Proceedings of the 17th World Congress
The International Federation of Automatic Control
Seoul, Korea, July 6-11, 2008

Design of a Robust Nonlinear Receding-Horizon Observer - Application to a biological system

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Abstract: The objective of this study is to design a robust receding-horizon observer for systems described by nonlinear models with uncertain parameters. Robustification in the presence of model uncertainties naturally leads to the formulation of a nonlinear min-max optimization problem, which can either be solved numerically or which can be converted to a simpler minimization problem using linearization along a nominal trajectory and recent results in linear robust receding-horizon estimation. This method is first evaluated in simulation and then with real-life experimental data collected from continuous cultures of phytoplankton.

Keywords: Software Sensors, State Estimation, Receding-horizon, Uncertain Systems, Bioprocesses, Phytoplankton

1. INTRODUCTION

Due to the lack of reliable on-line measurements of the key components of cell cultures, state estimation techniques (or software sensors) play an increasingly important role in bioprocess monitoring. A challenge to face in the design of software sensors is however the uncertainty associated with the underlying bioprocess model, which motivates in turn the development of robust state estimation techniques.

Receding-horizon estimation is a popular method where the most-likely initial conditions of a process model are estimated on the basis of data available in a moving time frame by solving a minimization problem. Using the estimated values and the process model, a state prediction can then be computed up to the next measurement time. Several applications of receding-horizon observers can be found in, e.g., Bogaerts and Hanus (2001), Alamir and Corriou (2003) and Valdéz-González et al. (2003).

In this study, we consider the classical situation where a process model has been previously identified based on experimental data, and where, besides the estimated values of the parameters, parameter bounds have been evaluated (these bounds result for instance from the analysis of the error covariance matrix and the consideration of confidence intervals). Our objective is to design a state estimation method, which would be robust to these parameter uncertainties. The formulation of this problem naturally leads to a nonlinear min-max optimization problem. The latter can either be solved numerically (at the price of expensive computation), or converted to a simpler minimization problem by using a model linearization along a nominal trajectory (defined by nominal parameter values and the most likely initial conditions) and recent results in linear robust receding-horizon estimation developed by Alessandri et al. (2005).

In order to demonstrate the performance of the method, the culture of phytoplankton in a bioreactor operated in chemostat mode is considered. Based on on-line measurements of the biovolume, the substrate concentration as well as the internal quota are reconstructed. The tests are first conducted in simulation using Droop model (Droop (1968)), then using real-life experimental data. In both cases, preliminary parameter identification provides bounds on the uncertain parameters, that can be exploited in the robust observer design.

Based on this application example, a comparison between the two numerical solution approaches to the min-max optimization problem is carried out. In order to reduce the computational expense required by the "brute-force" approach to the min-max optimization problem, advantage is taken from the monotonicity properties of the considered application example.

This paper is organized as follows. Section 2 describes the class of nonlinear systems that we are considering and formulates the optimization problems underlying receding-horizon estimation (classical formulation and robust formulation). In Section 3, the continuous culture of phytoplankton and Droop model are presented. Section 4 is devoted to the simplified numerical solution of the min-max optimization problem based on the monotonicity properties of the application example, whereas Section 5 makes uses of a model linearization and of recent results in linear robust estimation theory. Finally, Section 6 illustrates the algorithm performance with real-life experimental data and Section 7 is dedicated to conclusions and future work.

2. PROBLEM STATEMENT

Let us first assume that the system can be modelled by the following equations for $t \geq t_0$:

$$(\Sigma): \quad \begin{cases} \dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), \boldsymbol{\theta}) & \mathbf{x}(t_0) = \mathbf{x}_0 \\ \mathbf{y}_k = \mathbf{y}(t_k) = C\mathbf{x}(t_k) + \boldsymbol{\epsilon}(t_k) \end{cases} \quad (1)$$

The first part in (Σ) is the evolution equation and is represented by continuous-time differential equations where $\mathbf{f}$ is a vector of nonlinear functions, $\mathbf{x}(t) \in \Re^{n_x}$ is the vector of state variables, $\mathbf{u}(t) \in \mathcal{U} \subset \Re^{n_u}$ is the vector of inputs with $\mathcal{U}$ the set of admissible controls, a subset of the space of measurable bounded functions and $\boldsymbol{\theta} \in \Re^{n_\theta}$ is a parameter vector.

The second part is the observation equation and is modelled by linear discrete-time equations where C is the measurement matrix, $\mathbf{y}(t_k) \in \Re^{n_y}$ is the vector of sampled measurements and $\boldsymbol{\epsilon}(t_k) \in \Re^{n_y}$ is the measurement noise vector described by a Gaussian white noise sequence with a zero mean and a covariance matrix $Q(t_k)$.

$\mathbf{x}_0$ is the initial state vector containing values of the state at the initial time t_0. As usual in practice, the parameter vector has been estimated using some identification procedure based on experimental data, and parameter intervals $[\boldsymbol{\theta}^- \boldsymbol{\theta}^+]$ have been evaluated (for instance using the knowledge about the error covariance matrix) which are likely to enclose the unknown vector $\boldsymbol{\theta}$.

Finally, let us define $\mathbf{x}(t)$

$$\mathbf{x}(t) = \mathbf{g}(t, \mathbf{x}_0, \mathbf{u}_{t_0 \to t}, \boldsymbol{\theta}) \quad (2)$$

the state vector computed thanks to (1) from the initial state $\mathbf{x}(t_0)$ and corresponding to the input trajectory from $\mathbf{u}(t_0)$ to $\mathbf{u}(t)$.

2.1 Receding-horizon estimation

The basic principle of the receding-horizon observer is to compute estimates of the state trajectories using the process model and the best knowledge of the initial state vector resulting from an optimization procedure. Typically, the initial condition of a moving time frame is computed by minimizing the distance between the measurement data collected in the considered time frame and model prediction.

Let us consider a time interval containing $N+1$ measurements instants $\{t_{n-N}, \ldots, t_n\}$ with measurements $\{\mathbf{y}_{n-N}, \ldots, \mathbf{y}_n\}$. The typical receding-horizon optimization problem computes an estimation of the initial state $\hat{\mathbf{x}}_{n-N,n}$:

$$\hat{\mathbf{x}}^\circ_{n-N,n} = \arg \min_{\hat{\mathbf{x}}_{n-N,n}} J_{n,N}(\hat{\mathbf{x}}_{n-N,n}) \quad \text{with} \quad (3)$$

$$J_{n,N}(\hat{\mathbf{x}}_{n-N,n}) = \left\| \hat{\mathbf{x}}_{n-N,n} - \bar{\mathbf{x}}^\circ_{n-N} \right\|^2_M \quad (4)$$
$$+ \sum_{k=n-N}^{n} \left\| C\mathbf{g}\left(t_k, \hat{\mathbf{x}}_{n-N,n}, \mathbf{u}_{t_{n-N} \to t_k}, \boldsymbol{\theta}\right) - \mathbf{y}_k \right\|^2_{Q^{-1}(t_k)}$$

where $\|v\|_P = \left(v^T P v\right)^{1/2}$ is a weighted norm.

The first term in (4), weighted by the matrix M, expresses the belief in the estimation of the initial conditions of the moving-time horizon obtained from all the information collected up to n. In other words, this term has a recall action towards the solution of the optimization problem obtained in the previous estimation step. The matrix M is assumed to be positive definite and can be considered as a design parameter. This matrix determines the recall strength towards previously obtained initial conditions (note the analogy with nonlinear predictive control, where a reference optimal input trajectory is sometimes used in the same way). Depending on the value of n, two situations can be distinguished (Bogaerts and Hanus (2001)):

• $n \leq N$:

A full-horizon estimation scheme is then applied. In the $N_{min} - 1$ first steps ($(N_{min} - 1)n_y < n_x$), $\mathbf{x}^\circ_0 = \bar{\mathbf{x}}_0$, i.e., the best a priori available initial guess, in the absence of more on-line information. For $N_{min} \leq n \leq N$, $\hat{\mathbf{x}}^\circ_{0,n-1}$ is the solution of the optimization problem (4) on a time window of $n \leq N$ time instants.

$$\bar{\mathbf{x}}^\circ_0 = \begin{cases} \bar{\mathbf{x}}_0 & n < N_{min} \\ \hat{\mathbf{x}}^\circ_{0,n-1} & n \geq N_{min} \end{cases} \quad (5)$$

• $n > N$:

In this case, enough information is available and a receding-horizon can be used. The "reference" initial conditions of the time window $\bar{\mathbf{x}}^\circ_{n-N}$ are based on the optimal estimation computed in the previous step (t_{n-N-1}) and a one-step prediction computed with the system model.

$$\bar{\mathbf{x}}^\circ_{n-N} = \mathbf{g}\left(t_{n-N}, \hat{\mathbf{x}}^\circ_{n-N-1,n-1}, \mathbf{u}_{t_{n-N-1} \to t_{n-N}}, \boldsymbol{\theta}\right) \quad (6)$$

Knowing $\hat{\mathbf{x}}^\circ_{n-N,n}$, estimations until the next measurement time t_{n+1} are given by :

$$\hat{\mathbf{x}}(t) = \mathbf{g}\left(t, \hat{\mathbf{x}}^\circ_{n-N,n}, \mathbf{u}_{t_{n-N} \to t}, \boldsymbol{\theta}\right) \qquad \forall t \in [t_n, t_{n+1}] \quad (7)$$

2.2 Robust receding-horizon estimation

The previous procedure assumes a perfect knowledge of the parameter vector $\boldsymbol{\theta}$. In the case of uncertain parameters (a parameter interval $[\boldsymbol{\theta}^-, \boldsymbol{\theta}^+]$ is nonetheless available), an alternative procedure can be proposed implying a min-max optimization (see Alessandri et al. (2005) for the linear formulation):

$$\hat{\mathbf{x}}^\circ_{n-N,n} = \arg \min_{\hat{\mathbf{x}}_{n-N,n}} \max_{\hat{\boldsymbol{\theta}} \in [\boldsymbol{\theta}^-, \boldsymbol{\theta}^+]} \bar{J}_{n,N}\left(\hat{\mathbf{x}}_{n-N,n}, \hat{\boldsymbol{\theta}}\right) \quad (8)$$

$$\text{with } \bar{J}_{n,N}\left(\hat{\mathbf{x}}_{n-N,n}, \hat{\boldsymbol{\theta}}\right) = \left\| \hat{\mathbf{x}}_{n-N,n} - \bar{\mathbf{x}}^\circ_{n-N} \right\|^2_M +$$
$$\sum_{k=n-N}^{n} \left\| C\mathbf{g}\left(t_k, \hat{\mathbf{x}}_{n-N,n}, \mathbf{u}_{t_{n-N} \to t_k}, \hat{\boldsymbol{\theta}}\right) - \mathbf{y}_k \right\|^2_{Q^{-1}(t_k)} \quad (9)$$

and

• If $n \leq N$:

$$\bar{\mathbf{x}}^\circ_0 = \begin{cases} \bar{\mathbf{x}}_0 & n < N_{min} \\ \hat{\mathbf{x}}^\circ_{0,n-1} & n \geq N_{min} \end{cases}$$

• If $n > N$:

$$\bar{\mathbf{x}}^\circ_{n-N} = \mathbf{g}\left(t_{n-N}, \hat{\mathbf{x}}^\circ_{n-N-1,n-1}, \mathbf{u}_{t_{n-N-1} \to t_{n-N}}, \boldsymbol{\theta}_{nom}\right)$$

with

$$\boldsymbol{\theta}_{nom} = \frac{\boldsymbol{\theta}^- + \boldsymbol{\theta}^+}{2}$$

When $n > N$, the "reference" initial conditions of the time window $\bar{\mathbf{x}}^\circ_{n-N}$ are based on the optimal estimation computed in the previous step (t_{n-N-1}) and a one-step prediction computed with a nominal model (nominal means here using average parameters in the interval $[\boldsymbol{\theta}^-, \boldsymbol{\theta}^+]$). Indeed, as the time window is shifted of one step ahead, the previous knowledge on the initial conditions is "extrapolated" using an average process model.

3. ESTIMATION OF BIOLOGICAL VARIABLES IN CONTINUOUS CULTURES OF PHYTOPLANKTON

In order to test our algorithms, we consider the estimation of biological variables in the culture in chemostat of the green algae *Dunaliella tertiolecta*. This algae is a phytoplanktonic cell, i.e. a microscopic plant living in aquatic environment. Like terrestrial plants, it requires light for photosynthesis and essential substrates for growth, such as nitrate. As this kind of processes is difficult to study in the open sea, the phytoplanktonic growth analysis is carried out in a photobioreactor. A more complete description of the experimental setup can be found in Bernard et al. (2001).

In the following, three state variables will be considered to describe the growth of phytoplankton :

- The biovolume X which is the total amount of biomass per unit of volume ($\mu m^3/L$).
- The internal quota Q which is defined as the quantity of nitrogen per unit of biomass ($\mu mol/\mu m^3$).
- The substrate (nitrate) concentration S ($\mu mol/L$).

A mathematical model is used to predict the temporal evolution of the three above-mentioned state variables.

In contrast with the Monod model (Monod (1942)) which assumes that the consumed substrate is directly transformed in biomass, the Droop model (Burmaster (1979) and Droop (1968)) uncouples the growth rate from the uptake rate by introducing an intracellular store of nutrients.

The time-varying evolution equations resulting from mass balances are given by:

$$\begin{cases} \dot{X}(t) = -D(t)X(t) + \mu(Q)X(t) \\ \dot{Q}(t) = \rho(S) - \mu(Q)Q(t) \\ \dot{S}(t) = D(t)(S_{in}(t) - S(t)) - \rho(S)X(t) \end{cases} \tag{10}$$

with $\rho(S) = \rho_m \frac{S(t)}{S(t)+k_S}$ the uptake rate, and $\mu(Q) = \bar{\mu}\left(1 - \frac{k_Q}{Q(t)}\right)$, the growth rate. Moreover, if $Q(t) < k_Q$, $\mu(Q) = 0$.

In these equations, D represents the dilution rate (1/d, d : day), S_{in} the input substrate concentration ($\mu mol/L$), ρ the uptake rate ($\mu mol/(\mu m^3 d)$) and μ the growth rate ($1/d$). For the uptake rate, k_S and ρ_m represent respectively a half-saturation constant ($\mu mol/L$) for the substrate and the maximum uptake rate ($\mu mol/(\mu m^3 d)$). Concerning the growth rate, $\bar{\mu}$ is the theoretical maximum growth rate ($1/d$), obtained for an infinite internal quota and k_Q the minimum internal quota allowing growth ($\mu mol/\mu m^3$).

Moreover, to complete the model description, an observation equation is defined by measurements of biovolume X at measurement time t_k:

$$y_k = X(t_k) \tag{11}$$

An observability analysis of Droop model (10) shows that it is uniformly input observable with $y = X$ if $X \neq 0$ (see Bernard et al. (1998) for further details).

4. NUMERICAL SOLUTION OF THE MIN-MAX PROBLEM USING MONOTONICITY PROPERTIES

The solution of the min-max optimization problem (8)-(9) is not a trivial task especially in the nonlinear case where many local minima can occur.

In the considered application (continuous culture of phytoplankton described by Droop model), the monotonicity properties of the model can be used to simplify the maximization problem. Indeed, the worst-case biomass prediction (that will be compared to biomass measurements) can be generated using parameter bounds $\{\boldsymbol{\theta}^-, \boldsymbol{\theta}^+\}$ only, rather than by exploring the full parameter space $[\boldsymbol{\theta}^-, \boldsymbol{\theta}^+]$.

This simplification is approximate only since the biomass evolution is coupled with other component concentrations. However, Monte-Carlo simulations show that the computed trajectories are close to the worst ones, especially with small substrate concentrations.

The simplified algorithm is tested in simulation and compared with a standard receding-horizon observer, as described by (3)-(4), using an erroneous system model. The reference model parameters are the following : $\rho_m = 9.3 \times 10^{-9}$ $\mu mol \mu m^{-3} d^{-1}$, $k_S = 0.105$ $\mu mol L^{-1}$, $\bar{\mu} = 2$ d^{-1} and $k_Q = 1.8 \times 10^{-9}$ $\mu mol \mu m^{-3}$. The uncertain parameter subspace $[\boldsymbol{\theta}^-, \boldsymbol{\theta}^+]$ is given by $[0.8\boldsymbol{\theta}, 1.5\boldsymbol{\theta}]$. The input parameters (S_{in}, D) are assumed perfectly known. The parameters of the erroneous model are chosen on the parameter subspace border ($\boldsymbol{\theta}_{false} = [\boldsymbol{\theta}^+(1), \boldsymbol{\theta}^-(2), \boldsymbol{\theta}^+(3), \boldsymbol{\theta}^-(4)]^T$) and correspond thus to one of the worst-case model. The initial estimation $\bar{\mathbf{x}}_0$ is randomly chosen in the interval $[0.7\mathbf{x}_0, 3\mathbf{x}_0]$ with $\mathbf{x}_0$, the reference initial conditions given in Table 1.

	Unit	Reference value
X_0	$\mu m^3/L$	0.1×10^9
Q_0	$\mu mol/\mu m^3$	4.5×10^{-9}
S_0	$\mu mol/L$	50

Table 1. Reference values for initial conditions

The measurements of biovolume y_k (11), are corrupted by a Gaussian white noise (with a relative standard deviation of 0.08 and a minimum s.d. of 1.5×10^9 $\mu m^3/L$). Figure 1 compares the performance of the two above-mentioned algorithms. Expectedly, an erroneous model usually provides poor state estimates. In contrast, robust estimation provides satisfactory estimation, despite the uncertainty on the parameter values. Note however that this estimation is biased as the average model provides biased prediction of the real system state.

The main disadvantage of the present formulation is its high computational demand (even with the simplification based on the monotonicity properties of the considered example). This motivates the use of other artifices to obtain a more computationally effective formulation.

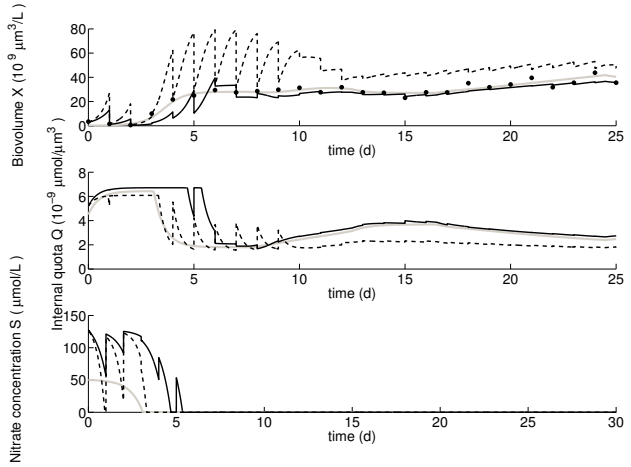

Fig. 1. Estimation performance in simulation. Real trajectories are in grey and estimations are in black. Biomass measurements are used for estimation. Simplified robust receding-horizon (−) and receding-horizon with an erroneous model (−−) have both as tuning parameters $N = 4$ and $M = 0.1I$ (I is the identity matrix).

5. ON THE USE OF LINEARIZATION TECHNIQUES

To reduce the computational demand, we will make use of linearization techniques. Indeed, in the linear case, a theorem is available (Sayed et al. (2002)) for converting a min-max optimization problem into a standard minimization problem λ°:

Theorem 1. Consider a regularized robust least-squares problem of the form :

$$\min_{\mathbf{z}} \max_{\|\mathrm{S}\|\leq 1} \|\mathbf{z}\|_V^2 + \|[D + \delta D(S)]\,\mathbf{z} - [\mathbf{e} + \delta\mathbf{e}\,(S)]\|_W^2 \quad (12)$$

$$\delta D(S) = HSE_d, \quad V > 0, \quad \delta\mathbf{e}(S) = HSE_e, \quad W \geq 0$$

Problem (12) has a unique global minimum $\mathbf{z}^\circ$ given by:

$$\mathbf{z}^\circ = \left(\hat{V} + D^T\hat{W}D\right)^{-1}\left(D^T\hat{W}\mathbf{e} + \lambda^\circ E_d^T E_e\right)$$

where

$$\hat{V} = V + \lambda^\circ E_d^T E_d$$
$$\hat{W} = W + WH\left(\lambda^\circ I - H^T W H\right)^\dagger H^T W$$

and the scalar parameter λ° is determined as

$$\lambda^\circ = \arg \min_{\lambda \geq \|H^T W H\|} \|\mathbf{z}(\lambda)\|_V^2 + \lambda \|E_d\mathbf{z}(\lambda) - E_e\|^2 + \|D\mathbf{z}(\lambda) - \mathbf{e}\|_{\hat{W}(\lambda)}^2$$

$$\mathbf{z}\,(\lambda) = \left(\hat{V}(\lambda) + D^T\hat{W}(\lambda)D\right)^{-1}\left(D^T\hat{W}(\lambda)\mathbf{e} + \lambda E_d^T E_e\right)$$
$$\hat{V}(\lambda) = V + \lambda E_d^T E_d$$
$$\hat{W}(\lambda) = W + WH\left(\lambda I - H^T W H\right)^\dagger H^T W$$

Matrix norm, like e.g. $\|P\|$ is related to the maximum singular value of the corresponding matrix i.e. $\|P\| = \left(\bar{\sigma}\left(P^T P\right)\right)^{1/2}$ with $\bar{\sigma}(P)$ the maximum eigenvalue of P. † denotes the left pseudoinverse.

Proof. A demonstration can be found in Sayed et al. (2002).

In order to apply this theorem to the min-max optimization problem (8)-(9), we have to linearize the nonlinear function $\mathbf{g}$ with respect to $\bar{\mathbf{x}}_{n-N}^\circ$ and $\boldsymbol{\theta}_{nom}$ (Bogaerts and Hanus (2000)):

$$\mathbf{g}\left(t_k, \hat{\mathbf{x}}_{n-N,n}, \mathbf{u}_{t_{n-N}\to t_k}, \hat{\boldsymbol{\theta}}\right) \approx \mathbf{g}_{nom}\,(n-N, t_k) + G_x\,(n - N, t_k)\left(\hat{\mathbf{x}}_{n-N,n} - \bar{\mathbf{x}}_{n-N}^\circ\right) + G_\theta\,(n-N, t_k)\left(\boldsymbol{\theta} - \boldsymbol{\theta}_{nom}\right)$$

with

$$\mathbf{g}_{nom}\,(n-N, t_k) = \mathbf{g}\left(t_k, \bar{\mathbf{x}}_{n-N}^\circ, \mathbf{u}_{t_{n-N}\to t_k}, \boldsymbol{\theta}_{nom}\right) \quad (13)$$

$$G_x\,(n-N, t_k) = \left.\frac{\partial \mathbf{g}\left(t_k, \mathbf{x}_{n-N}, \mathbf{u}_{t_{n-N}\to t_k}, \boldsymbol{\theta}\right)}{\partial \mathbf{x}_{n-N}}\right|_{\substack{\mathbf{x}_{n-N}=\bar{\mathbf{x}}_{n-N}^\circ \\ \boldsymbol{\theta}=\boldsymbol{\theta}_{nom}}} \quad (14)$$

$$G_\theta\,(n-N, t_k) = \left.\frac{\partial \mathbf{g}\left(t_k, \mathbf{x}_{n-N}, \mathbf{u}_{t_{n-N}\to t_k}, \boldsymbol{\theta}\right)}{\partial \boldsymbol{\theta}}\right|_{\substack{\mathbf{x}_{n-N}=\bar{\mathbf{x}}_{n-N}^\circ \\ \boldsymbol{\theta}=\boldsymbol{\theta}_{nom}}} \quad (15)$$

Jacobian matrices defined in (14) and (15) can be computed for time $t \in [t_{n-N}, t_n]$ by solving numerically the following differential equations together with (1) (for $\mathbf{x}(t_{n-N}) = \bar{\mathbf{x}}_{n-N}^\circ$ and $\boldsymbol{\theta} = \boldsymbol{\theta}_{nom}$):

$$\dot{G}_x\,(n-N, t) = \frac{\partial \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), \boldsymbol{\theta}_{nom})}{\partial \mathbf{x}} G_x\,(n-N, t) \quad (16)$$

$$\dot{G}_\theta\,(n-N, t) = \frac{\partial \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), \boldsymbol{\theta}_{nom})}{\partial \mathbf{x}} G_\theta\,(n-N, t) + \left.\frac{\partial \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), \boldsymbol{\theta})}{\partial \boldsymbol{\theta}}\right|_{\boldsymbol{\theta}=\boldsymbol{\theta}_{nom}} \quad (17)$$

The initial conditions of (16) and (17) are respectively given by :

$$G_x\,(n-N, t_{n-N}) = \frac{\partial \mathbf{g}\left(t_{n-N}, \mathbf{x}_{n-N}, \mathbf{u}_{t_{n-N}\to t_{n-N}}, \theta\right)}{\partial \mathbf{x}_{n-N}} = \frac{\partial \mathbf{x}_{n-N}}{\partial \mathbf{x}_{n-N}} = I_{n_x}$$
$$G_\theta\,(n-N, t_{n-N}) = 0_{n_x}$$

A new cost function can now be given by the following expression :

$$\check{J}_{n,N}\left(\hat{\mathbf{x}}_{n-N,n},\hat{\boldsymbol{\theta}}\right) = \left\|\hat{\mathbf{x}}_{n-N,n}-\bar{\mathbf{x}}^{\circ}_{n-N}\right\|^2_M + \left\|\bar{G}^n_{x,n-N}\left(\hat{\mathbf{x}}_{n-N,n}-\bar{\mathbf{x}}^{\circ}_{n-N}\right) - \left(Y^n_{n-N}-\bar{\mathbf{g}}^n_{n-N}-\bar{G}^n_{\theta,n-N}S\Delta\boldsymbol{\theta}_{max}\right)\right\|^2_W \quad (18)$$

with $\Delta\boldsymbol{\theta}_{max}=\frac{\boldsymbol{\theta}^+-\boldsymbol{\theta}^-}{2}$ and $\|S\|\leq 1$,

$W= diag\left(Q^{-1}(t_{n-N}),\ldots,Q^{-1}(t_n)\right)$ a bloc diagonal matrix, $Y^n_{n-N}=\left[\,y^T(t_{n-N})\ldots y^T(t_n)\,\right]^T$ the column vector containing all the measurements,

$\bar{\mathbf{g}}^n_{n-N}=\begin{bmatrix} C\mathbf{g}_{nom}\left(n-N,t_{n-N}\right)\\ \vdots \\ C\mathbf{g}_{nom}\left(n-N,t_n\right)\end{bmatrix}$ the column vector containing the measured state vectors for the "nominal" case $\left(\bar{\mathbf{x}}^{\circ}_{n-N},\boldsymbol{\theta}_{nom}\right)$, $\bar{G}^n_{x,n-N}=\begin{bmatrix} CG_x\left(n-N,t_{n-N}\right)\\ \vdots \\ CG_x\left(n-N,t_n\right)\end{bmatrix}$ regrouping the Jacobian matrices related to the initial state variables and $\bar{G}^n_{\theta,n-N}=\begin{bmatrix} CG_\theta\left(n-N,t_{n-N}\right)\\ \vdots \\ CG_\theta\left(n-N,t_n\right)\end{bmatrix}$ regrouping the Jacobian matrices related to the parameters.

Finally, the solution of the robust receding-horizon problem can be summarized as :

Theorem 2. The problem defined by (8) with (18) as cost function has a unique solution given by

$$\hat{\mathbf{x}}^{\circ}_{n-N,n}=\bar{\mathbf{x}}^{\circ}_{n-N}+\left(M+\bar{G}^{n,T}_{x,n-N}\hat{W}\bar{G}^n_{x,n-N}\right)^{-1}\bar{G}^{n,T}_{x,n-N}\hat{W}\left(Y^n_{n-N}-\bar{\mathbf{g}}^n_{n-N}\right)$$
$$with\quad \hat{W}(\lambda^\circ)=W+W\bar{G}^n_{\theta,n-N}\left(\lambda^\circ I-\bar{G}^{n,T}_{\theta,n-N}W\bar{G}^n_{\theta,n-N}\right)^{\dagger}\bar{G}^{n,T}_{\theta,n-N}W$$

λ° is computed from the following minimization

$$\lambda^\circ=\arg\min_{\lambda\geq\left\|G^{n,T}_{\theta,n-N}WG^n_{\theta,n-N}\right\|}\left\|\bar{\mathbf{z}}(\lambda)\right\|^2_M+\lambda\left\|\Delta\boldsymbol{\theta}_{max}\right\|^2 + \left\|G^n_{x,n-N}\bar{\mathbf{z}}(\lambda)-\left(Y^n_{n-N}-\bar{\mathbf{g}}^n_{n-N}\right)\right\|^2_{\hat{W}(\lambda)}$$

with

$$\bar{\mathbf{z}}(\lambda)=\left(M+\bar{G}^{n,T}_{x,n-N}\hat{W}(\lambda)G^n_{x,n-N}\right)^{-1}\bar{G}^{n,T}_{x,n-N}\hat{W}(\lambda)\left(Y^n_{n-N}-\bar{\mathbf{g}}^n_{n-N}\right)$$
$$\hat{W}(\lambda)=W+W\bar{G}^n_{\theta,n-N}\left(\lambda^\circ I-\bar{G}^{n,T}_{\theta,n-N}W\bar{G}^n_{\theta,n-N}\right)^{\dagger}\bar{G}^{n,T}_{\theta,n-N}W$$

Proof : a trivial application of Theorem 1.

Of course, the linearized model differs from the nonlinear one, and results in an approximation of the original problem. The quality of this approximation will mostly depend on the fact that the worst linearized model is as worse as the worst nonlinear one. This algorithm is now applied in simulation to the continuous culture of phytoplankton with the same parameter subspace, input values and initial conditions as in Section 4. Figure 2 shows these results obtained with the linearized robust receding-horizon observer. The design parameters are a time window with 6 instants ($N=5$) and a weighting matrix for the initial guess $M=I$. The algorithm performs well and offers a very significant computational load reduction (from several hours to a few minutes using Matlab on a standard PC).

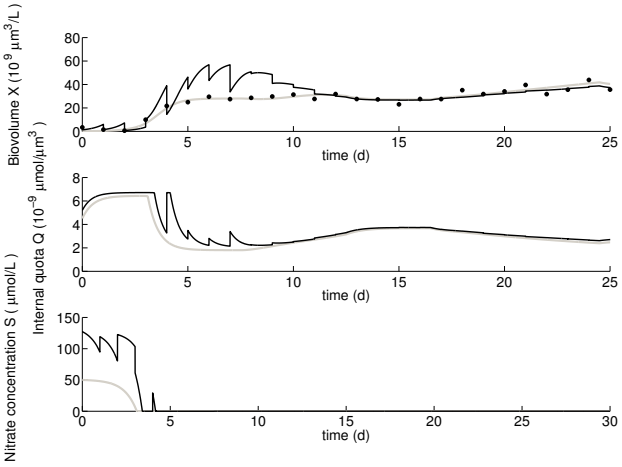

Fig. 2. Robust receding-horizon (linearized version) in simulation with $N=5$ and $M=I$. Real trajectories are in grey and estimations are in black. Biomass measurements are used for estimation.

6. EXPERIMENTAL APPLICATION

We now turn our attention to an experimental study, i.e., phytoplanktonic growth in the chemostat. In this setup, the culture volume is kept constant by continuously introducing and removing culture elements. According to an experimental protocol presented in Bernard et al. (2001), several measurements are collected, including biovolume or substrate concentration. The results of parameter identification are given in Table 2 for the model parameters and in Table 3 for the initial conditions. The experiments are carried out with a time-varying dilution rate $D(t)$, whose intervals of variation are given in Table 4.

Parameter	Unit	Interval value
S_{in}	$\mu mol/L$	[95 , 105]
k_S	$\mu mol/L$	[0.01 , 0.20]
$\bar{\mu}$	$1/d$	[1.70 , 2.30]
k_Q	$\mu mol/\mu m^3$	$[1.60\ ,\ 2.00]\times 10^{-9}$
ρ_m	$\mu mol/(\mu m^3 d)$	$[9.25\ ,\ 9.55]\times 10^{-9}$

Table 2. Interval values for model parameters

	Unit	Interval value
X_0	$\mu m^3/L$	$\left[10^{-6}\ ,\ 0.2\right]\times 10^9$
Q_0	$\mu mol/\mu m^3$	$[1\ ,\ 8]\times 10^{-9}$
S_0	$\mu mol/L$	[40 , 60]

Table 3. Interval values for initial conditions

A relative standard deviation of 0.03 is considered for the measurement errors related to the biomass concentration. Figure 3 shows some estimation results with the linearized

$[t]$ (d)	$[0, 7.5]$	$[7.5, 12.5]$	$[12.5, 16.5]$
D $(1/d)$	0	$[0.92, 1.02]$	$[0.9, 1.02]$
$[t]$ (d)	$[16.5, 17.5]$	$[17.5, 24.5]$	$[24.5, 33.5]$
D $(1/d)$	0.9	$[0.45, 0.51]$	$[0.51, 0.59]$

Table 4. Interval values for the dilution rate

robust receding-horizon algorithm described in Section 5. Design parameters are a time window with 11 instants and a weighting matrix for the initial estimation $M = 50I$.

A small value of N involves a higher sensitivity to measurements errors whereas a larger one induces longer computational efforts. Furthermore, a large time window penalizes prediction when the considered model is far from the real one. Moreover, M is related to the a priori initial estimation. A small value induces a higher confidence in the measurements and estimation is sensitive to outliers, whereas a larger one means a high confidence in the system model.

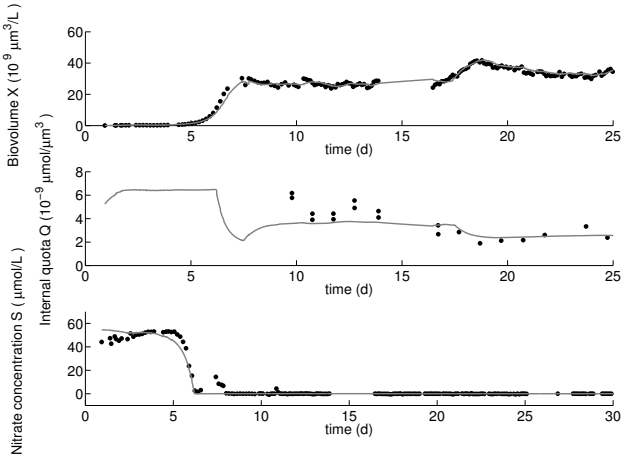

Fig. 3. Robust receding-horizon (linearized version) with experimental data. Design parameters are $N = 10$ and $M = 50I$.

7. CONCLUSIONS

In this study, we consider the situation where a process model has been identified and parameters are known to lie within some confidence intervals. Receding-horizon observers are then design in order to take account of these parameter uncertainties.

The underlying min-max optimization problem is solved in two ways. First, a brute-force numerical strategy is used, in which the monotonicity properties of the considered application example (continuous culture of phytoplankton described by Droop model) are exploited to reduce the search in the whole parameter subspace to a search on its boundary. Even though this simplification is very effective, the resulting computational demand is still excessive. Second, a linearization of the process model is effected along a nominal trajectory (defined by nominal parameters and most-likely initial conditions) and recent results in linear robust estimation are used to turn the original min-max problem to a simple minimization problem.

Tests in simulation and with real-life experimental data show good performance of robust methods with respect to model uncertainties. Moreover, in the second version, linearization allows to significantly reduce the computational load. Tuning is easy because it is only related to the length of the time window and to the weighting matrix of the a priori initial estimation.

Future work will be directed to the determination of min-max estimation bounds and their validation with other biological applications and real data.

8. ACKNOWLEDGMENT

This paper presents research results of the Belgian Network DYSCO (Dynamical Systems, Control, and Optimization), funded by the Interuniversity Attraction Poles Programme, initiated by the Belgian State, Science Policy Office. The scientific responsibility rests with its authors. The authors gratefully acknowledge the Laboratoire d'Océanographique in Villefranche-sur-Mer (France) for providing the experimental data.

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