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A New Critical Case for Adaptive Nonlinear Stabilization ⋆

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Abstract: It is fairly well known that there are fundamental differences between adaptive control of continuous-time and discrete-time nonlinear systems. In fact, even for the seemingly simple SISO control system $y_{t+1} = \theta_1 f(y_t) + u_t + w_{t+1}$ with a scalar unknown parameter θ_1 and noise disturbance $\{w_t\}$, and with a known function $f(\cdot)$ having possible nonlinear growth rate characterized by $|f(x)| = \Theta(|x|^b)$ with $b \geq 1$, the system is globally stabilizable by adaptive feedback **if and only if** $b < 4$**.** This was first found and proved by Guo (1997) for the Gaussian white noise case, and then proved by Li and Xie (2006) for the bounded noise case. Recently, a number of other type of "critical values" or "impossibility theorems" on the maximum capability of adaptive feedback were also found, mainly for systems with known control parameter as in the above model. In this paper, we will study the above basic model again but with additional unknown control parameter θ_2, i.e., u_t is replaced by $\theta_2 u_t$ in the above model. Interestingly, it turns out that the system is globally stabilizable **if and only if** $b < 3$**.** This is a new critical case for adaptive nonlinear stabilization, which has meaningful implications for the control of more general uncertain systems.

1. INTRODUCTION

It is well known that a fairly complete theory exists for adaptive control of linear systems in both continuous-time and discrete-time cases (cf. e.g., Aström and Wittenmark (1995)-Ioannou and Sun (1996)). Extensions of the existing results on linear systems to nonlinear systems with nonlinearity having linear growth rate are also possible (cf. e.g. Tao and Kokotovic (1996)). However, fundamental differences emerge between adaptive control of continuous- and discrete-time systems when the nonlinearities are allowed to have a nonlinear growth rate. In fact, in this case, it is still possible to design globally stablizing adaptive controls for a wide class of nonlinear continuous-time systems (cf. Krstic, Kanellakopoulos and Kokotovic (1995)), but fundamental difficulties exist for adaptive control of nonlinear discrete-time systems, partly because the high gain or nonlinear damping methods that are so powerful in the continuous-time case are no longer effective in the discrete-time case. Similarly, for sampled-data control of nonlinear uncertain systems, the design of stabilizing sampled-data feedback is possible if the sampling rate is high enough (cf.e.g., Mareels, Penfold and Evans (1992) and Skafidas, Fradkov, Evans and Mareels (1998)). However, if the sampling rate is a prescribed value, then difficulties again emerge in the design and analysis of globally stabilizing sampled-data feedbacks even for nonlinear systems with the nonlinearity having a linear growth rate. The fact that sampling usually destroys many helpful properties is one of the reasons why most of the existing design methods for nonlinear control remain in the continuous-time even in the nonadaptive case (cf. Kokotovic and Arcak (1999)), albeit many results in continuous-time still have their discrete-time counterparts (cf.e.g., Jiang and Wang (1999)).

Knowing the above difficulties that we encountered in the adaptive control of discrete-time (or sampled-data) nonlinear systems, one may naturally ask the following question: Are the difficulties mainly caused by our incapability in designing or analyzing the adaptive control systems, or by the inherent limitations on the capability of the feedback principle ? To answer this fundamental question, we have to place ourselves in a framework that is somewhat beyond those of the classical robust control and adaptive control. We need not only to answer what adaptive control can do, but also to answer the more difficult question what adaptive control cannot do. This means we need to study the maximum capability of the full feedback mechanism which includes all (nonlinear and time-varying) causal mappings from the data space to the control space, and are not only restricted to a fixed feedback law or a class of specific feedback laws.

A first step in this direction was made in Guo (1997), which considers the following system:

$$y_{t+1} = \theta_1 f(y_t) + u_t + w_{t+1}, \tag{1}$$

where θ_1 is an unknown parameter, $\{w_t\}$ is Gaussian white noise sequence, and where $f(\cdot)$ is a known nonlinear function possibly having a nonlinear growth rate characterized by

$$|f(x)| = \Theta(|x|^b) \quad \text{with} \quad b \geq 1.$$

It was found and proved that the system is a.s. globally adaptively stabilizable if and only if $b < 4$ (see, Guo (1997)). This result is also true if the Gaussian noise is replaced by bounded noises (see, Li and Xie (2006)). It goes without saying that this critical case on the feedback

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capability naturally gives an "impossibility result" on the maximum capability of feedback for the case where $b \geq 4$. It is worth pointing out that such "impossibility result" obviously holds also for any (more general) class of uncertain systems, which includes the basic system class described by as a subclass.

Later on, the above "impossibility result" was extended in to systems with multiple unknown parameters and with Gaussian white noise sequence by providing a polynomial rule (see, Xie and Guo (1999)). Similar results can also be obtained for the case where the uncertain parameters are lie in a bounded known region with Gaussian white noises again, but with a more general system structure (see, Xie and Guo (2000b)). More recently, Li, Xie and Guo (2006) proved that the polynomial rule of Xie and Guo (1999) does indeed gives a necessary and sufficient condition for global feedback stabilization of a wide class of nonlinear systems with multiple unknown parameters and with bounded noises.

It is worth pointing out that, for nonlinear systems with nonparametric uncertainties, fundamental limitations on the capability of adaptive feedback may still exist even for the case when the nonlinearities have a linear growth rate. For example, for the following first-order nonparametric control system:

$$y_{t+1} = f(y_t) + u_t + w_{t+1}; \quad t \geq 0; \quad y_0 \in \mathcal{R}^1,$$

where the unknown function $f(\cdot)$ belongs to the class of standard Lipschitz functions defined by:

$$\mathcal{F}(L) = \{f(\cdot) : |f(x) - f(y)| \leq L|x-y|, \quad \forall x, y\}$$

and where the noise sequence is bounded. It was found and proved by Xie and Guo (2000a) that the maximum "uncertainty ball" that can be stabilized by adaptive feedback is $\mathcal{F}(L)$ with $L = \frac{3}{2} + \sqrt{2}$. This critical case again gives an "impossibility result" for the case where $f \in \mathcal{F}(L)$ with $L > \frac{3}{2} + \sqrt{2}$. Related "impossibility results" are also found for sampled-data adaptive control of nonparametric nonlinear systems in Xue and Guo (2000).

However, all the results mentioned above assume that the parameter in front of the control law is known. It would certain be interesting to ask what will happen if the control parameter is also unknown ? The main purpose of this paper is to give an answer to this problem for a basic class of uncertain nonlinear systems, which will naturally has meaningful implications for more general uncertain systems as explained before.

2. MAIN RESULTS

In this paper, we consider adaptive control of the following basic uncertain system

$$y_{t+1} = \theta_1 f(y_t) + \theta_2 u_t + w_{t+1}, \tag{2}$$

where $\{u_t\}$ and $\{y_t\}$ are the system input and output processes, both θ_1 and θ_2 are unknown parameters, and $\{w_t\}$ is a disturbance process. To study the capability of adaptive feedback, we need the following assumptions:

A1) The unknown parameter vector $\theta = (\theta_1, \theta_2)^\tau$ belongs to a bounded domain $[\underline{\theta}_1, \overline{\theta}_1] \times [\underline{\theta}_2, \overline{\theta}_2] \subset \mathbb{R} \times \mathbb{R}$, and the interval for θ_2 does not contain 0.

A2) The noise sequence $\{w_t\}$ belongs to a class of bounded sequences with an unknown bound $w > 0$, i.e.,

$$\sup_{t \geq 1} |w_t| \leq w. \tag{3}$$

A3) The nonlinear function satisfies $|f(x)| = \Theta(|x|^b)$ as $|x| \to \infty$, in the sense that there exist some constants $x' > 0$ and $c_2 > c_1 > 0$ such that

$$c_1 \leq \frac{|f(x)|}{|x|^b} \leq c_2, \quad \forall |x| > x', \tag{4}$$

where $b \geq 1$ is a constant reflecting the rate of nonlinear growth.

We are interested in designing a feedback control law which robustly stabilizes the system (2) with respect to any possible θ and $\{w_t\}$ under the assumptions **A1**)-**A2**).

First, we restate the definition of a feedback control law (cf, Xie and Guo (2000b)).

Definition 2.1. A sequence $\{u_t\}$ is called a feedback control law if at any time $t \geq 0$, u_t is a (causal) function of all the observations up to the time t: $\{y_i,\ i \leq t\}$, i.e.,

$$u_t = h_t(y_0, \cdots, y_t) \tag{5}$$

where $h_t(\cdot) : I\!R^{t+1} \to I\!R^1$ can be any Lebesgue measurable (nonlinear) mapping.

We also need a definition of adaptive stabilizability in the sense of bounded input and bounded output.

Definition 2.2. The system (2) under the assumptions **A1)-A3)** is said to be adaptively stabilizable, if there exists a feedback control law $\{u_t\}$ such that for any $y_0 \in \mathbb{R}^1$, any θ, any $\{w_t\}$ satisfying **A1**)-**A2**), the outputs of the closed-loop system are bounded as follows:

$$\sup_{t \geq 0} |y_t| < \infty. \tag{6}$$

The main result of this paper is as follows:

Theorem 2.1. The system (2) under the assumptions **A1**)-**A3**) is adaptive stabilizable **if and only if** $b < 3$.

Remark 2.1. In comparison with the related results established in Guo (1997) and Li and Xie (2006) as explained in the Introduction, we see that the critical nonlinear growth rate reflecting the maximum capability of the feedback mechanism is reduced from $b = 4$ to the current $b = 3$, due to the additional uncertainty in the input channel.

3. THE PROOF OF SUFFICIENCY

The proof of sufficiency is constructive. We will design a simple adaptive control law, which robustly stabilizes the system (2) for any $b < 3$. We will also see that to implement the control algorithm, the bounds $[\underline{\theta}_1, \overline{\theta}_1] \times [\underline{\theta}_2, \overline{\theta}_2]$ and w need not to be known.

3.1 The Estimates of the Parameters

Without loss of generality, we can suppose that $|y_0| > x'$ is large enough. This is because we can let $u_t = 0$, $t = 0, 1, \cdots$ until there exists some $y_{t'}$ large enough, and then we can take $y_{t'}$ as y_0. Otherwise, if we can not find such $y_{t'}$, the sufficiency part is proven trivially.

Now, we denote $\vartheta_1 = \frac{\theta_1}{\theta_2}, \vartheta_2 = \frac{1}{\theta_2}$. Without loss of generality, suppose $\theta_2 > 0$. By **A1)**, it is easy to see that

$\vartheta_1 \in [\frac{\underline{\theta}_1}{\overline{\theta}_2}, \frac{\overline{\theta}_1}{\underline{\theta}_2}]$ and $\vartheta_2 \in [\frac{1}{\overline{\theta}_2}, \frac{1}{\underline{\theta}_2}]$. For the convenience of later use, we also denote $\underline{\vartheta}_1 = \frac{\underline{\theta}_1}{\overline{\theta}_2}, \overline{\vartheta}_1 = \frac{\overline{\theta}_1}{\underline{\theta}_2}, \underline{\vartheta}_2 = \frac{1}{\overline{\theta}_2}, \overline{\vartheta}_2 = \frac{1}{\underline{\theta}_2}$, which are obviously all positive numbers.

Let us take $u_0 = 0$ and $u_1 = -2\overline{\theta}_1 f(y_1)$, and rewrite the system (2) into the following form:

$$y_{t+1} = \varepsilon_t f(y_t) + w_{t+1}, \tag{7}$$

where by definition

$$\varepsilon_t = \theta_1 - \theta_2\beta_t = \theta_2(\vartheta_1 - \beta_t) \quad \text{and} \quad \beta_t = -\frac{u_t}{f(y_t)}.$$

Now for any $t \geq 2$, let

$$i_t := \operatorname*{argmax}_{0 \leq i \leq t-1} |f(y_i)|, \tag{8}$$

$$j_t := \operatorname*{argmax}_{0 \leq i < i_t} |f(y_i)|, \tag{9}$$

we can then define the parameter estimate for $(\vartheta_1, \vartheta_2)$ at time $t \geq 2$ as

$$\hat{\vartheta}_{1,t} = \frac{\begin{vmatrix} -u_{i_t} & -y_{i_t+1} \\ -u_{j_t} & -y_{j_t+1} \end{vmatrix}}{\begin{vmatrix} f(y_{i_t}) & -y_{i_t+1} \\ f(y_{j_t}) & -y_{j_t+1} \end{vmatrix}}; \quad \hat{\vartheta}_{2,t} = \frac{\begin{vmatrix} f(y_{i_t}) & -u_{i_t} \\ f(y_{j_t}) & -u_{j_t} \end{vmatrix}}{\begin{vmatrix} f(y_{i_t}) & -y_{i_t+1} \\ f(y_{j_t}) & -y_{j_t+1} \end{vmatrix}} \tag{10}$$

This estimate is defined through solving the following system equation for $(\vartheta_1, \vartheta_2)$:

$$\begin{cases} -u_{i_t} = \vartheta_1 f(y_{i_t}) + (w_{i_t+1} - y_{i_t+1})\vartheta_2 \\ -u_{j_t} = \vartheta_1 f(y_{j_t}) + (w_{j_t+1} - y_{j_t+1})\vartheta_2 \end{cases}. \tag{11}$$

by setting the noise to be zero. The error of the parameter estimate at time $t \geq 2$ are denoted by

$$\tilde{\vartheta}_{1,t} = \vartheta_1 - \hat{\vartheta}_{1,t}, \qquad \tilde{\vartheta}_{2,t} = \vartheta_2 - \hat{\vartheta}_{2,t}. \tag{12}$$

Now, notice that by (11)

$$\frac{y_{j_t+1}}{f(y_{j_t})} \cdot \frac{y_{i_t+1}}{f(y_{i_t})} = \left(\varepsilon_{j_t} + \frac{w_{j_t+1}}{f(y_{j_t})}\right)\left(\varepsilon_{i_t} + \frac{w_{i_t+1}}{f(y_{i_t})}\right), \tag{13}$$

hence

$$\left(\varepsilon_{j_t} + \frac{w_{j_t+1}}{f(y_{j_t})}\right)\left(\varepsilon_{i_t} + \frac{w_{i_t+1}}{f(y_{i_t})}\right) < 0 \tag{14}$$

will imply that $\frac{y_{j_t+1}}{f(y_{j_t})}$ and $\frac{y_{i_t+1}}{f(y_{i_t})}$ have different signs. The following lemma gives the range of the estimate error in this situation.

Lemma 3.1. If (14) holds, then for $t \geq 2$,

$$|\tilde{\vartheta}_{1,t}| \leq \overline{\vartheta}_2 w \frac{|y_{i_t+1}| + |y_{j_t+1}|}{|f(y_{i_t})y_{j_t+1}|}.$$

Proof. First, the equation

$$\begin{cases} y_{i_t+1} = \theta_1 y_{i_t}^b + \theta_2 u_{i_t} + w_{i_t+1} \\ y_{j_t+1} = \theta_1 y_{j_t}^b + \theta_2 u_{j_t} + w_{j_t+1} \end{cases},$$

can be rewritten as (11). Solving (11), we get

$$\begin{cases} \vartheta_1 = \dfrac{\begin{vmatrix} -u_{i_t} & w_{i_t+1} - y_{i_t+1} \\ -u_{j_t} & w_{j_t+1} - y_{j_t+1} \end{vmatrix}}{\begin{vmatrix} f(y_{i_t}) & w_{i_t+1} - y_{i_t+1} \\ f(y_{j_t}) & w_{j_t+1} - y_{j_t+1} \end{vmatrix}} \\ \vartheta_2 = \dfrac{\begin{vmatrix} f(y_{i_t}) & -u_{i_t} \\ f(y_{j_t}) & -u_{j_t} \end{vmatrix}}{\begin{vmatrix} f(y_{i_t}) & w_{i_t+1} - y_{i_t+1} \\ f(y_{j_t}) & w_{j_t+1} - y_{j_t+1} \end{vmatrix}} \end{cases}. \tag{15}$$

Then by (10) and (12), we can compute that

$$\begin{aligned} \tilde{\vartheta}_{1,t} &= \frac{\begin{vmatrix} -u_{i_t} & w_{i_t+1} - y_{i_t+1} \\ -u_{j_t} & w_{j_t+1} - y_{j_t+1} \end{vmatrix}}{\begin{vmatrix} f(y_{i_t}) & w_{i_t+1} - y_{i_t+1} \\ f(y_{j_t}) & w_{j_t+1} - y_{j_t+1} \end{vmatrix}} - \frac{\begin{vmatrix} -u_{i_t} & -y_{i_t+1} \\ -u_{j_t} & -y_{j_t+1} \end{vmatrix}}{\begin{vmatrix} f(y_{i_t}) & -y_{i_t+1} \\ f(y_{j_t}) & -y_{j_t+1} \end{vmatrix}} \\ &= \frac{f(y_{j_t})u_{i_t} - f(y_{i_t})u_{j_t}}{f(y_{i_t})(y_{j_t+1} - w_{j_t+1}) - f(y_{j_t})(y_{i_t+1} - w_{i_t+1})} \cdot \\ &\quad \frac{y_{i_t+1}w_{j_t+1} - y_{j_t+1}w_{i_t+1}}{f(y_{i_t})y_{j_t+1} - f(y_{j_t})y_{i_t+1}} \\ &= \vartheta_2 \frac{y_{i_t+1}w_{j_t+1} - y_{j_t+1}w_{i_t+1}}{f(y_{i_t})y_{j_t+1} - f(y_{j_t})y_{i_t+1}}, \end{aligned} \tag{16}$$

where the last equality follows from the expression of ϑ_2 in (15).

Now, by (13) and the argument above, we have

$$\begin{aligned} &|f(y_{i_t})y_{j_t+1} - f(y_{j_t})y_{i_t+1}| \\ &= \left|\frac{y_{j_t+1}}{f(y_{j_t})} - \frac{y_{i_t+1}}{f(y_{i_t})}\right| \cdot |f(y_{j_t})f(y_{i_t})| \\ &\geq |f(y_{i_t})y_{j_t+1}|. \end{aligned} \tag{17}$$

Hence by (16) and (17), we have

$$|\tilde{\vartheta}_{1,t}| \leq \overline{\vartheta}_2 w \frac{|y_{i_t+1}| + |y_{j_t+1}|}{|f(y_{i_t})y_{j_t+1}|}.$$

The proof is thus completed. ■

3.2 The Design of adaptive Control

In this subsection, we will discuss the design of adaptive control and prove the sufficiency part of Theorem 2.1.

To design the control which can stabilize the system (2), we need to define the subscript t_k for the output sequence $\{y_t\}$:

$$\begin{cases} t_0 = 0 \\ t_{k+1} = \inf\{t > t_k : |f(y_t)| \geq |f(y_{t_k})|\} \end{cases}. \tag{18}$$

then, we have

$$|f(y_t)| < |f(y_{t_k})| \leq |f(y_{t_{k+1}})|, \quad \text{for any } t_k < t < t_{k+1}.$$

Now, let $\Delta_t := \overline{\vartheta}_2 w \frac{|y_{i_t+1}| + |y_{j_t+1}|}{|f(y_{i_t})y_{j_t+1}|}$, for $k = 1, 2, \cdots$ where $\overline{\vartheta}_2$ and w are defined in subsection 2.1 and (3) respectively. We can define

$$\beta_t = \begin{cases} 0, & 0 \leq t < t_1 \\ 2\overline{\vartheta}_1, & t_1 \leq t < t_2 \\ \hat{\vartheta}_{1,t} - 2\Delta_t, & t_{2k} \leq t < t_{2k+1} \\ \hat{\vartheta}_{1,t} + 2\Delta_t, & t_{2k+1} \leq t < t_{2(k+1)} \end{cases}, \tag{19}$$

then the control can be designed by

$$u_t = -\beta_t f(y_t). \tag{20}$$

Remark 3.1. Notice that by (8), (9) and the definition of Δ_t, we know

$$\begin{cases} \hat{\vartheta}_{1,t} - 2\Delta_t = \hat{\vartheta}_{1,t_{2k}} - 2\Delta_{t_{2k}} & t_{2k} \le t < t_{2k+1} \\ \hat{\vartheta}_{1,t} + 2\Delta_t = \hat{\vartheta}_{1,t_{2k+1}} + 2\Delta_{t_{2k+1}} & t_{2k+1} \le t < t_{2(k+1)} \end{cases}.$$

To prove that the controller designed by (19) and (20) can stabilize the system (2), we proceed to analyze the closed-loop system.

Proposition 3.1. For the system (2) with the controller designed by (19) and (20), the following statements hold for all $k \ge 2$.
(i) $|y_{t_k}| \le 2c_2|\varepsilon_{t_{k-1}}||y_{t_{k-1}}|^b$.
(ii) $|y_{t_{k-2}+1}| \ge \frac{1}{4}|y_{t_{k-1}}|$.
(iii) $\left(\varepsilon_{t_{k-1}} + \frac{w_{t_{k-1}+1}}{f(y_{t_{k-1}})}\right)\left(\varepsilon_{t_{k-2}} + \frac{w_{t_{k-2}+1}}{f(y_{t_{k-2}})}\right) < 0$.
(iv) $|\varepsilon_{t_k}| = O\left(\left|\frac{y_{t_k}}{y_{t_{k-1}}^{b+1}}\right|\right)$.

Proof. (i) For $|y_{t_k}|$ large enough, by (7) we have

$$\begin{aligned} \frac{1}{2}|y_{t_k}| &\le |y_{t_k}| - |w_{t_k}| \le |y_{t_k} - w_{t_k}| \\ &= |\varepsilon_{t_k-1}||f(y_{t_k-1})|. \end{aligned} \tag{21}$$

Moreover, by (19) and Remark 3.1 we know that $\beta_{t_k-1} = \beta_{t_{(k-1)}}$ for all $k \ge 0$, which implies that $\varepsilon_{t_k-1} = \varepsilon_{t_{(k-1)}}$. Hence (21) gives

$$\begin{aligned} |y_{t_k}| &\le 2|\varepsilon_{t_k-1}||f(y_{t_k-1})| \le 2|\varepsilon_{t_{k-1}}||f(y_{t_{k-1}})| \\ &\le 2c_2|\varepsilon_{t_{k-1}}||y_{t_{k-1}}|^b. \end{aligned} \tag{22}$$

(ii) By (22), we have

$$\frac{1}{2}|y_{t_{k-1}}| \le |\varepsilon_{t_{k-2}}||f(y_{t_{k-2}})| \le |y_{t_{k-2}+1}| + w,$$

which gives (ii) for sufficiently large $|y_{t_{k-1}}|$.

(iii) In fact, we need only to show for any $k \ge 0$,

$$\varepsilon_{t_{2k}} + \frac{w_{t_{2k}+1}}{f(y_{t_{2k}})} > 0; \tag{23}$$

$$\varepsilon_{t_{2k+1}} + \frac{w_{t_{2k+1}+1}}{f(y_{t_{2k+1}})} < 0. \tag{24}$$

We will prove it by induction. First we verify the cases when $t = t_0 = 0$ and $t = t_1$ respectively.

For $t = 0$, by (19) and the definition of ε_t, we have

$$\varepsilon_{t_0} = \theta_2(\vartheta_1 - \beta_{t_0}) \ge \underline{\theta}_2\underline{\vartheta}_1. \tag{25}$$

Then, for $|y_{t_0}|$ large enough, the above inequality gives

$$\varepsilon_{t_0} + \frac{w_{t_0+1}}{f(y_{t_0})} \ge \underline{\theta}_2\underline{\vartheta}_1 - \frac{w}{|f(y_{t_0})|} > 0.$$

For the case of $t = t_1$, it can be proven similarly as that for $t = 0$.

Now, suppose (23) and (24) hold for some $k \ge 0$. For $t = t_{2(k+1)}$, by Lemma 3.1, we have

$$\begin{aligned} \varepsilon_{t_{2(k+1)}} &= \theta_2(\vartheta_1 - \beta_{t_{2(k+1)}}) \\ &= \theta_2(\vartheta_1 - \hat{\vartheta}_{1,t_{2(k+1)}} + 2\Delta_{t_{2(k+1)}}) \\ &= \theta_2(\tilde{\vartheta}_{1,t_{2(k+1)}} + 2\Delta_{t_{2(k+1)}}) \\ &\ge \underline{\theta}_2\Delta_{t_{2(k+1)}}. \end{aligned}$$

Note that for $t = t_{2(k+1)}$, according to (8) and (9) we have $i_t = t_{2k+1}$ and $j_t = t_{2k}$. Hence

$$\begin{aligned} &\varepsilon_{t_{2(k+1)}} + \frac{w_{t_{2(k+1)}+1}}{f(y_{t_{2(k+1)}})} \ge \underline{\theta}_2\Delta_{t_{2(k+1)}} - \frac{w}{|f(y_{t_{2(k+1)}})|} \\ &\ge w\frac{|y_{t_{2k+1}+1}| + |y_{t_{2k}+1}|}{|f(y_{t_{2k+1}})y_{t_{2k}+1}|} - \frac{w}{|f(y_{t_{2(k+1)}})|} \\ &> \frac{w}{|f(y_{t_{2k+1}})|} - \frac{w}{|f(y_{t_{2(k+1)}})|} > 0. \end{aligned}$$

Similarly, the result also holds for $t = t_{2k+3}$ by a similar reasoning as that for $t = t_{2(k+1)}$.

Hence, by induction we know that (iii) is true.

(iv) At time $t = t_k$, it is easy to see that $i_t = t_{k-1}$ and $j_t = t_{k-2}$. Then by (ii) and (iii) ,

$$\begin{aligned} \Delta_{t_k} &= \overline{\vartheta}_2 w\frac{|y_{i_t+1}| + |y_{j_t+1}|}{|f(y_{i_t})y_{j_t+1}|} \\ &= \overline{\vartheta}_2 w\frac{|y_{t_{k-1}+1}| + |y_{t_{k-2}+1}|}{|f(y_{t_{k-1}})y_{t_{k-2}+1}|} \\ &= O\left(\left|\frac{y_{t_k}}{y_{t_{k-1}}^{b+1}}\right|\right). \end{aligned}$$

Hence, by Lemma 3.1, we have for $k \ge 2$

$$\begin{aligned} |\varepsilon_{t_k}| &= \theta_2|\vartheta_1 - \beta_t| = \theta_2|\vartheta_1 - \hat{\vartheta}_{1,t_k} \pm 2\Delta_{t_k}| \\ &\le \theta_2(|\tilde{\vartheta}_{1,t_k}| + 2\Delta_{t_k}) \le 3\theta_2\Delta_{t_k} \\ &= O\left(\left|\frac{y_{t_k}}{y_{t_{k-1}}^{b+1}}\right|\right). \end{aligned}$$

This completes the proof. ■

The sufficiency proof of Theorem 2.1. We use a contradiction argument to prove that $\sup_{t\ge0}|y_t| < \infty$. Suppose there exist some $y_0 \in \mathbb{R}^1$, some $\{\theta_1, \theta_2\}$ and some sequence of $\{w_t\}$, such that for the control defined in (20), $\sup_{t\ge0}|y_t| = \infty$. Then for the subscript sequence $\{t_k\}$ defined in (18), we have $k \to \infty$.

Also note that, by Proposition 3.1 (i) and (iv), the system (2) at time t_{k+1} satisfies

$$|y_{t_{k+1}}| \le 2c_2|\varepsilon_{t_k}||y_{t_k}|^b = O\left(\left|\frac{y_{t_k}}{y_{t_{k-1}}}\right|^{b+1}\right). \tag{26}$$

To apply Lemma 3.5 in Xie and Guo (2000a), we take $a_k = \log|y_{t_k}|$, then the outputs will be bounded when $b + 1 < 4$, which contradicts to our assumption. Hence, the sufficiency is proved. ■

4. THE PROOF OF NECESSITY

We introduce a stochastic imbedding approach to the proof of necessity. Let $(\Omega, \mathcal{F}, P)$ be a probability space, and let $\theta \in \mathbb{R}^2$ be a random vector and $\{w_t\}_{t=1}^{\infty}$ be a stochastic process on this probability space respectively. (In fact, θ and $\{w_t\}_{t=1}^{\infty}$ are different from those defined in the assumptions **A1) - A2)**, we use the same notation just for convenience.) We consider the stochastic system in the form (2).

Assume that θ has a spherical p.d.f. $p(\theta)$, which satisfies

$$p(\theta) = \begin{cases} c(2^{-1}R^2 - \|\tilde{\theta}^c\|^2) & \text{if } 0 \le \|\tilde{\theta}^c\| \le R/2; \\ c(R - \|\tilde{\theta}^c\|)^2 & \text{if } R/2 \le \|\tilde{\theta}^c\| \le R; \\ 0 & otherwise \end{cases} \quad (27)$$

where $\tilde{\theta}^c = \theta - \theta^c$ with $\theta^c = \left(\frac{\underline{\theta}_1 + \overline{\theta}_1}{2}, \frac{\underline{\theta}_2 + \overline{\theta}_2}{2}\right)^T$ being the center of the uncertain domain, and

$$R = \min\{\frac{\overline{\theta}_1 - \underline{\theta}_1}{2}, \frac{\overline{\theta}_2 - \underline{\theta}_2}{2}\},$$

and where c is some constant to make $\int_{\|\tilde{\theta}^c\| \le R} p(\theta) d\theta = 1$.

Assume also that $\{w_k\}$ is an i.i.d random sequence independent of θ with truncated Gaussian p.d.f. $q(z)$:

$$q(z) = M \exp\left(-\frac{z^2}{2}\right), \qquad -w \le z \le w \qquad (28)$$

where w is defined in **A2)** and M satisfies

$$\int_{-w}^{w} M \exp\left(-\frac{z^2}{2}\right) dz = 1.$$

Remark 4.1. We need to note that $\{\theta : \|\tilde{\theta}^c\| \le R\} \subset [\underline{\theta}_1, \overline{\theta}_1] \times [\underline{\theta}_2, \overline{\theta}_2]$ by (27) and the definition of R, see Fig 1. The distribution of the noise in (28) also shows that $|w_t| \le w$ for any $t \ge 1$.

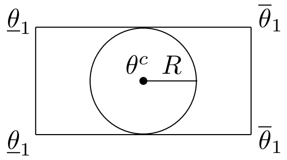

Fig.1. The area of θ

We will first show that in the above stochastic framework, if $b \ge 3$, then for any feedback control $u_t \in \mathcal{F}_y^t \triangleq \sigma\{y_i, 0 \le i \le t\}$, there always exists an initial condition y_0 and a set D_1 with positive probability such that the output signal y_t of the closed-loop control system tends to infinity at a rate faster than exponential on D_1. Then in the last part of this subsection, we will find a point in D_1 which corresponds to some values of θ and $\{w_t\}_{t=1}^{\infty}$, and we will see that these deterministic values are sufficient for the proof of necessity of Theorem 2.1. Thus by imbedding a random distribution, we are able to solve the problem in the original deterministic framework.

To prove the above fact, we first give a lemma which can be obtained by a little modification of the proof of [Xie and Guo (2000a), Theorem 3.2.2-Theorem 3.2.6 and Remark 3.2.3]. We omit the details here.

Lemma 4.1. Consider the following dynamical system:

$$y_{k+1} = \theta^T \phi_k + w_{k+1}, \quad k = 0, 1, \cdots,$$

where $\phi_k \triangleq (f(y_k), u_k)^T$, $y_0 > x'$ is deterministic and $y_i = 0$, $\forall i < 0$; the unknown parameter vector θ with p.d.f. $p(\theta)$ defined in (27) is independent of $\{w_k\}$ which is i.i.d. random sequence with distribution defined in (28). Then there exists some $D_0 \subset \Omega$ with $\text{Prob}(D_0) > 0$ such that for $t = 1, 2, \cdots$,

$$E[(\theta - \hat{\theta}_t)(\theta - \hat{\theta}_t)^T | \mathcal{F}_t^y] \ge \left\{\sum_{k=0}^{t-1} \phi_k \phi_k^T + K t^4 I\right\}^{-1} \quad on\ D_0,$$

where $\hat{\theta}_t \triangleq E\{\theta | \mathcal{F}_t^y\}, t = 1, 2, \cdots$; and $K > 0$ is some constant; I denotes the identity matrix. Furthermore, there exists some $D_1 \subset D_0$ with $\text{Prob}(D_1) > 0$ such that on D_1 for $t = 1, 2, \cdots$,

$$E[y_{t+1}^2 | \mathcal{F}_t^y] \le (K_1 t^4 + 4) y_{t+1}^2 + (K_1 t^4 + 4)(t+1)^2 + \sigma_w^2,$$

where $K_1 > 0$ is some constant and $\sigma_w^2 \triangleq E w_t^2$

In the following lemma, we will estimate the determinants of two matrices which will appear in the proof of the next proposition. It is easy to see that the two are modifications of the information matrices in Least Square-algorithm.

Lemma 4.2. Assume that for some $\delta > 0$ and $t \ge 1$, $|y_i| \ge |y_{i-1}|^{1+\delta}, i = 1, 2, \cdots, t$, and that the initial condition $y_0 \ge \max\{1, x'\}$ is sufficiently large, then the determinants of the matrices

$$P_{t+1}^{-1} \triangleq K(t+1)^4 I + \sum_{i=0}^{t} \phi_i \phi_i^T \qquad (29)$$

$$Q_{t+1}^{-1} \triangleq K t^4 I + \sum_{i=0}^{t} \phi_i \phi_i^T \qquad (30)$$

satisfy

$$|P_{t+1}^{-1}| \le K_2 \cdot \max\left\{|y_t^{b+1}|^2, |y_{t-1}^b y_{t+1}|^2\right\};$$
$$|Q_{t+1}|^{-1} \ge \underline{\vartheta}_2^2 \cdot \left(|f(y_t) y_t - f(y_{t-1}) y_{t+1}| - 2w|f(y_t)|\right)^2,$$

where $K > 0$ is defined in Lemma 4.1 and $K_2 > 1$ is some constant.

Remark 4.2. If $|f(y_{t-1}) y_{t+1}| < \frac{1}{2}|f(y_t) y_t|$, then we have for large enough $|y_t|$,

$$\begin{aligned} |Q_{t+1}|^{-1} &\ge \underline{\vartheta}_2^2 \cdot \left(\frac{1}{2}|f(y_t) y_t| - 2w|f(y_t)|\right)^2 \\ &\ge \underline{\vartheta}_2^2 \cdot \left(\frac{c_1}{2}|y_t^{b+1}| - 2wc_2|y_t|^b\right)^2 \\ &\ge \frac{\underline{\vartheta}_2^2 c_1}{4}|y_t|^{2(b+1)}. \end{aligned}$$

On the other hand, if $|f(y_{t-1}) y_{t+1}| \ge \frac{1}{2}|f(y_t) y_t|$, then we have $c_2|y_{t-1}^b y_{t+1}| \ge \frac{c_1}{2}|y_t^{b+1}|$. Moreover, if $1 + \delta = \frac{b+1}{2}$, we have for large $|y_t|$ and $b \ge 3$,

$$\begin{aligned} |y_{t+1}| &\ge \frac{c_1}{2c_2}\frac{|y_t^{b+1}|}{|y_{t-1}^b|} \ge \frac{c_1}{2c_2}|y_t|^{b+1-\frac{2b}{b+1}} \\ &\ge |y_t|^{b-1} \ge |y_t|^{\frac{b+1}{2}}. \end{aligned}$$

Proposition 4.1. Assume that the conditions of Lemma 4.1 hold. Then for any $u_t \in \mathcal{F}_y^t$, there always exists a y_0 and a set $D_1 \subset \Omega$ with positive probability such that the output signal $|y_t| \nearrow \infty$ on D_1 whenever $b \geq 3$.

Proof. It is easy to see that $E[w_{t+1}|\mathcal{F}_t^y] = Ew_{t+1} = 0$ by (28). By (2) we know that

$$y_{t+1} = \phi_t^T \tilde{\theta}_t + (\phi_t^T \hat{\theta}_t + u_t) + w_{t+1}, \tag{31}$$

where $\tilde{\theta}_t \triangleq \theta_t - \hat{\theta}_t$. Consequently, by the fact $E[\tilde{\theta}_t|\mathcal{F}_t^y] = 0$ and $E[w_{t+1}|\mathcal{F}_t^y] = 0$ it follows that for any $u_t \in \mathcal{F}_t^y$,

$$\begin{aligned} E[y_{t+1}^2|\mathcal{F}_t^y] &= \phi_t^T E[\tilde{\theta}_t\tilde{\theta}_t^T|\mathcal{F}_t^y]\phi_t + (\phi_t^T\hat{\theta}_t + u_t)^2 + E[w_{t+1}^2|\mathcal{F}_t^y] \\ &\geq \phi_t^T E[\tilde{\theta}_t\tilde{\theta}_t^T|\mathcal{F}_t^y]\phi_t + E[w_{t+1}^2|\mathcal{F}_t^y]. \end{aligned} \tag{32}$$

Let $\sigma^2 = \min\{\sigma_w^2, 1\}$, where $\sigma_w^2 = Ew_{t+1}^2$ defined in Lemma 4.1, then by Lemma 4.1, we have on D_1,

$$\begin{aligned} E[y_{t+1}^2|\mathcal{F}_t^y] &\geq \phi_t^T \left\{ \sum_{k=0}^{t-1} \phi_k\phi_k^T + Kt^4 I \right\}^{-1} \phi_t + \sigma_w^2 \\ &= \phi_t^T P_t \phi_t + \sigma_w^2 \geq \sigma^2 \frac{|P_t^{-1} + \phi_t^T\phi_t|}{|P_t^{-1}|} \qquad (33) \\ &= \sigma^2 \frac{|Q_{t+1}^{-1}|}{|P_t^{-1}|}, \qquad t \geq 1, \qquad (34) \end{aligned}$$

where P_t, Q_t are defined by (29) and (30).

Hence by (29) we have for $t \geq 1$,

$$y_{t+1}^2 \geq \frac{1}{K_1 t^4 + 4} \left[\sigma^2 \frac{|Q_{t+1}^{-1}|}{|P_t^{-1}|} - (K_1 t^4 + 4)(t+1)^2 - \sigma_w^2 \right],$$

Now, we proceed to show that on D_1 for large y_0,

$$|y_i| \geq |y_{i-1}|^{\frac{b+1}{2}}, \qquad i = 1, 2, \cdots$$

First, notice that when $b \geq 3$, we have $x^2 - (b+1)x + (b+1) \leq 0$ for $x = \dfrac{b+1}{2}$, and then

$$x^3 - bx^2 + b \leq 0.$$

Now, let $P(x) = x^2 - (b+1)x + (b+1)$ or $P(x) = x^3 - bx^2 + b$, then $P(x) \leq 0$ for $x = \dfrac{b+1}{2}$. By a similar argument as the proof in [Xie and Guo (2000a) , Theorem 3.3.1] from (83)-(85) and Remark 4.2, we know that the conclusion is true. ■

The proof of the necessity of Theorem 2.1. In the stochastic framework, note that any controller $u_t = h_t(y_0, \cdots, y_t)$ is measurable to $\mathcal{F}_y^t$. By Proposition 4.1, for any given control law $\{u_t\}$, there at least exists a sample point $\omega^* \in D_1 \subset \Omega$ with $\theta(\omega^*) = \theta^*$ and $w_t(\omega^*) = w_t^*$ for any $t \geq 1$ such that for some y_0^*, the absolute values of the output $|y_t(\omega^*)| = |y_t^*| \nearrow \infty$.

That is, for any given Borel function $h(\cdot)$, there exist some θ^* and $\{w_t^*\}$ satisfying assumptions **A1)-A2)** and a y_0^* such that the absolute values of the outputs

$$y_{t+1}^* = \theta_1^* {y_t^*}^b + \theta_2^* h_t(y_0^*, \cdots, y_t^*) + w_{t+1}^*$$

monotonously increase to infinity, which proves the necessity conclusion of Theorem 2.1. ■

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