambda \begin{pmatrix} I_\rho \\ \mathbf{K}\tilde{\Gamma}_C \end{pmatrix} \succ 0.$$

Hence, $n_+(\Lambda) \geq \rho$. But since Λ is obtained by restricting $\tilde{\Omega}_{cl_e}$ to a certain linear subspace, we have $\rho = n_+(\tilde{\Omega}_{cl_e}) \geq n_+(\Lambda)$. We conclude that $\mathbf{in}(\Lambda) = (\rho, \mu, 0)$ and we can then dualize the last inequality above as

$$(\star)^T \Lambda^{-1} \begin{pmatrix} -\tilde{\Gamma}_C^T \mathbf{K}^T \\ I_\mu \end{pmatrix} = (\star)^T \Xi \begin{pmatrix} -\mathbf{K}^T \\ I_\mu \end{pmatrix} \prec 0,$$

where $\Xi := (\star)^T \Lambda^{-1} \begin{pmatrix} -\tilde{\Gamma}_C^T & 0 \\ 0 & I_\mu \end{pmatrix}$. Hence, $n_-(\Xi) \geq \mu$. Similar to the previous case, since $\mu = n_-(\Lambda) \geq n_-(\Xi)$, we conclude that $\mathbf{in}(\Xi) = (\nu, \mu, 0)$. Therefore, one can always find a matrix $V = \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} \in \mathbb{R}^{(\mu+\nu)\times\mu}$ such that $V^T \Xi V \prec 0$ and V_2 is non-singular. Then, $\mathbf{K} = -\left(V_1 V_2^{-1}\right)^T$ satisfies (25). The order of the resulting controller is equal to the sum of the orders of the plant and the multiplier.

Finally, to prove that $\mathbf{A}_C$ is Hurwitz, consider the coupling conditions in (17). We can equivalently write

$$T^T Y T - \begin{pmatrix} X & 0 \\ 0 & 0 \end{pmatrix} - T^T (Y - Z) T \succ 0 \quad \text{and} \quad (Y - Z)^{-1} \succ 0.$$

By the Schur complement formula, this is equivalent to

$$\left(\begin{array}{c|c} T^T Y T - \begin{pmatrix} X & 0 \\ 0 & 0 \end{pmatrix} & T^T \\ \hline T & (Y - Z)^{-1} \end{array}\right) =: \left(\begin{array}{cc|c} Y_{22} - X & Y_{2G} & Y_{2C} \\ \star & Y_{GG} & Y_{GC} \\ \hline \star & \star & Y_{CC} \end{array}\right) \succ 0.$$

Hence, the constructed $\mathbf{A_C}$ is guaranteed to be Hurwitz by Theorem 3 and the $\mathcal{L}_2$-gain from w to z is less than γ for all admissible Δ blocks.

4. SUMMARY

We have given a finite-dimensional, convex solution to the problem of robust $\mathcal{L}_2$-gain feedforward control problem where the uncertainties affecting the system are described by IQCs involving dynamic multipliers. Our solution builds on the dual formulation of the stability characterization we have given in Scherer and Köse [2008] and uses a generalization of the quadratic elimination lemma that is applicable to the case involving dynamic multipliers. Our main result is stated as an existence condition for robust $\mathcal{L}_2$-gain feedforward controllers. Therefore, it differs from the results of Köse and Scherer [2007a], where the controller appears implicitly in the solvability conditions. Applications of the main technical tools used here to the problem of gain-scheduled controllers are reported in Scherer and Köse [2007b]. Lastly, the presented formulation can be adapted to the problem of robust $\mathcal{L}_2$-gain estimation in a fairly straightforward manner.

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Appendix A. QUADRATIC CONSTRAINTS AND ELIMINATION

Lemma 5. [Scherer, 2001] Let $S \in \mathbb{C}^{(m+n)\times m}$ have full column-rank and $M = M^* \in \mathbb{C}^{(m+n)\times(m+n)}$ be such that $\mathbf{in}(M) = (m, n, 0)$. Then, $S^*MS \succ 0$ if and only if $S_\perp^* M^{-1} S_\perp \prec 0$, where $S_\perp$ forms a basis for the orthogonal complement of the image of S.

A generalization of Lemma 5 to operators acting on $\mathcal{L}_2$ can be given as in the next lemma. The proof follows ideas similar to the proof of Lemma 5 and is omitted for brevity.

Lemma 6. Let $\Delta : \mathcal{L}_2^\rho \to \mathcal{L}_2^\mu$ be linear and suppose $\Pi = \Pi^* \in R\mathcal{L}_\infty^{(\rho+\mu)\times(\rho+\mu)}$ is such that $\mathbf{in}(\Pi) = (\rho, \mu, 0)$ on $\mathbb{C}^0$. Then, the following statements are equivalent:

(i) $\left\langle \begin{pmatrix} v \\ \Delta v \end{pmatrix}, \Pi \begin{pmatrix} v \\ \Delta v \end{pmatrix} \right\rangle \geq 0 \quad \forall v \in \mathcal{L}_2^\rho.$

(ii) $\left\langle \begin{pmatrix} -\Delta^* w \\ w \end{pmatrix}, \Pi^{-1} \begin{pmatrix} -\Delta^* w \\ w \end{pmatrix} \right\rangle \leq 0 \quad \forall w \in \mathcal{L}_2^\mu.$

Finally, the derivation of the existence conditions of the feedforward controller is based on the following elimination lemma.

Lemma 7. [Helmersson, 1999] Let $\mathcal{A} \in \mathbb{R}^{(k+n)\times n}$, $\mathcal{B} \in \mathbb{R}^{(k+n)\times m}$, $\mathcal{C} \in \mathbb{R}^{p\times n}$ and $\Omega = \Omega^T \in \mathbb{R}^{(k+n)\times(k+n)}$ be given. Assume $\mathbf{in}(\Omega) = (k, n, 0)$. Then, there exists a $K \in \mathbb{R}^{m\times p}$ such that

$$(\mathcal{A} + \mathcal{B}K\mathcal{C})^T \Omega (\mathcal{A} + \mathcal{B}K\mathcal{C}) \prec 0 \qquad \text{(A.1)}$$

if and only if

$$\left(\mathcal{C}^T\right)_\perp^T \mathcal{A}^T \Omega \mathcal{A} \left(\mathcal{C}^T\right)_\perp \prec 0 \qquad \text{(A.2a)}$$

$$\left(\, \mathcal{A} \;\; \mathcal{B} \,\right)_\perp^T \Omega^{-1} \left(\, \mathcal{A} \;\; \mathcal{B} \,\right)_\perp \succ 0. \qquad \text{(A.2b)}$$