Proceedings of the 17th World Congress
The International Federation of Automatic Control
Seoul, Korea, July 6-11, 2008

Analysis of the Energy Based Swing-up Control for a Double Pendulum on a Cart ⋆

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Abstract: Designing and analyzing controllers for underactuated systems with *underactuation degree greater than one* is still an open and challenging problem. In this paper, we study an unsolved problem of analyzing energy based swing-up control for a double pendulum on a cart, which has three degrees of freedom and only one control input. We present an original analysis of the convergence of the energy of the cart-double pendulum system. We show that for all initial states of the cart-double pendulum system, if the convergent value of the energy is not equal to the energy at the upright (up-up) equilibrium point, then the cart-double pendulum remains at its up-down, down-up, and down-down equilibrium points. Moreover, we show that these three equilibrium points are unstable. This shows that for almost all initial states of the cart-double pendulum system, as time approaches infinity, the energy of the cart-double pendulum system can be controlled to its energy at the upright equilibrium point. This paper provides insight into the energy based control approach to underactuated systems with underactuation degree greater than one.

1. INTRODUCTION

The recent years have witnessed an increasing interest in studying underactuated mechanical systems, which possess fewer actuators than degrees of freedom. For a class of underactuated systems with underactuation degree one, that is, the number of control inputs is one less than that of degrees of freedom, several approaches have been shown effective for designing and analyzing controllers for these systems, see e.g., Acosta et al. [2005], Grizzle et al. [2005], Fantoni et al. [2000], Kolesnichenko and Shiriaev [2002].

However, designing and analyzing controllers for underactuated systems with *underactuation degree greater than one* is still an open and challenging problem. Though the energy based control approach developed in the seminal works of Fantoni et al. [2000], Kolesnichenko and Shiriaev [2002], Spong [1996] can be used to design controllers for these systems, the analysis of the convergence of energy or the analysis of the closed-loop solution of these systems has not been reported much yet.

On the other hand, the control of pendulum(s) on a cart has been studied greatly for investigating effectiveness of various kinds of control schemes and demonstrating ideas emerging in the area of nonlinear control, see e.g., Lin et al. [1996], Wei et al. [1995]. Recently, the cart-pendulum systems have been treated as underactuated systems.

In this paper, we investigate how to analyze energy based swing-up control for a double pendulum on a cart, which has three degrees of freedom and only one control input. Due to the cascade structure of two pendulums, to the best of our knowledge, strict analysis has not been reported yet.

Before describing the main contribution of this paper, we recall some existing results related to the swing-up control of pendulum(s). For swinging up the pendulum(s), the energy based control approach is often adopted, see e.g., Åström et al. [1999], Åström and Furuta [2000], Chung and Hauser [1995], Lozano et al. [2000], Yamakita et al. [1995]. On the one hand, without controlling the displacement of the cart, for the single pendulum-cart system, Åström and Furuta [2000] designed an energy based control to drive the energy of the pendulum to its potential energy at the vertical; Xin and Kaneda [2005] extended the results of Åström and Furuta [2000] to two parallel pendulums on a cart and proved the convergence of the energy of each pendulum to its desired value. On the other hand, for the single pendulum-cart system, Lozano et al. [2000] and Shiriaev et al. [2000] reported how to design and analyze the energy based control to drive *both* the energy of the pendulum-cart system to its potential energy of pendulum at the vertical *and* the displacement of the cart to a desired value.

In this paper, we apply directly the procedures in Lozano et al. [2000], Kolesnichenko and Shiriaev [2002] to derive an energy based swing-up controller for the cart-double pendulum system. Different from Åström et al. [1999], Zhong and Röck [2001], the main contribution of this paper is the presentation of an original analysis of the convergence of the energy of the cart-double pendulum system. Specially, under the designed controller, we show that for all initial states of the cart-double pendulum system, if the convergent value of its energy is not equal

⋆ This work was supported in part by a Grant-in-aid Scientific Research (C) under grant no. 19560452.

to the energy at the upright (up-up) equilibrium point, then the cart-double pendulum remains at its up-down, down-up, and down-down equilibrium points. Moreover, we show that these 3 equilibrium points are unstable. This shows that for almost all initial states of the cart-double pendulum system, as time approaches infinity, the energy of the cart-double pendulum system can be controlled to its energy at the upright equilibrium point.

2. PRELIMINARIES AND PROBLEM FORMULATION

2.1 Preliminaries

The cart-double pendulum system shown in Fig. 1 consists of two-linked pendulums on a cart. For pendulum i, θ_i is the angle between pendulum i and the vertical, I_i is the inertia moment with respect to the center of mass (COM), l_{ci} is the distance between the COM and joint i, l_i is its length, and m_i is its mass. For the cart, m_c is its mass, and x is its displacement; f is a force to move the cart which is the control input.

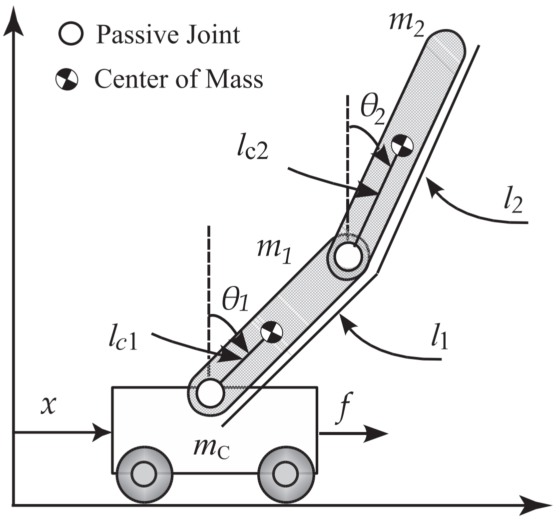

Fig. 1. A double pendulum on a cart.

With notations shown in Fig. 1, the motion equation of the double pendulum on a cart is:

$$M(\theta)\ddot{q} + C(\theta,\dot{\theta})\dot{q} + G(q) = Bf, \tag{1}$$

where

$$q = \begin{bmatrix} x \\ \theta_1 \\ \theta_2 \end{bmatrix}, \ \theta = \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix}, \ B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \tag{2}$$

and

$$M(\theta) = \begin{bmatrix} \alpha_0 & \frac{\beta_1}{g}\cos\theta_1 & \frac{\beta_2}{g}\cos\theta_2 \\ \frac{\beta_1}{g}\cos\theta_1 & \alpha_1 & \alpha_3\cos(\theta_1-\theta_2) \\ \frac{\beta_2}{g}\cos\theta_2 & \alpha_3\cos(\theta_1-\theta_2) & \alpha_2 \end{bmatrix}, \tag{3}$$

$$C(\theta,\dot{\theta}) = \begin{bmatrix} 0 & -\frac{\beta_1}{g}\dot{\theta}_1\sin\theta_1 & -\frac{\beta_2}{g}\dot{\theta}_2\sin\theta_2 \\ 0 & 0 & \alpha_3\dot{\theta}_2\sin(\theta_1-\theta_2) \\ 0 & -\alpha_3\dot{\theta}_1\sin(\theta_1-\theta_2) & 0 \end{bmatrix}, \tag{4}$$

$$G(\theta) = \begin{bmatrix} 0 \\ -\beta_1\sin\theta_1 \\ -\beta_2\sin\theta_2 \end{bmatrix}, \tag{5}$$

$$\begin{cases} \alpha_0 = m_c + m_1 + m_2, \\ \alpha_1 = m_1 l_{c1}^2 + m_2 l_1^2 + I_1, \\ \alpha_2 = m_2 l_{c2}^2 + I_2, \\ \alpha_3 = m_2 l_1 l_{c2}, \\ \beta_1 = (m_1 l_{c1} + m_2 l_1)g, \\ \beta_2 = m_2 l_{c2} g, \end{cases} \tag{6}$$

and g is the acceleration of the gravity.

The energy of the cart-double pendulum system is

$$E(q,\dot{q}) = \frac{1}{2}\dot{q}^{\mathrm{T}} M(\theta)\dot{q} + P(q), \tag{7}$$

where $P(q)$ is the potential energy and is defined as

$$P(q) = \beta_1\cos\theta_1 + \beta_2\cos\theta_2. \tag{8}$$

Since in (1) and (7) each θ_i $(i = 1,2)$ only appears as an argument of periodic functions of period 2π, this paper takes θ_i modulo 2π, i.e., $\theta_1 \times \theta_2$ is defined over a torus $S^1 \times S^1$, where S^1 denotes a unit circle.

We recall the following lemma presented in Xin and Kaneda [2007] which is important in analyzing the energy based swing-up control of the cart-double pendulum system.

LEMMA 1. α_1, α_2, α_3, β_1, and β_2 in (6) satisfy the following relations:

$$\alpha_2\beta_1 > \alpha_3\beta_2, \tag{9}$$

$$\alpha_3\beta_1 \geq \alpha_1\beta_2. \tag{10}$$

2.2 Problem Formulation

Consider the following upright (up-up) equilibrium point:

$$q = 0, \ \ \dot{q} = 0. \tag{11}$$

The objective of this paper is to design and analyze a control law such that

$$\lim_{t\to\infty} E(q,\dot{q}) = E_{uu}, \quad \lim_{t\to\infty} \dot{x} = 0, \quad \lim_{t\to\infty} x = 0, \tag{12}$$

where $E_{uu} := \beta_1 + \beta_2$ is the energy of the cart-double pendulum system at the upright equilibrium point.

3. DESIGN OF SWING-UP CONTROLLER

In this section, we apply directly the procedures in Lozano et al. [2000], Kolesnichenko and Shiriaev [2002] to derive an energy based swing-up controller for the cart-double pendulum system.

Define the following Lyapunov function candidate:

$$V = \frac{1}{2}(E - E_{uu})^2 + \frac{1}{2}k_D\dot{x}^2 + \frac{1}{2}k_P x^2, \tag{13}$$

where k_D and k_P are positive constants.

Taking the time derivative of V along (1) and using $\dot{E} = \dot{q}^{\mathrm{T}} B f = \dot{x} f$, we obtain

$$\dot{V} = \dot{x}((E - E_{uu})f + k_D \ddot{x} + k_P x).$$

If we can choose f such that

$$(E - E_{uu})f + k_D \ddot{x} + k_P x = -k_V \dot{x}, \tag{14}$$

where k_V is a positive constant, then

$$\dot{V} = -k_V \dot{x}^2 \leq 0. \tag{15}$$

In what follows, we study when (14) is solvable with respect to f. To this end, we obtain $\ddot{x}$ from (1) as

$$\ddot{x} = B^{\mathrm{T}} \ddot{q} = B^{\mathrm{T}} M^{-1}(Bf - C\dot{q} - G). \tag{16}$$

Putting (16) into (14), we have

$$\lambda(q, \dot{q}) f = k_D B^T M^{-1}(C\dot{q} + G) - k_P x - k_V \dot{x}, \tag{17}$$

where

$$\lambda(q, \dot{q}) := E(q, \dot{q}) - E_{uu} + k_D B^T M^{-1} B. \tag{18}$$

If

$$\lambda(q, \dot{q}) \neq 0, \quad \text{for } \forall q,\ \forall \dot{q} \tag{19}$$

holds, then we can derive f from (17) as

$$f = \left(k_D B^T M^{-1}(C\dot{q} + G) - k_P x - k_V \dot{x}\right)/\lambda. \tag{20}$$

Since $M(\theta) > 0$, we obtain

$$B^{\mathrm{T}} M^{-1} B = \left(\alpha_1 \alpha_2 - \alpha_3^2 \cos(\theta_1 - \theta_2)\right)/\det(M) > 0.$$

Thus, (19) is equivalent to

$$k_D \neq \left(E_{uu} - E(q, \dot{q})\right)(B^{\mathrm{T}} M^{-1} B)^{-1}, \quad \text{for } \forall q,\ \forall \dot{q}. \tag{21}$$

Using $E(q, \dot{q}) \geq P(\theta)$, $E_{uu} \geq P(\theta)$, and $k_D > 0$, we can see that (19) holds if and only if

$$k_D > \max_{\theta_1, \theta_2} \left(E_{uu} - P(\theta)\right)(B^T M^{-1} B)^{-1}. \tag{22}$$

Now, we use LaSalle's theorem, see e.g., Khalil [2002], to analyze the motion of the closed-loop system. Suppose (22) holds. Under the controller (20), owing to $\dot{V} \leq 0$ in (15), V is bounded. Define

$$\Psi = \{(q, \dot{q}) \mid V(q, \dot{q}) \leq c\}, \tag{23}$$

where c is a positive constant. Then any closed-loop solution $(q(t), \dot{q}(t))$ starting in Ψ remains in Ψ for all $t \geq 0$. Let W be the largest invariant set in

$$\Gamma = \{(q, \dot{q}) \mid \dot{V} = 0\}. \tag{24}$$

Using LaSalle's theorem, we know that every $(q(t), \dot{q}(t))$ starting in Ψ approaches W as $t \to \infty$. Since $\dot{V} = 0$ holds identically in W, therefore, V and x are some constants in W. Moreover, using (13), we know that E is also a constant in W. Therefore,

$$\lim_{t \to \infty} E = E^*, \quad \lim_{t \to \infty} \dot{x} = 0, \quad \lim_{t \to \infty} x = x^*, \tag{25}$$

where E^* and x^* are some constants. In W, substituting $x \equiv x^*$ and $E \equiv E^*$ into (7) yields

$$\frac{1}{2}\alpha_1 \dot{\theta}_1^2 + \frac{1}{2}\alpha_2 \dot{\theta}_2^2 + \alpha_3 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2)$$
$$+ \beta_1 \cos\theta_1 + \beta_2 \cos\theta_2 \equiv E^*. \tag{26}$$

The largest invariant set W can be expressed as:

$$W = \{(q, \dot{q}) \mid (\theta, \dot{\theta}) \text{ satisfies (26) and } x \equiv x^*\}. \tag{27}$$

The obtained results in this section are summarized by the following proposition.

Proposition 1. Consider the cart-double pendulum system in (1). Suppose that $k_P > 0$, $k_D > 0$ and $k_V > 0$ hold. Then the control law (20) has no singular point for the cart-double pendulum system starting from any initial state if and only if k_D satisfies (22). In this case, (25) holds for some constants E^* and x^*, and every closed-loop solution $(q(t), \dot{q}(t))$ approaches the invariant set W defined in (27) as $t \to \infty$.

4. CONVERGENCE OF ENERGY OF THE CART-DOUBLE PENDULUM SYSTEM

Comparing our objective expressed in (12) and our obtained result in (25), we need to study the convergent value of the energy E^* and the displacement of the cart x^*.

In the invariant set W, substituting $E \equiv E^*$ and $x \equiv x^*$ into (14) yields

$$(E^* - E_{uu})f + k_P x^* \equiv 0. \tag{28}$$

We address two cases of $E^* = E_{uu}$ and $E^* \neq E_{uu}$, separately.

Case 1: $E^* = E_{uu}$

For this case, $x^* = 0$ follows directly from (28). From (26), this yields

$$\frac{1}{2}\alpha_1 \dot{\theta}_1^2 + \frac{1}{2}\alpha_2 \dot{\theta}_2^2 + \alpha_3 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2)$$
$$+ \beta_1 \cos\theta_1 + \beta_2 \cos\theta_2 = \beta_1 + \beta_2. \tag{29}$$

Thus, as $t \to \infty$, the closed-loop solution $(q(t), \dot{q}(t))$ approaches the following invariant set:

$$W_r = \{(q, \dot{q}) \mid (\theta, \dot{\theta}) \text{ satisfies (29) and } x \equiv 0\}. \tag{30}$$

Case 2: $E^* \neq E_{uu}$

For this case, we present the following proposition, which is a main result of this paper.

Proposition 2. Consider the cart-double pendulum system in (1). Suppose that k_D satisfies (22), $k_P > 0$ and $k_V > 0$ hold. If $E^* \neq E_{uu}$, then

1) *invariant set W in (26) contains only the following three equilibrium points:*
   Up-down equilibrium point: $(0, 0, \pi, 0, 0, 0)$,
   Down-up equilibrium point: $(0, \pi, 0, 0, 0, 0)$,
   Down-down equilibrium point: $(0, \pi, \pi, 0, 0, 0)$;
2) *these three equilibrium points are unstable.*

We only present the proof of Statement 1); due to page limitations, we omit the proof of Statement 2) which is similar to that of Theorem 4 in Xin and Kaneda [2005].

Proof. Using $E^* \neq E_{uu}$, it follows from (28) that f is a constant f^* satisfying

$$(E^* - E_{uu})f^* + k_P x^* \equiv 0. \tag{31}$$

In the invariant set W, by using $x \equiv x^*$ and defining

$$a = \frac{\beta_1}{\beta_2},\ \ b = \frac{\alpha_1}{\alpha_3},\ \ d = \frac{\beta_2}{\alpha_3},\ \ e = \frac{\alpha_2}{\alpha_3}, \tag{32}$$

we can rewrite (1) as

$$a\ddot{\theta}_1 \cos\theta_1 + \ddot{\theta}_2 \cos\theta_2 - a\dot{\theta}_1^2 \sin\theta_1 - \dot{\theta}_2^2 \sin\theta_2 \equiv \frac{f^* g}{\beta_2}, \tag{33}$$

$$b\ddot{\theta}_1 + \ddot{\theta}_2 \cos(\theta_1 - \theta_2) + \dot{\theta}_2^2 \sin(\theta_1 - \theta_2) - ad\sin\theta_1 \equiv 0, \tag{34}$$

$$\ddot{\theta}_1 \cos(\theta_1 - \theta_2) + e\ddot{\theta}_2 - \dot{\theta}_1^2 \sin(\theta_1 - \theta_2) - d\sin\theta_2 \equiv 0. \tag{35}$$

By exploiting (33)-(35), we carry out the proof via the following four steps:

Step 1: Show that both the constant force f^* acting on the cart and the displacement of the cart x^* are 0, i.e.,

$$f^* = 0, \quad x^* = 0. \tag{36}$$

Step 2: Show

$$a\dot{\theta}_1 \cos\theta_1 + \dot{\theta}_2 \cos\theta_2 \equiv 0 \tag{37}$$

and

$$a\sin\theta_1 + \sin\theta_2 \equiv 0. \tag{38}$$

This means that the COM of two-linked pendulums is on the vertical line.

Step 3: By using the properties of mechanical parameters shown in Lemma 1, we show the following relationship about the angular accelerations between two pendulums:

$$\ddot{\theta}_2 \equiv h\ddot{\theta}_1, \tag{39}$$

where

$$h = \frac{a(a-b)}{ae-1} = \frac{\beta_1(\alpha_3\beta_1 - \alpha_1\beta_2)}{\alpha_2\beta_1 - \alpha_3\beta_2} \geq 0. \tag{40}$$

Step 4: By contradiction, we prove

$$\dot{\theta}_1 \equiv 0, \text{ and } \dot{\theta}_2 \equiv 0. \tag{41}$$

On the contrary, if (41) does not hold, then

$$a = 1 \text{ and } h = 1 \tag{42}$$

must hold. We show a contradiction by proving that the both equations in (42) can not hold simultaneously for any double pendulum.

In what follows, we present some details of Steps 1 to 4. As to **Step 1**, integrating (33) with respect to time t yields

$$a\dot{\theta}_1 \cos\theta_1 + \dot{\theta}_2 \cos\theta_2 = \frac{f^* g}{\beta_2} t + \gamma_1, \tag{43}$$

where γ_1 is a constant. Since E^* is bounded, from in (26), we can see that $\dot{\theta}_1$ and $\dot{\theta}_2$ are bounded; this shows that the left-hand side of (43) is bounded. Thus, $f^* = 0$ must hold; otherwise, the absolute value of the right-hand side of (43), that is, $|f^* gt/\beta_2 + \gamma_1|$, will go to infinity as $t \to \infty$. From (31), we show $x^* = 0$.

As to **Step 2**, using $f^* = 0$ and integrating (43) with respect to time t gives

$$a\sin\theta_1 + \sin\theta_2 \equiv \gamma_1 t + \gamma_2. \tag{44}$$

Thus $\gamma_1 = 0$ must hold; otherwise, the absolute value of the right-hand side of (44) will go to infinity as $t \to \infty$, this contradicts the fact that the left-hand side of (44) is bounded for all t. This completes the proof of (37).

Next, using $\gamma_1 = 0$ and (44), we obtain

$$a\sin\theta_1 + \sin\theta_2 \equiv \gamma_2. \tag{45}$$

To show $\gamma_2 = 0$, we write the sum of (34) and (35) as

$$\frac{d}{dt}(b\dot{\theta}_1 + e\dot{\theta}_2 + (\dot{\theta}_1 + \dot{\theta}_2)\cos(\theta_1 - \theta_2)) = d\gamma_2. \tag{46}$$

Integrating (46) with respect to time t, we have

$$b\dot{\theta}_1 + e\dot{\theta}_2 + (\dot{\theta}_1 + \dot{\theta}_2)\cos(\theta_1 - \theta_2) \equiv d\gamma_2 t + \gamma_3. \tag{47}$$

From the boundedness of the left-hand side of the above equation, we can see $\gamma_2 = 0$. This together with (45) shows (38).

As to **Step 3**, putting $\gamma_2 = 0$ into (47) gives

$$b\dot{\theta}_1 + e\dot{\theta}_2 + (\dot{\theta}_1 + \dot{\theta}_2)\cos(\theta_1 - \theta_2) = \gamma_3. \tag{48}$$

For the simplicity of notations in what follows, we denote

$$S_1 := \sin\theta_1,\ S_2 := \sin\theta_2,\ C_1 := \cos\theta_1,\ C_2 := \cos\theta_2,$$
$$C_{12} := \cos(\theta_1 - \theta_2), \quad S_{12} := \sin(\theta_1 - \theta_2).$$

To simplify (48), by using (37) and (38), we have

$$\begin{aligned}
&(\dot{\theta}_1 + \dot{\theta}_2)\cos(\theta_1 - \theta_2) = (\dot{\theta}_1 + \dot{\theta}_2)(C_1C_2 + S_1S_2) \\
&= (\dot{\theta}_1 C_1 C_2 + \dot{\theta}_2 S_1 S_2) + (\dot{\theta}_1 S_1 S_2 + \dot{\theta}_2 C_1 C_2) \\
&= -(\dot{\theta}_2 C_2^2 + \dot{\theta}_2 S_2^2)/a - a(\dot{\theta}_1 S_1^2 + \dot{\theta}_1 C_1^2) \\
&= -\dot{\theta}_2/a - a\dot{\theta}_1.
\end{aligned} \tag{49}$$

This reduces (48) to

$$(ae-1)\dot{\theta}_2 - a(a-b)\dot{\theta}_1 = a\gamma_3. \tag{50}$$

Using Lemma 1, we have

$$ae - 1 = \frac{\alpha_2\beta_1\alpha_3\beta_2}{\alpha_3\beta_2} > 0, \quad a - b = \frac{\alpha_3\beta_1 - \alpha_1\beta_2}{\alpha_3\beta_2} \geq 0.$$

which yields

$$\dot{\theta}_2 \equiv h\dot{\theta}_1 + \gamma, \tag{51}$$

where

$$\gamma = \frac{a\gamma_3}{ae-1}. \tag{52}$$

Differentiating (51) yields (39).

As to **Step 4**, first, we show that (42) does not hold for any pendulum system. On the contrary, suppose that (42) holds. From h in (40), using $a = 1$ and $h = 1$ yields $b + e = 2$. However, this contradicts

$$b + e = \frac{\alpha_1}{\alpha_3} + \frac{\alpha_2}{\alpha_3} \geq 2\sqrt{\frac{\alpha_1\alpha_2}{\alpha_3^2}} > 2.$$

owing to the fact that $\alpha_1\alpha_2 > \alpha_3^2$ which can be checked directly using (6).

Second, to complete **Step 4**, we now only need to show by contradiction that if (41) does not holds, then (42) holds.

By using (37), we know that $\dot{\theta}_1 \equiv 0$ is equivalent to $\dot{\theta}_2 \equiv 0$; therefore, if (41) does not hold, then

$$\dot{\theta}_1 \not\equiv 0 \quad \text{and} \quad \dot{\theta}_2 \not\equiv 0. \tag{53}$$

We proceed the proof by treating γ in (51) by the case $\gamma = 0$ and the case $\gamma \neq 0$ separately.

Case 2.1 $\gamma = 0$ in (51)

In this case, from (37) and (53), we obtain

$$aC_1 \equiv -hC_2. \tag{54}$$

This with (38) yields

$$(h^2 - 1)C_2^2 + 1 - a^2 \equiv 0.$$

Owing to (53), we can see that C_2 ($\cos\theta_2$) is not a constant. This, with the fact that θ_2 is continuously differentiable, implies that the above equation has infinite number of solutions. Thus, $h^2 = 1$ and $a^2 = 1$. With $a > 0$ and $h \geq 0$, we know that (42) must hold.

Case 2.2 $\gamma \neq 0$ in (51)

To start with, by using (38) and (51), we eliminate $\ddot{\theta}_1$ and $\ddot{\theta}_2$ from the motion equations of two-linked pendulums (34) and (35), and we obtain the following new and key relation about the angular velocities between two pendulums:

$$(b + hC_{12})\dot{\theta}_1^2 + (eh + C_{12})\dot{\theta}_2^2 \equiv d(aC_1 + hC_2). \tag{55}$$

The detail derivation of (55) is omitted due to page limitations.

Next, with (37), (38), and (51), this equation helps us to eliminate $\dot{\theta}_1$ and $\dot{\theta}_2$, we obtain a 7th-order polynomial of C_1 as follows:

$$\sum_{j=0}^{7} \psi_j C_1^j \equiv 0, \tag{56}$$

where ψ_j $(0 \leq j \leq 7)$ are constants. The detail expression of each ψ_j is omitted due to page limitations. Owing to (53), we can see that C_1 ($\cos\theta_1$) is not a constant. This, with the fact that θ_1 is continuously differentiable, implies that the above equation has infinite number of solutions. Thus,

$$\psi_j = 0, \quad \text{for } 0 \leq j \leq 7. \tag{57}$$

We can show (57) holds if and only if (42) holds.

In conclusion, if (53) holds, then for both Case 2.1 of $\gamma = 0$ and Case 2.2 of $\gamma \neq 0$ (42) must hold. Since we have proved that (42) does not hold for any double pendulum, this raises a contradiction. Thus, (41) holds. From (34) and (35), we have

$$\sin\theta_1 \equiv 0, \quad \sin\theta_2 \equiv 0.$$

With $E^* \neq E_{uu}$, we obtain

$$(\theta_1, \theta_2) \equiv (0, \pi),\ (\theta_1, \theta_2) \equiv (\pi, 0),\ (\theta_1, \theta_2) \equiv (\pi, \pi).$$

This completes the proof of Statement 1). ■

We give the following proposition to summarize the obtained results for **Cases 1** and **2**.

Proposition 3. Consider the cart-double pendulum system in (1). Suppose that k_D satisfies (22), $k_P > 0$ and $k_V > 0$ hold. Under the controller (20), as $t \to \infty$ the closed-loop solution $(q(t), \dot{q}(t))$ approaches either the invariant set W_r defined in (30) or one of the three equilibrium points described in Proposition 2. Moreover, these three equilibrium points are unstable.

5. SIMULATION RESULTS

The validity of the developed theoretical results was verified via numerical simulation investigation for a cart-double pendulum system with following mechanical parameters $m_1 = 1.0$ [kg], $m_1 = 0.5$ [kg], $m_2 = 0.5$ [kg], $l_1 = 1$ [m], $l_2 = 1.0$ [m], $l_{c1} = 0.5$ [m], $l_{c2} = 0.5$ [m], $I_1 = m_1 l_{c1}^2/3$, $I_2 = m_2 l_{c2}^2/3$. We took the gravity acceleration $g = 9.81$ [m/s^2].

For an initial condition $q(0) = [\,1,\ 0,\ \pi/2,\ 0,\ \pi,\ 0.1\,]^{\mathrm{T}}$, we chose $k_D = 27.3$, $k_P = 47.2$, and $k_V = 65.0$. The simulation results are shown in Figs. 2 and (3), where we took θ_1 and θ_2 modulo 2π. We observe that V is non-increasing and approaches 0, and E and x approach E_{uu} and 0, respectively. The double pendulum moved close to $(\theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2) = (0, 0, 0, 0)$ at about $t = 9.8$ [s]. When the cart-double pendulum moved close to $q = [\,0,\ 0,\ 0,\ 0,\ 0,\ 0\,]^{\mathrm{T}}$, we switched the swing-up controller to a local stabilizing controller designed by the LQR method, to stabilize them about the vertical. See the simulation results of the swing-up and stabilizing control shown in Fig. 4.

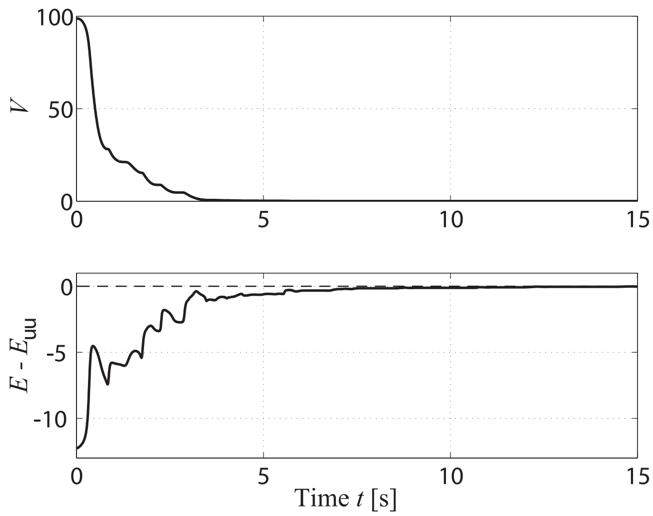

Fig. 2. Time responses of V and $E - E_{uu}$ under the swing-up control.

6. CONCLUSIONS

In this paper, we analyzed energy based swing-up control for a double pendulum on a cart. We presented an original analysis of the convergence of the energy of the cart-double

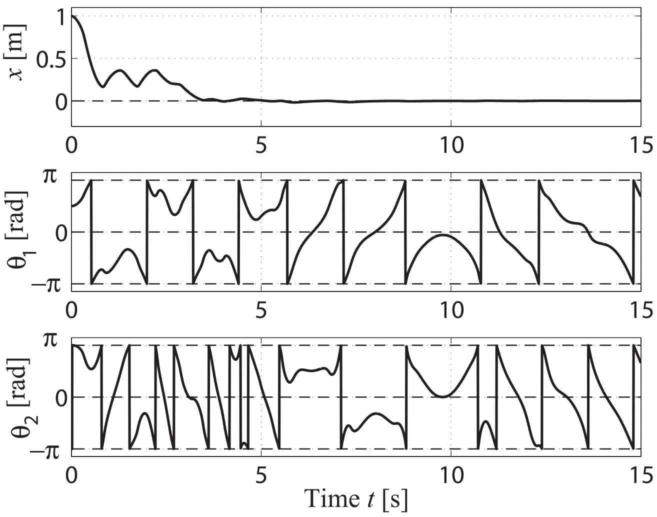

Fig. 3. Time responses of x, θ_1 and θ_2 under the swing-up control.

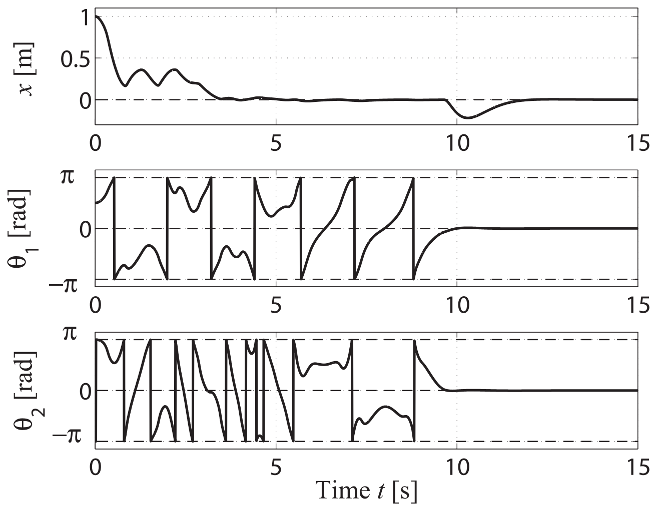

Fig. 4. Time responses of x, θ_1 and θ_2 under the swing-up and stabilizing control.

pendulum system. We showed that for all initial states of the cart-double pendulum system, if the convergent value of the energy is not equal to the energy at the upright (up-up) equilibrium point, then the cart-double pendulum remains at its up-down, down-up, and down-down equilibrium points. Moreover, we showed that these three equilibrium points are unstable. This shows that for almost all initial states of the cart-double pendulum system, as time approaches infinity, the energy of the cart-double pendulum system can be controlled to its energy at the upright equilibrium point. This paper provided insight into the energy based control approach to underactuated systems with underactuation degree greater than one.

REFERENCES

J.A. Acosta, R. Ortega, A. Astolfi, A. D. Mahindrakar, "Interconnection and damping assignment passivity-based control of mechanical systems with underactuation degree one," *IEEE Transactions on Automatic Control*, vol. 50, no. 12, pp. 1936–1955, 2005.

K. J. Åström, K. Furuta, M. Iwashiro and T. Hoshino, "Energy based strategies for swinging up a double pendulum," *Proceedings of the 14th World Congress of IFAC*, 1999.

K. J. Åström and K. Furuta, "Swinging up a pendulum by energy control," *Automatica*, vol. 36, no. 2, pp. 287–295, 2000.

C. C. Chung and J. Hauser, "Nonlinear control of a swinging pendulum," *Automatica*, vol. 31, no. 6, pp. 851–862, 1995.

I. Fantoni, R. Lozano, and M. W. Spong, "Energy based control of the Pendubot," *IEEE Transactions on Automatic Control*, vol. 45, no. 4, pp. 725–729, 2000.

J. W. Grizzle, C. H. Moog, and C. Chevallereau, "Nonlinear control of mechanical systems with an unactuated cyclic variable," *IEEE Transactions on Automatic Control*, vol. 50, no. 5, pp. 559–575, 2005.

H. K. Khalil, *Nonlinear Systems* (3rd ed.), Prentice-Hall, Englewood Cliffs, NJ, 2002.

O. Kolesnichenko and A. S. Shiriaev, "Partial stabilization of underactuated Euler-Lagrange systems via a class of feedback transformations," *Systems & Control Letters*, vol. 45, no. 2, pp. 121–132, 2002.

Z. Lin, A. Saberi, M. Gutmann and Y. A. Shamash, "Linear controller for an inverted pendulum having restricted travel: a high-and-low approach," *Automatica*, vol. 32, no. 6, pp. 933–937, 1996.

R. Lozano, I. Fantoni and D. J. Block, "Stabilization of the inverted pendulum around its homoclinic orbit," *Systems & Control Letters*, vol. 40, no. 3, pp. 197–204, 2000.

A. Shiriaev, A. Pogromsky, H. Ludvigsen and O. Egeland, "On global properties of passivity-based control of an inverted pendulum," *International Journal of Robust Nonlinear Control*, vol. 10, pp. 283–300, 2000.

M. W. Spong, "Energy based control of a class of underactuated mechanical systems," *Proceedings of 1996 IFAC World Congress*, vol. F, pp. 431–435, 1996.

Q. Wei, W. P. Dayawansa and W. S. Levine, "Nonlinear controller for an inverted pendulum having restricted travel," *Automatica*, vol. 31, no. 6, pp. 841–850, 1995.

X. Xin and M. Kaneda, "Analysis of the energy-based control for swinging up two pendulums," *IEEE Transactions on Automatic Control*, vol. 50, no. 5, pp. 679–684, 2005.

X. Xin and M. Kaneda, "Analysis of the energy based swing-up control of the Acrobot," *International Journal of Robust and Nonlinear Control*, vol. 17, pp. 1503–1524, 2007.

M. Yamakita, M. Iwashiro, Y. Sugahara and K. Furuta, "Robust swing up control double pendulum," *Proceedings of American Control Conference*, pp. 290–296, 1995.

W. Zhong and H. Röck: "Energy and passivity based control of the double inverted pendulum on a cart," *Proceedings of the 2001 IEEE International Conference on Control Applications*, pp. 896–900, 2001.