**Proceedings of the 17th World Congress**
**The International Federation of Automatic Control**
**Seoul, Korea, July 6-11, 2008**

# Optimal Finite-precision Implementations of Linear Parameter Varying Controllers

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**Abstract:** Digital computing devices have a finite precision. Hence when digital controllers are implemented, there is rounding on the variables and parameters resulting in the various finite-word-length effects on the closed-loop stability and performance of the system. In this paper we concentrate on the coefficient sensitivity problem. That is: to determine the controller realization that minimizes the sensitivity of the closed-loop system to small perturbations on the controller coefficients. The sensitivity minimization problem can be approximated by a stability radius maximization problem. In this paper we consider the coefficient sensitivity problem for digital implementations of linear parameter-varying controllers. The problem of maximizing the stability radius for the coefficient sensitivity problem for linear parameter-varying controllers reduces to the solution of a set of linear matrix inequalities. The approach is demonstrated on an example. Furthermore, the example shows that eigenvalue sensitivity measures are not generally suitable for linear-parameter-varying controller, finite-word-length problems.

Keywords: Finite-precision, digital controller, finite word length, digital implementation, linear parameter-varying systems, time-varying systems, linear matrix inequalities, eigenvalue sensitivity, FPGAs.

## 1. INTRODUCTION

Gain scheduling is a common method of control in a variety of practical applications. A particular framework for designing such controllers is provided by the use of Linear Parameter Varying (LPV) models. A number of analysis and controller design methods have been developed for such plants, generally resulting in LPV controllers. Reviews of gain-scheduling and LPV design methods can be found in Leith and Leithead [2000] and in Rugh and Shamma [2000].

Unlike for Linear Time Invariant (LTI) controllers, there has been little work on the implementation of LPV controllers. The design methods are often in continuous time [Gahinet et al., 1995], and so far there is no straightforward method for discretizing the controllers that guarantees performance and stability in the discrete time [Apkarian, 1997]. That however, is not the problem that is studied here. We concentrate only on the problem of the optimal quantization of the controller coefficients. This has been fairly well studied for linear time-invariant controllers [see Istepanian and Whidborne, 2001, for example] but almost totally ignored for LPV and gain-scheduling controllers. Kelly and Evers [1997] consider the problem of interpolating the controllers for gain-scheduling problems, and in order to get good numerical conditioning of the controllers, advise that balanced realizations be used. The example studied in this paper demonstrates that this is probably sound advice. The special case of periodically varying linear controllers has been studied by Farges et al. [2007], and this is for a state feedback controller case only.

The problem of quantization of the controller coefficients comes under the broader class of problems known as Finite-Word-Length (FWL) problems and arises because the set of real numbers that can be stored in a digital computer is a subset of the real space. To store all real numbers would require an infinite number of bits which is not physically possible — hence the moniker "Finite-Word-Length". As a consequence, constants and variables in a digital computer are subject to rounding and the subsequent computational errors well-known in numerical analysis. In addition, the range of possible numbers is also finite. Hence for the implementation of a controller (or filter) by a digital computer, consideration of the effects of the finite-precision and range (the FWL effects) is important.

There are three main FWL effects; (i) errors resulting from finite precision in the controller coefficients (the *coefficient sensitivity* problem), (ii) errors resulting from rounding of variables after each arithmetic computation (the *round-off noise* problem) and (iii) limitations imposed by the finite range of variables and constants (the *overflow/underflow* problem or the *scaling* problem). In this paper we confine ourselves to just the coefficient sensitivity problem for LPV digital controllers, and more particularly to the sensitivity of the stability.

In particular, it is well known that the FWL effects are strongly dependent on the controller realization. An LPV state-space controller has the form

$$x(k+1) = A(\theta(k))x(k) + B(\theta(k))y(k) \quad (1)$$

$$u(k) = C(\theta(k))x(k) + D(\theta(k))y(k). \quad (2)$$

By means of a similarity transformation, all equivalent state-space realizations are given by

$$\tilde{x}(k+1) = T^{-1}A(\theta(k))T\tilde{x}(k) + T^{-1}B(\theta(k))y(k) \quad (3)$$

$$u(k) = C(\theta(k))Tx(k) + D(\theta(k))y(k) \quad (4)$$

where $T$ is non-singular. The objective of this study is to determine realizations that have low coefficient sensitivity for the range of $\theta(k)$. Most previous studies of controller coefficient stability sensitivity for LTI controllers have considered minimization of eigenvalue sensitivity measures [e.g. Li, 1998, Istepanian et al., 1998, Wu et al., 2000a, Whidborne et al., 2001, Wu et al., 2001, 2000b, Yu and Ko, 2003, Hilaire et al., 2006]. As we will see in Section 5, such measures are not generally appropriate for LPV systems. Consequently, we will use a measure based on the complex stability radius. This measure can be related to the probability of closed-loop instability resulting from the FWL [Fialho and Georgiou, 1994]. Furthermore, the optimal realization problem using this measure can be posed as a Linear Matrix Inequality (LMI) problem and hence easily solved [Fialho and Georgiou, 2001]. In this paper we extend this approach to a class of LPV problems.

After an introduction to LPV control systems in the next section, the theory for LTI coefficient sensitivity minimization using a stability radius approach is outlined in Section 3. An eigenvalues sensitivity measure is also defined. In Section 4, the coefficient sensitivity minimization using a stability radius approach is extended to LPV systems. In Section 5, the method is illustrated with a simple example. The paper concludes with some comments.

## 2. LPV SYSTEMS

Consider the LPV plant with $m$ states, $\ell$ inputs and $q$ outputs that is given by

$$x_p(k+1) = A_p(\theta(k))x_p(k) + B_p u(k) \quad (5)$$

$$y(k) = C_p x_p(k) \quad (6)$$

where $A_p$ depends affinely on the time-varying parameter vector, $\theta(k)$, and $\theta(k)$ is known at the sample instant, $k$ (i.e. the measurement is available in real time). Note that here we restrict ourselves to plants where only the plant $A_p$ matrix is dependent on $\theta$. This should not be a great practical restriction [Gahinet et al., 1995, Apkarian et al., 1995b] since parameter dependence in the $B_p$ ($C_p$) matrix can be moved to the $A_p$ matrix by augmentation of $A_p$ with a high-bandwidth first-order lag inserted at the inputs (outputs).

We assume that an $n$-state LPV controller of the form

$$x(k+1) = A(\theta(k))x(k) + B(\theta(k))y(k) \quad (7)$$

$$u(k) = C(\theta(k))x(k) + D(\theta(k))y(k). \quad (8)$$

has been designed, and where $R(\theta(k))$ depends affinely on $\theta$ where

$$R := \begin{bmatrix} A(\theta(k)) & B(\theta(k)) \\ C(\theta(k)) & D(\theta(k)) \end{bmatrix}. \quad (9)$$

The closed loop system matrix is then given by

$$A_c = \begin{bmatrix} A(\theta(k)) & B(\theta(k))C_p \\ B_pC(\theta(k)) & A_p(\theta) + B_pD(\theta(k))C_p \end{bmatrix}. \quad (10)$$

Defining

$$A_0 := \begin{bmatrix} 0 & 0 \\ 0 & A_p \end{bmatrix}, \quad (11)$$

$$B_I := \begin{bmatrix} I & 0 \\ 0 & B_p \end{bmatrix}, \quad (12)$$

$$C_I := \begin{bmatrix} I & 0 \\ 0 & C_p \end{bmatrix}, \quad (13)$$

we get

$$A_c = A_0(\theta(k)) + B_I R(\theta(k)) C_I, \quad (14)$$

which is also affinely dependent on $\theta$.

## 3. FWL COEFFICIENT SENSITIVITY FOR LTI SYSTEMS

The FWL problem has been fairly extensively studied for LTI systems over the last 2 decades. In this paper we will concentrate on the stability problem, that is: what realization of the controller maximizes the stability margins in the face of controller perturbations resulting from the FWL? There have been two main approaches to the LTI FWL stability margin problem, those based on the eigenvalue sensitivity and those base on the stability radius. These approaches are compared by Chen et al. [2002].

In this paper we will use the complex stability radius, originally proposed for FWL analysis by Fialho and Georgiou [1994] who also related it to a probability measure of the required word-length for stable implementation. Note that even though the probability of stable implementation is not 100% guaranteed, this does not create a practical difficulty, because the stability of the implementation with a particular word length can be checked *a posteriori*. The problem of determining the optimal realization in terms of the complex stability radius can be posed as an LMI [Fialho and Georgiou, 1999, 2001] and can hence be completely solved.

A common approach to determining the controller realization that minimizes the coefficient sensitivity is by means of the closed loop pole sensitivity. This was first considered for LTI control systems by Li [1998] using the maximum of the 2-norms of the closed loop eigenvalue sensitivities, but the associated optimization problem was not completely solved [Whidborne et al., 2000]. An alternative approach using the weighted sum of the 2-norms of the closed loop eigenvalue sensitivities can be solved [Whidborne et al., 2001]. This sensitivity measure has nicer mathematical properties than that of Li [1998] and other less conservative measures [e.g. Wu et al., 2001] and hence is considered in this paper.

### 3.1 *Stability radius*

Let us consider the LTI versions of the set of equations (5)–(10). As a result of the FWL, a digital controller is not implemented exactly, but each of the coefficients of the $A, B, C, D$ matrices may be perturbed. The maximum perturbation will depend on the chosen representation schemes, such as fixed-point or floating-point, and on the number of wordlength. See Hilaire et al. [2007] for details.

Thus the controller matrix $R$ is perturbed to $R + \Delta$ and the closed loop system matrix is perturbed to $A_c + B_I \Delta C_I$. Let the maximum perturbation be given by the max norm

$$\| \Delta \|_{\max} := \max_{i,j} |\Delta_{i,j}| \tag{15}$$

and define the FWL stability margin as

$$\eta_0 := \inf \{ \| \Delta \|_{\max} : A_c + B_I \Delta C_I \text{ is unstable } \}. \tag{16}$$

However, $\eta_0$ is hard to compute. Fialho and Georgiou [1994] instead propose the use of the spectral norm to measure the perturbation

$$\| \Delta \|_2 := \max \left\{ \sqrt{\lambda_i} : \lambda_i \text{ are the eigenvalues of } \Delta^T \Delta \right\} \tag{17}$$

with the FWL stability margin given by the complex stability radius [Hinrichsen and Pritchard, 1986]

$$\eta_c := \inf \{ \| \Delta \|_2 : A_C + B_I \Delta C_I \text{ is unstable } \}, \tag{18}$$

which can be easily computed by

$$\eta_c = \frac{1}{\| C_I (zI - A_c)^{-1} B_I \|_\infty}, \tag{19}$$

and $\| \cdot \|_\infty$ denotes the $\mathcal{H}_\infty$-norm. Since $\| \Delta \|_{\max} \leq \| \Delta \|_2$, then $\eta_c$ provides an upper bound on $\eta_0$.

Now if we wish to find the realization of the controller that maximizes the stability radius, $\eta_c$, then we wish to find the non-singular transformation matrix, $T$, such that $\eta_c$ is maximized. The controller, and subsequently the closed loop state matrix, $A_c$, are both dependent on $T$. Define

$$A_T(T) := \begin{bmatrix} T^{-1} & 0 \\ 0 & I \end{bmatrix} A_c \begin{bmatrix} T & 0 \\ 0 & I \end{bmatrix} \tag{20}$$

then the problem to find the realization that maximizes $\eta_c$ is equivalent to [Fialho and Georgiou, 2001]

$$\min_{T \text{ non singular}} \| C_I (zI - A_T(T))^{-1} B_I \|_\infty \tag{21}$$

Denoting the optimal value of (21) as $\gamma_{\text{opt}}$ and the corresponding transformation matrix as $T_{\text{opt}}$, then Fialho and Georgiou [2001] provide the following proposition to solve the problem.

*Proposition 1.* The optimal value $\gamma_{\text{opt}}$ is the minimum $\gamma$ for which there exists a matrix of the form

$$P = \begin{bmatrix} P_1 & 0 & 0 \\ 0 & P_2 & 0 \\ 0 & 0 & I \end{bmatrix} \tag{22}$$

with $P_1 = P_1^T > 0$, $P_1 \in \mathbb{R}^{(n+m)\times(n+m)}$, $P_2 = P_2^T > 0$, $P_2 \in \mathbb{R}^{n \times n}$, $I \in \mathbb{R}^{q \times q}$, such that

$$M^T P M < P \tag{23}$$

where

$$M(\gamma) := \begin{bmatrix} A_c & B_I/\gamma \\ C_I & 0 \end{bmatrix} \tag{24}$$

and $P_2 = T_{\text{opt}}^T T_{\text{opt}}$.

The above LMI for a $\gamma > \gamma_{\text{opt}}$ can be solved using standard software, and so $\gamma_{\text{opt}}$ can be determined to an arbitrary accuracy using a bisection search [Fialho and Georgiou, 2001].

### *3.2 Eigenvalue sensitivity*

A measure of the closed-loop poles sensitivity [Whidborne et al., 2001] to controller coefficient perturbation for an LTI system is

$$\Psi = \sum_{k=1}^{n+m} w_k \Psi_k \tag{25}$$

where $n+m$ is the number of closed loop poles/eigenvalues, $w_k$ is a non-negative real scalar weighting and

$$\Psi_k = \sum_{i=1}^{n_x} \left( \frac{\partial \lambda_k}{\partial x_i} \right)^2 \tag{26}$$

where $\{\lambda_i : i = 1, \ldots, m+n\}$ represents the set of unique closed-loop poles/ eigenvalues and $\{x_i : i = 1, \ldots, n_x\}$ are the controller parameters. In this paper, we take

$$w_k := \frac{1}{1 - |\lambda_k|}. \tag{27}$$

The means to calculate $\Psi$ is provided in Whidborne et al. [2001].

## 4. FWL COEFFICIENT SENSITIVITY FOR LPV SYSTEMS

### *4.1 Quadratic $\mathcal{H}_\infty$ performance*

Following [Apkarian et al., 1995b,a], we will consider polytopic LPV systems. Firstly we define a matrix polytope as the convex hull of $r$ matrices, $N_1, N_2, \ldots, N_r$, that is

$$\text{Co}\{N_i, i = 1, \ldots, r\} := \left\{ \sum_{i=1}^r \alpha_i N_i : \alpha_i \geq 0, \sum_{i=1}^r \alpha_i = 1 \right\}. \tag{28}$$

We assume that the discrete time varying parameter, $\theta(k)$, is confined to the the polytope, $\Theta$, with vertices $\hat{\theta}_1, \hat{\theta}_2, \ldots, \hat{\theta}_r$, that is

$$\theta(k) \in \Theta, \tag{29}$$

where

$$\Theta := \text{Co}\{\hat{\theta}_1, \hat{\theta}_2, \ldots, \hat{\theta}_r\} \tag{30}$$

and that the dependence of the state space matrices on $\theta$ is affine. Such an LPV system is termed *polytopic*.

A polytopic system has quadratic $\mathcal{H}_\infty$ performance [Apkarian et al., 1995b] of $\gamma$ if and only if there exists a Lyapunov function $V(x) = x^T P x$ with $X > 0$ that establishes global stability and ensures that the $\mathcal{L}_2$ gain of the system is bounded by $\gamma$. That is $\| y \|_2 < \gamma \| u \|_2$ along all possible parameter trajectories $\theta(k) \in \Theta$.

It is shown in Apkarian et al. [1995b] that for polytopic LPV systems, the vertex property means that quadratic $\mathcal{H}_\infty$ performance is ensured for all $\theta(k) \in \Theta$ if $\mathcal{H}_\infty$ performance is ensured at all $r$ vertices of the polytopic LPV system.

### *4.2 Coefficient Sensitivity Minimization for LPV Systems*

Thus the way to determine the controller realization that maximizes the stability sensitivity (in one sense) of the closed loop polytopic LPV system is to determine the controller realization that minimizes the quadratic $\mathcal{H}_\infty$ performance of the closed loop polytopic LPV system.

From (11) and (10), the closed loop system is affine in $\theta$ and is hence a polytopic LPV system. So we just need to solve a system of LMIs that minimizes the $\mathcal{H}_\infty$ performance at each vertex using Proposition 1. Thus the following is proposed.

*Proposition 2.* The optimal quadratic $\mathcal{H}_\infty$ performance, $\gamma_{\text{opt}}$ is the minimum $\gamma$ for which there exists a $P = P^T > 0$ of the form

$$P = \begin{bmatrix} P_1 & 0 & 0 \\ 0 & P_2 & 0 \\ 0 & 0 & I \end{bmatrix} \tag{31}$$

such that

$$M_i^T(\gamma) P M_i(\gamma) < P, \text{ for } i = 1, 2, \ldots, r \tag{32}$$

where

$$M_i(\gamma) := \begin{bmatrix} A_c(\hat{\theta}_i) & B_I/\gamma \\ C_I & 0 \end{bmatrix}. \tag{33}$$

The optimal nonsingular transformation matrix is obtained from $P_2 = T_{\text{opt}}^T T_{\text{opt}}$.

## 5. EXAMPLE

To illustrate the method, we consider a simplified mass-spring-damper type model

$$J(t)\ddot{y}(t) = -0.5\dot{y}(t) - y(t) + \tau(t) \tag{34}$$

where $\tau$ is the control, and $0.5 \leq J \leq 5$ is the inertia term which is assumed to be time-varying and measurable. The control is augmented with a high frequency pole located at $-100$ to remove the dependency of the input matrix on $J$. We set $\theta(t) = 1/J(t)$, $\hat{\theta}_1 = 0.2$, $\hat{\theta}_2 = 2$ and define

$$\alpha_1 := \frac{\theta - \hat{\theta}_1}{\hat{\theta}_2 - \hat{\theta}_1}, \tag{35}$$

$$\alpha_2 := \frac{\hat{\theta}_2 - \theta}{\hat{\theta}_2 - \hat{\theta}_1}. \tag{36}$$

The resulting continuous-time state space description is given by

$$\dot{x} = A_g(\alpha_1, \alpha_2)x + B_g u, \tag{37}$$

$$y = C_g x, \tag{38}$$

$$\alpha_1 + \alpha_2 = 1, \ \alpha_1 \geq 0, \ \alpha_2 \geq 0, \tag{39}$$

with

$$A_g = \begin{bmatrix} -1/100 & 0 & 0 \\ 0 & 0 & 1 \\ (0.2\alpha_1 + 2\alpha_2) & -(0.2\alpha_1 + 2\alpha_2) & -(0.1\alpha_1 + \alpha_2) \end{bmatrix}, \tag{40}$$

$$B_g = \begin{bmatrix} 1/100 \\ 0 \\ 0 \end{bmatrix}, \tag{41}$$

$$C_g = [0 \ 1 \ 0]. \tag{42}$$

Weighting functions

$$W_1(s) = \frac{(s + 1/5)}{1.8(s + 1/5000)} \tag{43}$$

and

$$W_2(s) = \frac{(s/50 + 1)}{(s/10000 + 10)} \tag{44}$$

are defined. The MATLAB LMI Toolbox function, `hinfgs`, with the criterion

$$\left\| \begin{bmatrix} W_1 S \\ W_2 K S \end{bmatrix} \right\|_\infty < 1 \tag{45}$$

is used to obtain an LPV controller [Gahinet et al., 1995]. The routine performs some order reduction so the resulting controller is order 4.

The controller at each vertex is discretized using the Tustin transformation with a sampling rate of 500Hz giving

$$K(z, \alpha_1, \alpha_2) = B(zI - A)^{-1}C + D \tag{46}$$

where

$$A = (\alpha_1 A_1 + \alpha_2 A_2), \tag{47}$$

$$B = (\alpha_1 B_1 + \alpha_2 B_2), \tag{48}$$

$$C = (\alpha_1 C_1 + \alpha_2 C_2), \tag{49}$$

$$D = (\alpha_1 D_1 + \alpha_2 D_2) \tag{50}$$

and

$$A_1 = \begin{bmatrix} 701.85 & -3950.8 & 892.66 & 97.320 \\ -0.10476 & 969.03 & 11.910 & 25.007 \\ -0.35992 & 3.0329 & 998.02 & -6.4805 \\ 0.13622 & 0.22055 & -0.22193 & 996.96 \end{bmatrix} \times 10^{-3}, \tag{51}$$

$$A_2 = \begin{bmatrix} 677.14 & -3892.8 & 867.11 & 96.590 \\ 11.622 & 941.49 & 24.029 & 25.317 \\ -3.4822 & 10.366 & 994.79 & -6.5611 \\ 1.7715 & -3.6618 & 1.6399 & 997.78 \end{bmatrix} \times 10^{-3}, \tag{52}$$

$$B_1 = \begin{bmatrix} 18.355 \\ -8.7698 \\ 2.0605 \\ 0.90468 \end{bmatrix}, \tag{53}$$

$$B_2 = \begin{bmatrix} 12.055 \\ -5.7663 \\ 1.2560 \\ 1.0273 \end{bmatrix}, \tag{54}$$

$$C_1 = [1.2791 \ 43.488 \ -9.8262 \ -1.0630] \times 10^{-3}, \tag{55}$$

$$C_2 = [1.5547 \ 42.8403 \ -9.5412 \ -1.0551] \times 10^{-3}, \tag{56}$$

$$D_1 = -0.20511, \tag{57}$$

$$D_2 = -0.13473. \tag{58}$$

To determine the optimal realization of the discrete-time controller, the plant must be discretized. However, a straight-forward discretization of (37) will result in the discrete-time state-space input matrix, $B_p$, being dependent on $A_g$ and hence parameter dependent and time-varying. To overcome this, the unaugmented model was discretized and augmented with the filter $0.18127/(z - 0.8187)$; this being the filter $100/(s+100)$ discretized with a sampling rate of 500Hz and a zero-order hold.

The LMI

$$M_i^T(\gamma) P M_i(\gamma) < P, \text{ for } i = 1, 2 \tag{59}$$

is repeatedly solved with a bisection search to obtain $\gamma_{\text{opt}}$ to a tolerance of $10^{-3}$. The optimal value is $\gamma_{\text{opt}} = 2.736 \times 10^3$. The resulting optimal state transformation matrix is

$$T_{\text{opt}} = \begin{bmatrix} 154.9889 & -11.4332 & 0.1092 & -1.2230 \\ -11.4332 & 2.9358 & 4.7713 & 1.8837 \\ 0.1092 & 4.7713 & 20.7874 & 6.3868 \\ -1.2230 & 1.8837 & 6.3868 & 9.7101 \end{bmatrix}. \tag{60}$$

Denoting $C_I[zI - A_T(T_{\text{opt}})]^{-1}B_I$ by $M_{\text{opt}}$, the $\mathcal{H}_\infty$-norm of $M_{\text{opt}}$ at frozen values of $\alpha_1$ is shown in Figure 1 along with the optimum quadratic $\mathcal{H}_\infty$ performance, $\gamma_{\text{opt}}$. It can be seen that the optimum quadratic performance bounds the frozen-$\alpha_1$ $\mathcal{H}_\infty$-norm as expected.

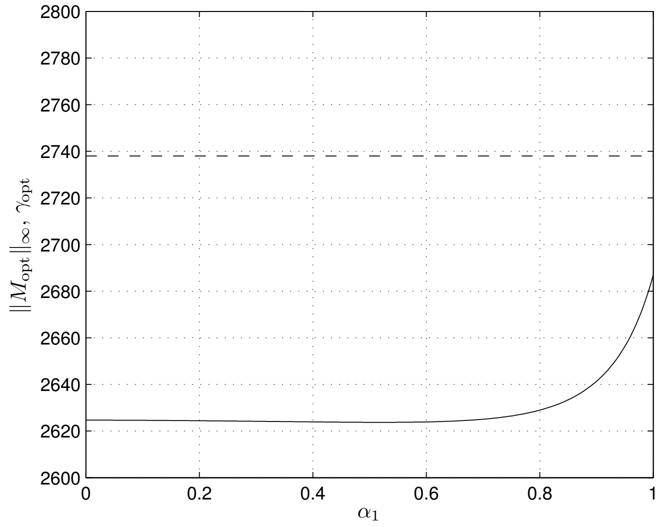


Fig. 1. Frozen-$\alpha_1$ $\mathcal{H}_\infty$-norm against $\alpha_1$. The optimum quadratic performance $\gamma_{\rm opt}$ is shown as the dashed line.

For the purpose of comparison, two other realizations are determined. The modal realization is calculated from the MATLAB function, `canon` and the balanced realization from the MATLAB function, `balreal`. Balanced realizations are known to have good open-loop FWL properties [Gevers and Li, 1993]. The required transformation matrix for both cases is calculated from the controller of (46) with $\alpha_1 = \alpha_2 = 0.5$.

The complex stability radius, $\eta_c$, at frozen values of $\alpha_1$ is shown in Figure 2 for these two realizations as well as for the optimal realization and the original realization. The optimal realization has the largest complex stability radius over all values of $\alpha_1$. Surprisingly, the modal form is not as good as the original realization.

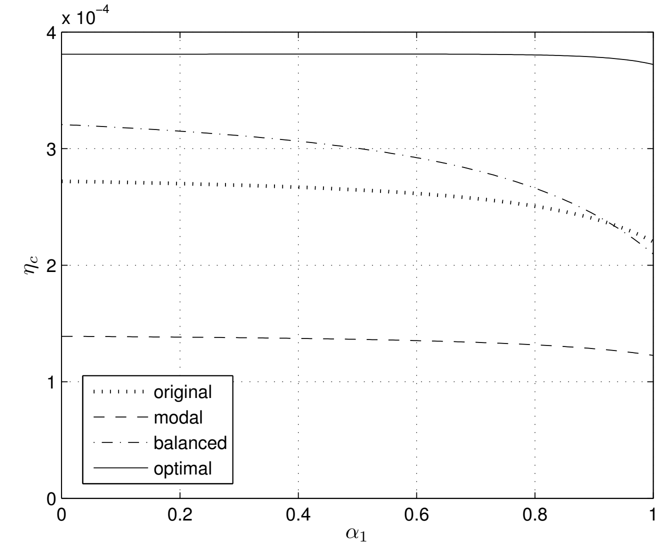


Fig. 2. Complex stability radius, $\eta_c$, against $\alpha_1$ for frozen $\alpha_1$.

The eigenvalue sensitivity measure, $\Psi$, at frozen values of $\alpha_1$ is shown in Figure 3 for the four realizations. The balanced realization is actually better than the optimal realization. What is more significant is that for all four realizations there is a singularity at $\alpha_1 \simeq 0.634$. This can be explained by looking at the closed loop eigenvalues. For $\alpha_1 = 0$, the closed loop eigenvalues are $\{0.7014, 0.8187, 0.9687, 0.9977, 0.9996, 0.9991 \pm 0.0013j\}$, and for $\alpha_2 = 0$, they are $\{0.8187, 0.9996, 0.9975, 0.8083 \pm 0.1757j, 0.9978 \pm 0.0024j\}$. Hence two of the real eigenvalues have migrated to complex conjugate positions with increasing $\alpha_1$. When $\alpha_1 = 0.634$, these two eigenvalues are real and equal. First-order eigenvalue sensitivity can only be determined for unique eigenvalues, hence the singularity. We can thus conclude that eigenvalue sensitivity is not generally suitable for LPV systems coefficient sensitivity.

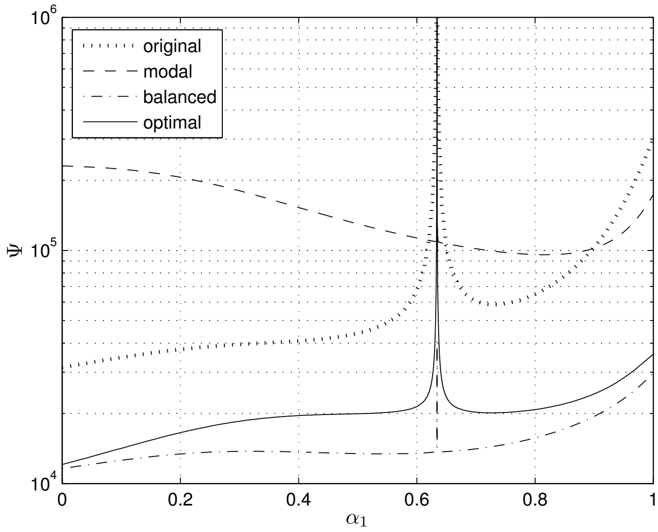


Fig. 3. Eigenvalue sensitivity measure, $\Psi$, against $\alpha_1$ for frozen $\alpha_1$.

## 6. CONCLUDING REMARKS

In this paper, a method for minimizing the coefficient sensitivity for FWL implementations of LPV controllers is proposed. The problem arises because digital computing devices have a finite precision. The problem is of particular importance for Field Programmable Gate Arrays (FPGA) which are increasingly being used for controller implementations [Fang et al., 2005].

The problem reduces to an LMI problem which can be easily solved using standard software. The method is illustrated with a simple example. The example problem is not one that is particularly sensitive to FWL effects, but it illustrates the method. More importantly, it demonstrates a limitation in eigenvalue sensitivity measures for LPV systems where the closed-loop eigenvalues may be non-unique for some values of the parameter, $\theta$. This is not a general concern for LTI systems, because generally engineers avoid designs with multiple closed-loop eigenvalues.

Apart from testing on harder problems, future work will include minimizing the FWL coefficient sensitivity for observer-controller structures. This is of particular importance because it aids the implementation of the parameter interpolation. The effect of rounding on the scheduling parameter. $\theta$, needs addressing. The problem of closed loop transfer function sensitivity will also be addressed, this can be done through use of the bounded real lemma. In addition, the round-off noise problem and the scaling problem [Boyd et al., 1993] can be reduced to an LMI for LTI systems, it is envisaged that this approach can be extended to LPV systems.

## ACKNOWLEDGEMENTS

With thanks to Ecole des Mines Nantes for supporting the visit of JF Whidborne to EMN where this work was conducted.

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