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Robust H_∞ filtering by means of lead-lag controller

Philippe Neveux * **Eric Blanco** **

* *Université d'Avignon et des Pays de Vaucluse*
UMR UAPV-INRA Environnement Méditerranéen et Modélisation des Agro-Hydrosystèmes
Faculté des sciences
84029 Avignon cedex 1 - France
email: philippe.neveux@univ-avignon.fr

** *Ecole Centrale de Lyon*
Laboratoire Ampère - UMR CNRS 5005
36 avenue Guy de Collongue
69134 Ecully cedex - France
email: eric.blanco@ec-lyon.fr

Abstract: In the present paper, a new strategy for robust filtering problem of linear time-invariant (LTI) continuous time system is proposed. The key idea consists in generalizing the structure of a linear state estimator of the Luenberger class. As a matter of fact, the closed loop form of this class of state estimator can be assimilated to a closed loop control problem. Then, the standard correction term can be viewed as a Proportional controller. In this paper, we propose a more general form of controller in order to obtain the robustness. An example shows the efficiency of the proposed approach.

Keywords: Filtering techniques, H-infinity optimization, signal processing, robustness, uncertainty.

1. INTRODUCTION

In engineering problems, there is a recurrent need of information concerning process state in order to control it. Unfortunately, in practice, it is not systematically possible to have access to these information. Consequently, from available measurements and process model, one should manage to estimate the physically unaccessible information. In the state-space representation of linear systems, this problem can be tackled by the Luenberger observer.

The robustness of standard state estimators of the Lunberger class is intimately related to the validity of the available model. As a matter of fact, errors in the model parameters lead to poor performances. In order to robustify this kind of linear estimators, various approaches have been used.

H_∞ robust estimators have been intensively developed in order to guaranty a constant level of performance over a range of possible models. Clearly, this means that the designer has to give a representation to the model errors (termed as model uncertainties) and then, minimize the effect of the worst case uncertainty on the estimation error (Petersen [1994], RezaMoheimani [1998], Souza [1995], Gao [2005]).

Another approach consists in considering that the poor performances are due to the lack of information on the model. This problem has been solved by introducing the sensitivity of the estimation error with respect the supposed uncertain parameters. In this context, no uncertainty modelling is required (Neveux [2001]).

Finally, a control-based approach has been considered. As a matter of fact, state estimation can be viewed as a state feedback control problem. Consequently, in order to solve the problem of biased estimation in presence of model uncertainty, an integral action as been added to the standard Luenberger structure (Lee [1979], Busawon [2000], Duan [2001]). The obtained filter is known as the Proportional-Integral Kalman filter (PI Kalman).

In the present paper, we adopt the latter approach. The basic idea consists in considering that even if the PI structure is of great interest, it can be improved by a more general form. In this work, it is shown that the original structure of the Luenberger observer can be modified in order to give a more general tuning framework. In that purpose, we propose in the Laplace domain to consider that the correcting term in the estimator is a lead-lag controller rather than a PI. The state-space representation is given and a H_∞ tuning criterion is given in order to ease the implementation of the proposed estimator. The optimization will be done by a Genetic Algorithm. This kind of heuristic optimization has been used with success in deconvolution (Chen [2000]) and control problems (Zhang [2005]).

The paper is organized as follows. In section 2, we present the system under consideration and give useful assumptions in order to solve the filtering problem. The structure of the proposed robust filter is given in section 3. A tuning criterion is given in order to ease the implementation of the filter. An example is presented in section 4 that shows the interest of such an approach. Finally, concluding remarks will be given in section 5.

2. POSITION OF THE PROBLEM

Consider the LTI multi-input multi-output system (Σ) represented by the set of equations:

$$\begin{cases} \dot{x}(t) = \tilde{A}x(t) + Bu(t) + Mw(t) \\ y(t) = Cx(t) + v(t) \\ z(t) = Lx(t) \end{cases} \quad (1)$$

with the state $x(t) \in \Re^n$, the deterministic input $u(t) \in \Re^m$, the measured output $y(t) \in \Re^p$ and the desired output $z(t) \in \Re^q$. All the matrices are real valued matrices with appropriate dimensions. Further, we assume that the matrix $\tilde{A}$ is uncertain. Hence, we decompose it as follows: $\tilde{A} = A + \Delta A$ where A is the nominal matrix that designer holds to design the state estimator, ΔA is the modeling error.

The filtering problem will be solved under the following assumptions:

(1) the system is detectable and stabilizable,
(2) $w(t)$ and $v(t)$ are L_2 noises.

The objective of this paper is to estimate the desired signal $z(t)$ from the known deterministic input $u(t)$ and the measured output $y(t)$ when the system is uncertain.

3. ROBUST FILTER DESIGN

3.1 Structural considerations

The conventional structure of the linear state estimator is the Luenberger one. Its state equation is given by:

$$\begin{cases} \dot{\hat{x}}_L(t) = A\hat{x}_L(t) + Bu(t) + K_L\epsilon \\ \epsilon(t) = y(t) - C\hat{x}_L(t) \\ \hat{z}_L(t) = L\hat{x}_L(t) \end{cases} \quad (2)$$

where the matrix $K_L \in \Re^{n\times p}$ is a tuning matrix that is generally obtained by pole assignment of the matrix $(A-K_LC)$, and $\hat{x}_L(t) \in \Re^n$ is the state estimate. The most popular state estimator of this class is the Kalman filter that sets K_L according to the minimum variance criterion (Gelb [1974]).

In presence of model uncertainty, three approaches have been developed in order to guaranty robust signal estimation. In the following, two of them, related to control approaches, will be developed. In order to explain their strategies, we give the general transfer function form of the Luenberger like estimator, that is:

$$\hat{x}_L(s) = P(s)Bu(s) + P(s)\mathcal{K}_L(s)\epsilon(s) \quad (3)$$

with

$$P(s) = (sI - A)^{-1} \quad (4)$$

The block diagram corresponding to this relation is given in figure 1. Clearly, it appears that this structure is a control one where an additive control $\zeta(t) \in \Re^n$ permits the estimator to fit the state of the system.

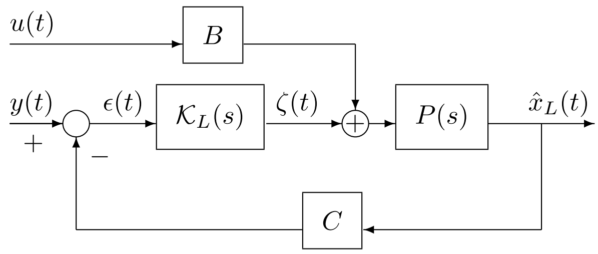

Fig. 1. Block diagram of the generalized structure of the filtering problem.

In presence of model uncertainty, the following control-based strategies are adopted in order to avoid biased estimation:

(1) increase the matrix gain K_L in order to enlarge the bandpass of the estimator. In the case of Kalman based design technique, the robust estimator is based on a modelling of the model uncertainties by means of a stochastic (Petersen [1994], RezaMoheimani [1998], Souza [1995]) or a polytopic decomposition (Gao [2005]).
(2) consider that $\mathcal{K}_L(s)$ is a Proportional Integral controller (Lee [1979], Busawon [2000], Duan [2001]). The state-space equations of the corresponding estimator is obtained from the relations in the Laplace domain and give rise to the PI Kalman filter.

Though efficient, the first class of solution introduces an important level of noise in the state estimate (due to an important enlargement of the bandpass in comparison with the H_2 Kalman filter) while the second class permits to overcome this problem (Busawon [2000]). Nevertheless, the stability of the PI Kalman filter should be carefully treated. This problem has been addressed in Duan [2001] and solved by means of eigenstructure assignment in the discrete-time context.

3.2 The lead-lag controller

The PI controller is a so-called universal controller. Consequently, it may not be perfectly adapted to the case under study. In order to overcome this drawback, we propose the following structure for $\mathcal{K}_L(s)$:

$$\mathcal{K}_L^{ij}(s) = \frac{1 + \pi_{ij}\kappa_{ij}\tau_j s}{1 + \tau_j s} \quad (5)$$

which represents the lead-lag controller. The tuning parameters are τ_j and π_{ij}. Parameters κ_{ij} are Kalman filter gains.

In order the give trends on the lead-lag controller, let introduce $\alpha_{ij} = \pi_{ij}\kappa_{ij}$. This parameter permits to define the controller properties as follows:

- if $\alpha_{ij} > 1$, then $\mathcal{K}_L^{ij}(s)$ is a lead controller. It permits to increase the rapidity of the system due to an increased bandpass of the closed-loop system.

- if $0 < \alpha_{ij} < 1$, then $\mathcal{K}_L^{ij}(s)$ is a lag controller. This action acts essentially on a very low frequency range. It does not influence the bandpass and the rapidity of the closed-loop system.

Remark 1. From the tuning point of view, we should systematically ensure that α_{ij} is positive. As a consequence, the parameters π_{ij} should have the same sign as the corresponding κ_{ij}. Furthermore, as the controller $\mathcal{K}_L^{ij}(s)$ should be stable, it is clear that τ_j is at least be positive.

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In the robust estimation problem, the objective is to enlarge the bandpass of the estimator in comparison to the standard Kalman filter. But a compromise should be found, between a too large bandpass that will entail an important level of noise on the state estimate and an unbiased (and reliable) state estimate. The proposed approach should permit such a compromise by an adapted frequency shaping of $\mathcal{K}_L(s)$.

3.3 The lead-lag H_∞ robust estimator

In order to obtain an estimate of the state of the system with a prescribed H_∞ level of performance, we should give the state equation of the proposed lead-lag estimator. The following lemma permits to obtain such an equation:

Lemma 1. The estimate $\hat{z}(t)$ of the signal $z(t)$ of the uncertain system (Σ) with $K_L(s)$ as a lead-lag controller of the form (5) is given by:

$$\begin{cases} \dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + \zeta(t) \\ \dot{\chi}(t) = F(\chi(t) - \epsilon(t)) \\ \zeta(t) = G\chi(t) + H\epsilon(t) \\ \epsilon(t) = y(t) - C\hat{x}(t) \\ \hat{z}(t) = L\hat{x}(t) \end{cases} \tag{6}$$

where

$$F = -\mathrm{diag}\left([1/\tau_1 \dots 1/\tau_p]\right); H = \Pi \star K; G = \mathbf{1} - H \tag{7}$$

with

- τ_j is a strictly positive real, that sets the high frequency bandpass of the controller related to the measured output j,
- $K = \{\kappa_{ij}\}$ is the $\Re^{n\times p}$ Kalman gain matrix,
- $\Pi = \{\pi_{ij}\}$ is a $\Re^{n\times p}$ weighting matrix,
- $\mathbf{1}$ is the $\Re^{n\times p}$ matrix with all elements equal to one,
- $\Pi \star K$ is the Kronecker product of Π with K.

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Proof 1. From (6), express the transfer matrix $\mathcal{K}_L(s)$ and verify that it coincides with (5). It comes, that:

$$\mathcal{K}_L(s) = \frac{\zeta(s)}{\epsilon(s)} = GT_{\chi\epsilon}(s) + H \tag{8}$$

with

$$T_{\chi\epsilon}(s) = \frac{\chi(s)}{\epsilon(s)} = -(sI - F)^{-1}F \tag{9}$$

Clearly, we have:

$$T_{\chi\epsilon}(s) = \mathrm{diag}\left(\left[\, \left(s + \tau_1^{-1}\right)^{-1} \dots \left(s + \tau_p^{-1}\right)^{-1} \,\right]\right)$$

Expressing H and G and replacing in (8) leads to (5). This completes the proof.

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Remark 2. The PI Kalman state equation can be easily derived form (6) by modifying the equation of the variable $\chi(t)$ and set $H_{PI} = H = K$. Further F and G become tuning matrices denoted F_{PI} and G_{PI} respectively. Concerning $\chi(t)$, we should write that:

$$\dot{\chi}(t) = F_{PI}\epsilon(t)$$

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Remark 3. By construction, it appears that the state $\chi(t)$ associated to the lead-lag controller is a vector of dimension p, the number of measured output. Hence, compared to standard approaches, the increase of the numerical burden is reasonable. Compared to the PI Kalman, as the order of $\mathcal{K}_L(s)$ remains unchanged, the two approaches are equivalent from this point of view.

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In order to find the tuning parameters such that the proposed estimator has a H_∞ performance, we have the result:

Theorem 1. The robust lead-lag estimator given in Lemma 1 achieves H_∞ performance if the tuning matrices F and Π are such that

- The estimation error $\tilde{z}(t) = z(t) - \hat{z}(t)$ satisfies
$$\|\phi_{\tilde{z}\tilde{z}}\|_\infty^2 = \sup_{\omega\in\Re} \|\phi_{\tilde{z}\tilde{z}}(j\omega)\|_2^2 < \gamma^2 \tag{10}$$
with
$$\phi_{\tilde{z}\tilde{z}} = T^*_{\tilde{z}\tilde{w}}(s)T_{\tilde{z}\tilde{w}}(s) \tag{11}$$
and
$$T_{\tilde{z}\tilde{w}}(s) = \frac{\tilde{z}(s)}{\tilde{w}(s)} = \overline{L}\left(sI - \overline{A}\right)^{-1}\overline{M} \tag{12}$$
$$\overline{A} = \begin{bmatrix} A - HC & -G \\ -FC & F \end{bmatrix}; \overline{M} = \begin{bmatrix} M & -H \\ 0 & F \end{bmatrix}; \overline{L} = [\, L \;\; 0 \,]$$
where $T^*(s)$ is the transpose complex conjugate of $T(s)$, i.e. $T^*(s) = T'(-s)$, and $\|\cdot\|_2$ is the Euclidian norm,
- The matrix $\overline{A}$ is a stable matrix.

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Proof 2. In order to express $\phi_{\tilde{z}\tilde{z}}$, we define the state estimation error $\tilde{x}(t) = x(t) - \hat{x}(t)$ then:

$$\dot{\tilde{x}}(t) = A\tilde{x}(t) + Mw(t) - \zeta(t)$$

Now, consider the state $\tilde{X}'(t) = [\tilde{x}'(t)\chi'(t)]$ then, we have:

$$\begin{cases} \dot{\tilde{X}}(t) = \overline{A}\tilde{X}(t) + \overline{M}\tilde{w}(t) \\ \tilde{z}(t) = \overline{L}\tilde{X}(t) \end{cases} \tag{13}$$

where

$$\tilde{w}(t) = \begin{bmatrix} w(t) \\ v(t) \end{bmatrix}$$

The result in Theorem 1 is obtain after trivial manipulations. This completes the proof.

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The minimization of the H_∞ scaling parameter γ will be done by means of a Genetic Algorithm (GA). This method has been used with success in control problems (Zhang [2005]) as well as in deconvolution (Chen [2000]) in order to optimize tuning parameters according to a given criterion. See references in the above mentioned papers for more information on GA. In the present case, we have to find the $(n+1)p$ parameters that constitute the matrices F and Π such that γ is minimized.

Remark 4. In complement to Remark 3, it should be noticed that the proposed approach requires the same number of parameters as the PI Kalman filter does.

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4. EXAMPLE

Let consider a single input-single output LTI system defined by the set of nominal matrices:

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -3 & -2 \end{bmatrix}; \; B = M = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$C = L = [\,10 \;\; 0 \;\; 0\,]$$

where the parameter a_0 is assumed to be uncertain. Its nominal value is $a_0 = 5$. In order to validate the proposed approach to standard robust H_∞ estimators, we consider that the uncertainty range represents $\delta = \pm 15\%$ of the nominal value. This example is interesting in the sense that some of its poles are located close to the imaginary axis. As a matter of fact, for the nominal model, we have the poles $[\,-1.843; -0.078 + 1.645j; -0.078 - 1.645j\,]$.

4.1 The protocol

In the following, we will consider that the true value of a_0 is varying in the proposed range. In figure 2, we have plotted the deterministic input $u(t)$ together with the measured output $y(t)$ for $\delta = +15\%$. The validation of the proposed approach will be done by inspecting the following aspects:

- the behavior of the estimation error with respect with the noises, i.e., we will plot $\|\phi_{\tilde{z}\tilde{z}}\|_2$.
- the relative root mean square error of each state $x_i(t)$ of the system, i.e.

$$RRMSE_i = \sqrt{\frac{E\{\tilde{x}_i^2(t)\}}{E\{x_i^2(t)\}}}$$

  evaluated over 200 realizations of the noises. For each state, we will plot the corresponding $RRMSE$ over the range $\delta = \pm 15\%$. $E\{\cdot\}$ stands for the mathematical expectation.

Each plot will permit to compare the proposed estimator to the standard H_∞ estimator (Souza [1995]) and to a PI Kalman filter designed under the terms in Theorem 1.

Remark 5. Note that the desired signal $z(t)$ is 10 times the state $x_1(t)$. Hence, $RRMSE_1$ is also the $RRMSE$ of the estimation error of the $z(t)$.

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4.2 The tuning of the estimators

The standard H_∞ estimator has been obtained after writing the uncertainty part as:

$$\Delta A = H_A \Delta(t) E_A$$

with

$$H_A = \begin{bmatrix} 0 \\ 0 \\ 0.15a_0 \end{bmatrix}; \; E = [\,1 \;\; 0 \;\; 0\,]$$

where $\Delta(t)$ is a matrix of Lebesgue measurable elements such that $\Delta(t)\Delta'(t) < I$.

The design of the standard H_∞ estimator requires the solution of two Riccati equations and the tuning of two real parameters. In the following the optimal tuning is used in order to evaluate the proposed approach.

The PI Kalman filter with H_∞ performance and the lead-lag robust H_∞ estimator will be tuned thanks to the Matlab® Genetic Algorithm Toolbox. In order to obtain the optimum, we have considered a population of 1000 with a crossover of 0.50 and a mutation rate of 0.15. Due to the size of the problem, 4 parameters ($n = 3$, $p = 1$) have to be tuned for each estimator. The results of the optimization are given below:

- the PI Kalman estimator

$$F_{PI} = 1.9061; G_{PI} = \begin{bmatrix} 0.3521 \\ -0.6822 \\ 0.3080 \end{bmatrix}; \gamma_{PI}^2 = 3.4084$$

- the Lead-lag estimator

$$\tau_1 = 10^{-5}; \Pi = 10^3 \begin{bmatrix} 2.7368 \\ 1770.5 \\ -1.5688 \end{bmatrix}; \gamma^2 = 2.7607 \cdot 10^{-2}$$

The Bode plots of $\mathcal{K}_L(s)$ are given in figure 3. It appears that the controllers have high-pass characteristic which implies an increase in the dynamics of the closed-loop that should guaranty the precision of the estimation. This should be confirmed by the simulation results.

4.3 Simulation results

From figure 4, it is clear that the proposed estimator outperforms the standard H_∞ estimator. Compared to the PI Kalman, the performances are equivalent. Even if for low $|\delta|$ the PI Kalman performs better. On the overall uncertainty range, the proposed estimator has a constant level of performance which clearly sets its robustness.

Furthermore, the plots of the $RRMSE$ of the states $x_2(t)$ and $x_3(t)$ for the 3 filters (see figures 5-6) show that the proposed estimator outperforms the two others. It appears that the PI Kalman does not guaranty a reliable state estimate while the standard H_∞ estimator gives reasonable estimates.

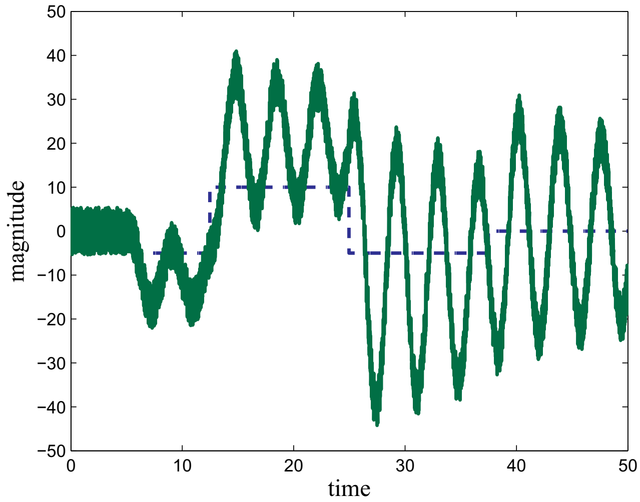

Fig. 2. The deterministic input $u(t)$ (dashed line) and the measured output $y(t)$ (solid line) for $\delta = +15\%$.

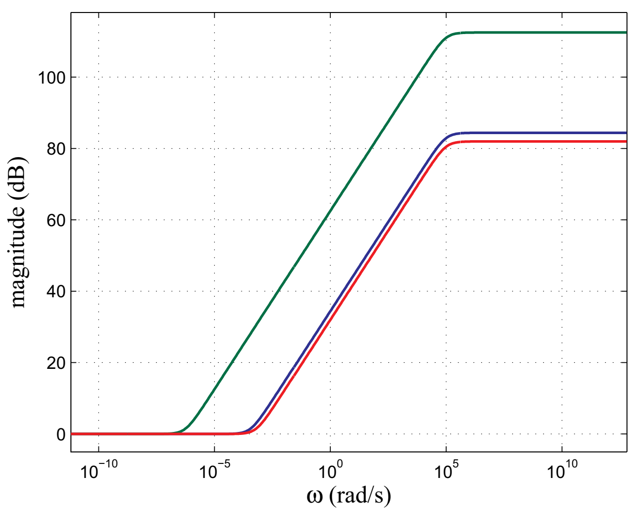

Fig. 3. Bode plots of the optimal $\mathcal{K}_L(s)$ guaranteeing the H_∞ performance of the proposed filter.

Clearly, the criterion in Theorem 1 permits to obtain a robust H_∞ lead-lag estimator. This example shows clearly the ability of the lead-lag controller to propose an adapted solution to the robust estimation problem while the PI Kalman fails.

From figure 7, it appears that the bandpass of the standard H_∞ estimator is the largest bandpass of the 3 estimators studied. Consequently, its estimates, though reliable, suffer from an important residual estimation noise. The bandpass of the PI Kalman and the proposed estimator are equivalent even if the rejection of the PI Kalman is stiffer which can be compared to a Chebyshev filter. Finally, the proposed estimator has a low gain in the bandpass. This means that it permits to reduce the level of noise even in its bandpass, while the other robust estimators do not attenuate the noises in their bandpass. The latter point explains that the optimal H_∞ lead-lag estimator outperforms the optimal standard H_∞ estimator.

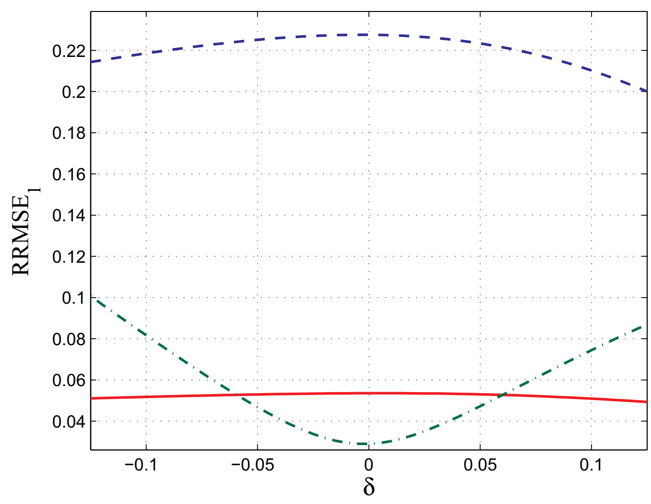

Fig. 4. $RRMSE$ of the proposed filter (solid line) vs standard H_∞ filter (dashed line) and the PI Kalman (dash-dotted line) for the state $x_1(t)$.

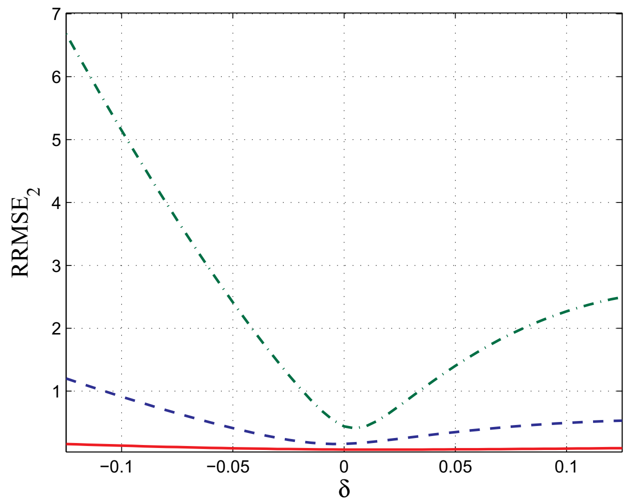

Fig. 5. $RRMSE$ of the proposed filter (solid line) vs standard H_∞ filter (dashed line) and the PI Kalman (dash-dotted line) for the state $x_2(t)$.

5. CONCLUSION

In the present paper, we have proposed a control point of view to the design of robust H_∞ estimator for LTI uncertain systems. The basic idea consists in consider that the estimator structure is a state feedback that aims at rejecting the effect of noises on the measured signal in order to estimate the state of the system. In presence of model uncertainty, the standard Luenberger or Kalman estimator has poor performance. As a matter of fact, they act as Proportional controllers. We propose to replace the proportional by a lead-lag controller. We have proposed a state-space realization of the corresponding estimator. The tuning parameters are obtained by optimizing a H_∞ criterion by means of a Genetic Algorithm. An example has shown the great ability of the proposed approach to

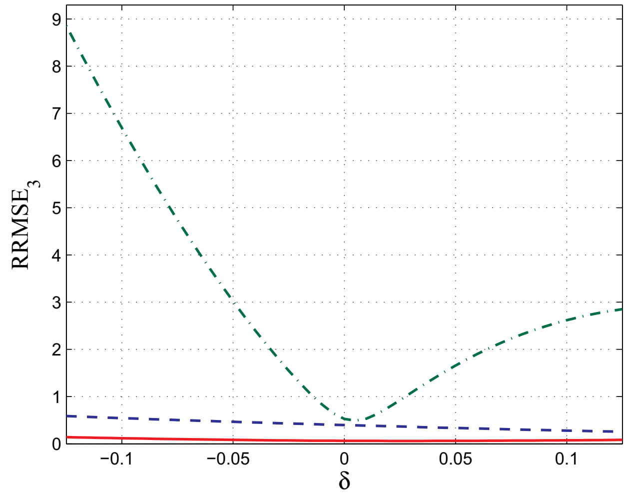

Fig. 6. *RRMSE* of the proposed filter (solid line) vs standard H_∞ filter (dashed line) and the PI Kalman (dash-dotted line) for the state $x_3(t)$.

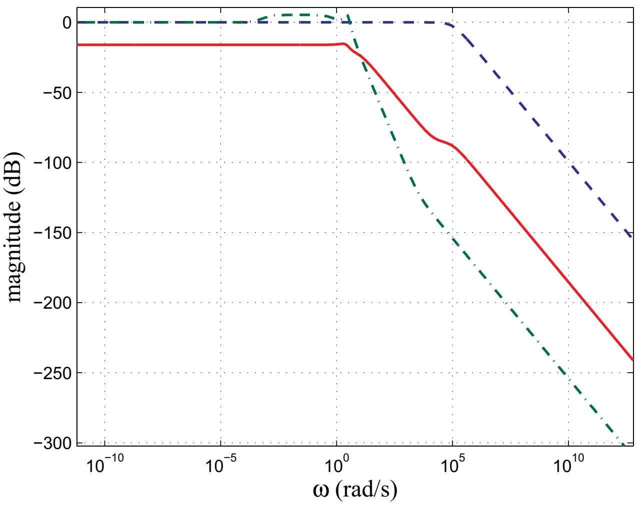

Fig. 7. Bode plots of $\|\phi_{\tilde{z}\tilde{z}}\|_2$ for the proposed filter (solid line) vs standard H_∞ filter (dashed line) and the PI Kalman (dash-dotted line).

guaranty a constant level of performance over a range of possible systems.

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