Proceedings of the 17th World Congress
The International Federation of Automatic Control
Seoul, Korea, July 6-11, 2008

Stabilizing reduced order model predictive control for constrained linear systems

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Abstract: This paper considers a stabilizing reduced order model predictive control for constrained linear discrete-time systems. By employing a system decomposition on the input-output function space, a reduced order model predictive control law, which guarantees closed-loop stability and feasibility, is obtained from a low dimensional on-line optimization problem. Numerical examples are provided to illustrate the proposed method.

Keywords: Model predictive control; Constrained systems; Stability; LQ control; Quadratic programming.

1. INTRODUCTION

Model predictive control (MPC) has become a standard control strategy for multivariable constrained systems (Maciejowski [2002], Morari and Lee [1999], Kwon [2005], Richalet et al. [1978]). In MPC, future behavior of a plant is predicted by employing a model of the plant and an optimal control sequence over a finite prediction horizon is computed based on an optimization problem. At each sampling time, the computation is repeated based on a new measurement and a shifted horizon and only the first control of an optimal control sequence is applied to the plant. Although this feature of MPC allows handling system constraints in a systematic manner, the requirement of the on-line optimization limits the applicable area of MPC especially for large-scale systems or plants with sufficiently large prediction horizon. Recently, design methods for reducing on-line computational complexity have been studied in some literature (Bemporad et al. [2002a,b], Rojas et al. [2004], Kojima and Morari [2004], Cagienard et al. [2007], Hara and Kojima [2007a,b]). Explicit solutions to constrained linear MPC, which are computed off-line, have been developed by multi-parametric programming (Bemporad et al. [2002a,b]). In Rojas et al. [2004], Kojima and Morari [2004], MPC methods based on a simplified predictive model have been reported by singular value decomposition approach for both continuous and discrete-time constrained systems. More recently a reduced order model predictive control has been derived based on a system decomposition approach (Hara and Kojima [2007a,b]). Although this approach enables us to obtain an MPC law via a low-dimensional on-line optimization problem, stability of the resulting closed-loop system is not necessarily guaranteed.

The advantage of the reduced order MPC is that the MPC law, especially with long prediction horizon, can be obtained from a reduced order on-line optimization problem, while the traditional MPC formulation requires solving the computationally expensive optimization problem on-line. In our previous work (Hara and Kojima [2007a]), we developed the system decomposition method for discrete-time linear systems, and based on this method, a reduced order MPC law is derived. Further, by exploring unforced dynamics of a plant, an alternative reduced order MPC law, which is improved in terms of control performance and feasibility, is obtained (Hara and Kojima [2007b]). In this paper, by employing the system decomposition method, we derive an alternative stabilizing reduced order MPC law such that feasibility and stability of the closed-loop system are guaranteed. Furthermore, we clarify a set of initial states for which the stability of the reduced order MPC is ensured.

The paper is organized as follows. In Section 2, we provide a fundamental idea of the system decomposition and formulate the problem. In Section 3, we introduce the system decomposition of discrete-time linear systems, which clarifies dominant input-output relations of the plant. In Section 4, we present a stabilizing reduced order model predictive control law, which guarantees feasibility and stability of the control law. In Section 5, numerical examples are provided to illustrate the obtained reduced order MPC law.

2. PROBLEM FORMULATION AND FUNDAMENTAL IDEA

Consider a discrete-time linear time-invariant system with constraints

$$\Sigma \;:\; x_{k+1} = Ax_k + Bu_k \tag{1}$$
$$u_k \in \mathbb{U} := \{u \in \mathbb{R}^m \mid G_U u \le E_U\} \tag{2}$$
$$x_k \in \mathbb{X} := \{x \in \mathbb{R}^n \mid G_X x \le E_X\} \tag{3}$$
$$G_U \in \mathbb{R}^{\bar{m}\times m}, E_U \in \mathbb{R}^{\bar{m}}, G_X \in \mathbb{R}^{\bar{n}\times n}, E_X \in \mathbb{R}^{\bar{n}}$$

where $x_k \in \mathbb{R}^n$, $u_k \in \mathbb{R}^m$ are the state and input respectively. We make the following assumptions for the system Σ:

(A1) A is nonsingular.
(A2) (A, B) is stabilizable and B has column fullrank.

(A3) The sets $\mathbb{U}$, $\mathbb{X}$ are bounded and $0 \in \mathrm{int}(\mathbb{U})$, $0 \in \mathrm{int}(\mathbb{X})$.

Define the constrained finite-time LQ control problem

$$\mathcal{P}_{LQ}: \min_{u_0,\cdots,u_{N-1}} J(x_0, u_0, \cdots, u_{N-1}) \tag{4}$$

$$\text{s.t. } u_k \in \mathbb{U},\ x_k \in \mathbb{X},\ (1),\ k = 0, 1, \cdots, N-1 \tag{5}$$

$$J(x_0, u_0, \cdots, u_{N-1}) := x_N^{\mathrm{T}} P x_N + \sum_{k=0}^{N-1} \left\{ x_k^{\mathrm{T}} Q x_k + u_k^{\mathrm{T}} R u_k \right\},\ Q > 0, R > 0 \tag{6}$$

where N is the time horizon and $P > 0$ is the stabilizing solution to the Riccati equation

$$P = A^{\mathrm{T}} P A + Q - A^{\mathrm{T}} P B (R + B^{\mathrm{T}} P B)^{-1} B^{\mathrm{T}} P A. \tag{7}$$

The control problem $\mathcal{P}_{LQ}$ can be reformulated as a quadratic programming problem (QP-problem). In the traditional MPC, a control law is obtained by solving the QP-problem at each sample time. However, the computational complexity increases as the prediction horizon N grows since the dominating factor of the complexity of the optimization problem depends on the number of decision variables. In order to attack the constrained LQ control problem, we introduce the system decomposition of finite-horizon discrete-time systems (Hara and Kojima [2007a]). This approach is based on the following fundamental idea. When no constraints are imposed on the system, the unconstrained LQ control and state response are given by

$$u_k = K_{LQ} A_c^k x_0, \tag{8}$$

$$x_k = A_c^k x_0 \tag{9}$$

$$K_{LQ} := -(R + B^{\mathrm{T}} P B)^{-1} B^{\mathrm{T}} P A, \tag{10}$$

$$A_c := A + B K_{LQ}. \tag{11}$$

Let us now decompose the initial state x_0 into

$$x_0 = \sum_{i=1}^{n} \alpha_i \cdot w_i^0,\ \alpha_i \in \mathbb{R} \tag{12}$$

where $\{w_i^0 \in \mathbb{R}^n : i = 1, 2, \cdots, n\}$ is a basis of $\mathbb{R}^n$, the LQ control (8) and the response (9) are equivalently written as follows:

$$u_k = \sum_{i=1}^{n} \alpha_i \cdot w_{i\,k}^1,\ k = 0, \cdots, N-1 \tag{13}$$

$$x_k = \sum_{i=1}^{n} \alpha_i \cdot w_{i\,k}^2,\ k = 0, \cdots, N \tag{14}$$

$$w_i^1 := \begin{bmatrix} w_{i\,0}^1 \\ w_{i\,1}^1 \\ \vdots \\ w_{i\,N-1}^1 \end{bmatrix},\ w_i^2 := \begin{bmatrix} w_{i\,0}^2 \\ w_{i\,1}^2 \\ \vdots \\ w_{i\,N}^2 \end{bmatrix},\ \begin{array}{l} w_{i\,k}^1 := K_{LQ} A_c^k w_i^0 \\ w_{i\,k}^2 := A_c^k w_i^0. \end{array}$$

The decomposition (12),(13),(14) characterizes the system behavior of the unconstrained LQ control by the triplets $W_i := (w_i^0, w_i^1, w_i^2)(i = 1, 2, \cdots, n)$, which are denoting the initial states(w_i^0), inputs(w_i^1), and state responses(w_i^2). This decomposition of the LQ control indicates that by introducing additional s-triplet $W_i := (w_i^0, w_i^1, w_i^2)(i = n+1, n+2, \cdots, n+s = N_s)$ which have dominant influence on the system response, it enables us to deal with the constrained LQ problem via a lower dimensional optimization problem. Based on the idea observed here, reduced order MPC laws are derived by employing the system decomposition method Hara and Kojima [2007a,b].

Although the reduced order MPC law can be obtained from the lower dimensional optimization problem, there is no guarantee that an optimization problem for the reduced order MPC law remains feasible at all time steps and the resulting closed-loop system is stabilized. In the sequel, we introduce the system decomposition method and derive a reduced order model predictive control that guarantees both feasibility and stability of the closed-loop system.

3. SYSTEM DECOMPOSITION OF DISCRETE-TIME LINEAR SYSTEMS

Let us begin with the system decomposition method for discrete-time systems (Hara and Kojima [2007a]).

Define spaces $\mathcal{U} := \mathbb{R}^{m \cdot N}, \mathcal{Z} := \mathbb{R}^{n \cdot (N+1)}$ and denote input-output responses as follows:

$$\hat{u} := \begin{bmatrix} R^{\frac{1}{2}} u_0 \\ R^{\frac{1}{2}} u_1 \\ \vdots \\ R^{\frac{1}{2}} u_{N-1} \end{bmatrix} \in \mathcal{U},\ \hat{z} := \begin{bmatrix} z^0 \\ z^1 \end{bmatrix} \in \mathcal{Z}$$

$$\begin{array}{c} z^0 = P^{\frac{1}{2}} x_N \\ z_k^1 = Q^{\frac{1}{2}} x_k \\ (k = 0, 1, \cdots, N-1) \end{array},\ z^1 = \begin{bmatrix} z_0^1 \\ \vdots \\ z_{N-1}^1 \end{bmatrix} \tag{15}$$

The input-output relation is described by

$$\hat{z} = F_a x_0 + G \hat{u} \tag{16}$$

$$F_a := \begin{bmatrix} P^{\frac{1}{2}} A^N \\ Q^{\frac{1}{2}} I \\ Q^{\frac{1}{2}} A \\ \vdots \\ Q^{\frac{1}{2}} A^{N-1} \end{bmatrix} \in \mathbb{R}^{n \cdot (N+1) \times n}, \tag{17}$$

$$G := \begin{bmatrix} P^{\frac{1}{2}} A^{N-1} B R^{-\frac{1}{2}} & P^{\frac{1}{2}} A^{N-2} B R^{-\frac{1}{2}} & \cdots & P^{\frac{1}{2}} B R^{-\frac{1}{2}} \\ 0 & 0 & \cdots & 0 \\ Q^{\frac{1}{2}} B R^{-\frac{1}{2}} & 0 & \cdots & 0 \\ Q^{\frac{1}{2}} A B R^{-\frac{1}{2}} & Q^{\frac{1}{2}} B R^{-\frac{1}{2}} & & 0 \\ \vdots & \vdots & & \vdots \\ Q^{\frac{1}{2}} A^{N-2} B R^{-\frac{1}{2}} & Q^{\frac{1}{2}} A^{N-3} B R^{-\frac{1}{2}} & \cdots & 0 \end{bmatrix} \in \mathbb{R}^{n \cdot (N+1) \times m \cdot N} \tag{18}$$

and the cost function (6) is written as follows:

$$J = \|\hat{u}\|_{\mathcal{U}}^2 + \|\hat{z}\|_{\mathcal{Z}}^2 = x_0^{\mathrm{T}} F_a^{\mathrm{T}} (I + G G^{\mathrm{T}})^{-1} F_a x_0 + \begin{bmatrix} x_0 \\ \hat{u} \end{bmatrix}^{\mathrm{T}} \Delta \begin{bmatrix} x_0 \\ \hat{u} \end{bmatrix} \tag{19}$$

$$\Delta := \begin{bmatrix} F_a & G \\ 0 & I \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} G \\ I \end{bmatrix} (I + G^{\mathrm{T}} G)^{-1} \begin{bmatrix} G \\ I \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} F_a & G \\ 0 & I \end{bmatrix} \in \mathbb{R}^{(n+m \cdot N) \times (n+m \cdot N)} \tag{20}$$

The eigenvalues and eigenvectors of the matrix Δ characterize the dominant pairs of the initial state and control signal. Next lemma provides a calculation method of the eigenvalues and eigenvectors by fixed size of matrix.

Lemma 1. (Hara and Kojima [2007a]) The eigenvalues of Δ are given by the roots of the following transcendental equation.

$$\det\left\{[-P,\ I]H_a(\lambda)^N \begin{bmatrix} I \\ \lambda I + P \end{bmatrix}\right\} = 0 \quad (21)$$

$$H_a(\lambda) := \begin{bmatrix} A + \frac{1}{1-\lambda}BR^{-1}B^{\mathrm{T}}A^{-\mathrm{T}}Q & -\frac{1}{1-\lambda}BR^{-1}B^{\mathrm{T}}A^{-\mathrm{T}} \\ -A^{-\mathrm{T}}Q & A^{-\mathrm{T}} \end{bmatrix} \quad (22)$$

The eigenvector $w_i = (w_i^0, w_i^1) \in \mathbb{R}^n \times \mathcal{U}$ corresponding to the eigenvalue λ_i is constructively given as follows:

$$w_i^0 \neq 0 : [-P,\ I]H_a(\lambda_i)^N \begin{bmatrix} I \\ \lambda_i I + P \end{bmatrix} w_i^0 = 0 \quad (23)$$

$$w_i^1 = \begin{bmatrix} w_{i0}^1 \\ w_{i1}^1 \\ \vdots \\ w_{iN-1}^1 \end{bmatrix}, \quad (24)$$

$$w_{ik}^1 = -\frac{1}{1-\lambda_i}R^{-\frac{1}{2}}B^{\mathrm{T}}[0,\ I]H_a(\lambda_i)^{k+1} \begin{bmatrix} I \\ \lambda_i I + P \end{bmatrix} w_i^0 \quad (25)$$
$$k = 0, 1, \cdots, N-1$$

Further, when $\hat{u} = w_i^1$ is applied to the system with the initial condition $x_0 = w_i^0$, the system response x_k is given by

$$x_k = w_{ik}^2 = [I,\ 0]H_a(\lambda_i)^k \begin{bmatrix} I \\ \lambda_i I + P \end{bmatrix} w_i^0, \quad k = 0, 1, \cdots, N. \quad (26)$$

□

Remark 1. In terms of the eigenvalue problem of Δ, the following properties are verified for the system Σ.

(a) The matrix Δ has eigenvalues at zero and the corresponding eigenvector yields the LQ control:

$$u_k = R^{-\frac{1}{2}}w_k^1 = K_{LQ}A_c^k w^0, w^0 \neq 0 \quad (k = 0, 1, \cdots, N-1). \quad (27)$$

Thus the eigenvalue problem of Δ characterizes a set of control sequences which naturally includes the LQ control.

(b) The eigenvectors $\{w_i\}$ of Δ are orthogonal in $\mathbb{R}^n \times \mathcal{U}$.

□

In Lemma 1, the assumption (A1) is necessary since the formula requires the symplectic matrix H_a, which involves the inverse of the state transition matrix A.

In the sequel, we normalize the eigenvectors as $\|(w_i^0, w_i^1)\|_{\mathbb{R}^n\times\mathcal{U}} = 1$ $(i = 1, 2, \cdots)$ and define w_i^2 by the corresponding state responses.

4. STABILIZING REDUCED ORDER MODEL PREDICTIVE CONTROL

In the previous section, the system decomposition of discrete-time linear systems is presented, which characterizes the dominant pairs of the initial state and control sequence. By employing the system decomposition, we derive a reduced order model predictive control law which guarantees the closed-loop stability.

Let $\lambda_i(\lambda_1 = \cdots = \lambda_n = 0, \lambda_{n+1} \geq \cdots \geq \lambda_{n+m\cdot N})$, $w_i = (w_i^0, w_i^1)(i = 1, 2, \cdots, n + m \cdot N)$ be the eigenvalues and eigenvectors of the matrix Δ. We denote the state and input at time t by x_t, u_t and the predicted state and input at time $t+k$ by $x_{k|t}$, $u_{k|t}$, which are computed based on the measurement $x_t(= x_{0|t})$ at time t. When we apply a control sequence:

$$u_{k|t} = \sum_{i=1}^{N_s} \alpha_i R^{-\frac{1}{2}} w_{ik}^1 + \alpha_{N_s+1} \cdot \bar{u}_k \ (k = 0, 1, \cdots, N-1), \quad (N_s = n + s) \quad (28)$$

the following system response is generated (Lemma 1).

$$x_{k|t} = \sum_{i=1}^{N_s} \alpha_i \cdot w_{ik}^2 + A^k\tilde{x}_0 + \alpha_{N_s+1} \cdot \bar{x}_k \quad (29)$$

$$\bar{x}_k := \sum_{i=0}^{k-1} A^i B\bar{u}_{k-1-i},\ \bar{x}_0 = 0,\ k = 1, 2, \cdots, N \quad (30)$$

$$\tilde{x}_0 := x_{0|t} - \sum_{i=1}^{N_s} \alpha_i \cdot w_i^0 \quad (31)$$

Here N_s is a design parameter, which represents the number of basis vectors in the input space. In the control (28), $\bar{u}_k \in \mathbb{R}^m (k = 0, 1, \cdots, N-1)$ denotes an auxiliary input sequence to be designed and is introduced to guarantee stability of the resulting closed-loop system. The cost function (6) corresponding to the input (28) is expressed as follows:

$$J\left(v, x_0, \bar{U}\right) = v^{\mathrm{T}}Hv + 2v^{\mathrm{T}}Fx_0 + x_0^{\mathrm{T}}Sx_0 \quad (32)$$
$$v := [\alpha_1, \alpha_2, \cdots, \alpha_{N_s+1}]^{\mathrm{T}} \quad (33)$$
$$\bar{U} = \{\bar{u}_0, \bar{u}_1, \cdots, \bar{u}_{N-1}\} \quad (34)$$

$$\begin{aligned} H :=& \tilde{I}^{\mathrm{T}}\Big\{\Lambda + W_0^{\mathrm{T}}(P + A^{N^{\mathrm{T}}}PA^N + \sum_{k=0}^{N-1} A^{k^{\mathrm{T}}}QA^k)W_0 \\ &- W_0^{\mathrm{T}}(A^{N^{\mathrm{T}}}PW_N^2 + \sum_{k=0}^{N-1} A^{k^{\mathrm{T}}}QW_k^2) \\ &- (A^{N^{\mathrm{T}}}PW_N^2 + \sum_{k=0}^{N-1} A^{k^{\mathrm{T}}}QW_k^2)^{\mathrm{T}}W_0\Big\}\tilde{I} \\ &+ \bar{I}\Big\{\bar{x}_N^{\mathrm{T}}P\bar{x}_N + \sum_{k=0}^{N-1}(\bar{x}_k^{\mathrm{T}}Q\bar{x}_k + \bar{u}_k^{\mathrm{T}}R\bar{u}_k)\Big\}\bar{I}^{\mathrm{T}} \\ &+ \bar{I}\Big\{\bar{x}_N^{\mathrm{T}}P(W_N^2 + A^NW_0) \\ &+ \sum_{k=0}^{N-1} \bar{x}_k^{\mathrm{T}}Q(W_k^2 - A^kW_0) + \bar{u}_k^{\mathrm{T}}R^{-\frac{1}{2}}W_k^1\Big\}\tilde{I} \\ &+ \tilde{I}^{\mathrm{T}}\Big\{\bar{x}_N^{\mathrm{T}}P(W_N^2 + A^NW_0) \\ &+ \sum_{k=0}^{N-1} \bar{x}_k^{\mathrm{T}}Q(W_k^2 - A^kW_0) + \bar{u}_k^{\mathrm{T}}R^{-\frac{1}{2}}W_k^1\Big\}^{\mathrm{T}}\bar{I}^{\mathrm{T}} \end{aligned}$$

$$\begin{aligned} F :=& (\tilde{I}^{\mathrm{T}}W_N^{2\,T} + \bar{I}\bar{x}_N^{\mathrm{T}})PA^N + \sum_{k=0}^{N-1}(\tilde{I}^{\mathrm{T}}W_k^{2\,\mathrm{T}} + \bar{I}\bar{x}_k^{\mathrm{T}})QA^k \\ &- \tilde{I}^{\mathrm{T}}W_0^{\mathrm{T}}S \end{aligned} \quad (35)$$

$$S := A^{N^{\mathrm{T}}} P A^N + \sum_{k=0}^{N-1} A^{k^{\mathrm{T}}} Q A^k \quad (36)$$

$$\Lambda := \mathrm{diag}(\lambda_1, \lambda_2, \cdots, \lambda_{N_s}) \in \mathbb{R}^{N_s \times N_s}$$

$$W_0 := [w_1^0, w_2^0, \cdots, w_{N_s}^0] \in \mathbb{R}^{n \times N_s}$$

$$W_k^1 := [w_{1k}^1, w_{2k}^1, \cdots, w_{N_s k}^1] \in \mathbb{R}^{m \times N_s}, k = 0, \cdots, N$$

$$W_k^2 := [w_{1k}^2, w_{2k}^2, \cdots, w_{N_s k}^2] \in \mathbb{R}^{n \times N_s}, k = 0, \cdots, N$$

$$\tilde{I} := [\mathbb{I}_{N_s}, 0] \in \mathbb{R}^{N_s \times (N_s+1)}$$

$$\bar{I} := [0, \cdots, 0, 1]^{\mathrm{T}} \in \mathbb{R}^{N_s+1} \quad (37)$$

We will solve the following optimization problem $\mathcal{P}$, which is obtained by reformulating the constrained LQ control problem $\mathcal{P}_{LQ}$ (4), (5) with additional conditions for closed-loop stability.

$$\mathcal{P} \ : V^*(x_t) = \min_{v \in \mathbb{R}^{N_s+1}} J(v, x_t, \bar{U}_t) \quad (38)$$

$$\text{s.t.} \sum_{i=1}^{N_s} \alpha_i R^{-\frac{1}{2}} w_{ik}^1 + \alpha_{N_s+1} \cdot \bar{u}_k \in \mathbb{U} \quad (39)$$

$$\sum_{i=1}^{N_s} \alpha_i \cdot w_{ik}^2 + A^k \tilde{x}_0 + \alpha_{N_s+1} \cdot \bar{x}_k \in \mathbb{X} \quad (40)$$

$$k = 0, 1, \cdots, N-1$$

$$\sum_{i=1}^{N_s} \alpha_i \cdot w_{iN}^2 + A^N \tilde{x}_0 + \alpha_{N_s+1} \cdot \bar{x}_N \in \mathbb{X}_f \quad (41)$$

$$\alpha_{N_s+1} = 0 (t = 0) \quad (42)$$

$$\bar{U}_0 := \{0, 0, \cdots, 0\} \, (t = 0) \quad (43)$$

$$\bar{U}_t := \left\{ u_{1|t-1}^*, u_{2|t-1}^*, \cdots, u_{N-1|t-1}^*, K_{LQ} x_{N|t-1}^* \right\} \quad (t \geq 1) \quad (44)$$

Here $u_{k|t}^*$, $x_{k|t}^*$ denote the optimal controls and states computed by (28),(29) with an optimal coefficient vector obtained from the optimization problem $\mathcal{P}$ at time t. The conditions (41)–(44) are introduced to guarantee recursive feasibility of the optimization problem and stability of the closed-loop system. The condition (41) represents an invariant terminal set, which is the maximal output admissible set $O_\infty(\{x \in \mathbb{R}^n : K_{LQ} A_c^k x \in \mathbb{U}, \ A_c^k x \in \mathbb{X}, \forall k \in \mathbb{Z}^+\})$(Gilbert and Tan [1991]) expressed as

$$\mathbb{X}_f := \{x \in \mathbb{R}^n | Mx \leq m\}, M \in \mathbb{R}^{\tilde{n} \times n}, m \in \mathbb{R}^{\tilde{n}}. \quad (45)$$

The control sequence defined in (44) implies that the additional input sequence $\bar{u}_k$ introduced in (28) varies at each sampling time and plays a role in providing a feasible solution for the optimization problem $\mathcal{P}$. Note that the matrices $H \geq 0$ and F in the objective function are time-varying as their elements contain $\bar{u}_k, \bar{x}_k$. The reduced order model predictive control law is obtained by solving this optimization problem with $N_s < m \cdot N$ at each sample time $t = 0, 1, \cdots$. Next theorem guarantees the recursive feasibility of the problem $\mathcal{P}$ and the closed-loop stability of the reduced order model predictive control law.

Theorem 2. Assume that the optimization problem $\mathcal{P}$ is feasible at initial time $t = 0$. Then $\mathcal{P}$ is feasible for all times $t = 1, 2, \cdots$ and the control law $u_t = u_{0|t}^*$ asymptotically stabilizes the system (1).

Proof. (Feasibility) The optimization problem $\mathcal{P}$ is feasible at $t = 0$ by the assumption. At time $t = 1$, we consider the input sequence $\bar{U}_1 = \{u_{1|0}^*, \cdots, u_{N-1|0}^*, K_{LQ} x_{N|0}^*\}$, which consists of the abbreviated input sequence obtained at $t = 0$ and the LQ control. Since the first $N-1$ elements of $\bar{U}_1$ steers $x_1 = x_{1|0}$ to $x_{N|0}^* \in \mathbb{X}_f$ while satisfying the input and state constraints and the last element $K_{LQ} x_{N|0}^*$ keeps the state in $\mathbb{X}_f$, $\bar{U}_1$ is a feasible input sequence at $t = 1$. This input sequence is realizable by the optimization problem $\mathcal{P}$ if the coefficient vector $v = [0, 0, \cdots, 0, 1]^{\mathrm{T}}$ is chosen as the solution to the problem $\mathcal{P}$. Thus the optimization problem has at least one feasible solution at $t = 1$. Feasibility after $t \geq 2$ follows from the same argument.

(Stability) The proof is similarly obtained from Lyapunov stability arguments (Mayne et al. [2000]). Let us consider a candidate Lyapunov function $V^*(x)$. We will show that $\Delta V := V^*(x_{t+1}) - V^*(x_t)$ is negative for $\forall x_t \neq 0$. At time $t + 1$, let us assume that the coefficient vector $v = [0, \cdots, 0, 1]^{\mathrm{T}}$ is employed for the solution to the problem $\mathcal{P}$, which implies that $\bar{U}_{t+1}$ is chosen as a predicted input sequence. Then this sequence is feasible and we have

$$\begin{aligned} V(x_{t+1}) - V^*(x_t) &= x_{N|t+1}^{\mathrm{T}} P x_{N|t+1} + x_{N-1|t+1}^{\mathrm{T}} Q x_{N-1|t+1} \\ &- u_{0|t}^* R u_{0|t}^* + (K_{LQ} x_{N|t}^*)^{\mathrm{T}} R (K_{LQ} x_{N|t}^*) \\ &- x_{N|t}^{*\ \mathrm{T}} P x_{N|t}^* - x_{0|t}^{*\ \mathrm{T}} Q x_{0|t}^* \end{aligned} \quad (46)$$

$$V(x_{t+1}) := J([0, \cdots, 0, 1]^{\mathrm{T}}, x_{t+1}, \bar{U}_{t+1}) \quad (47)$$

where $x_{\cdot|t+1}$ denotes the predicted state to the input sequence $\bar{U}_{t+1}$. Since the equality

$$\begin{aligned} &x_{N|t+1}^{\mathrm{T}} P x_{N|t+1} - x_{N|t}^{*\ \mathrm{T}} P x_{N|t}^* \\ &= -(K_{LQ} x_{N|t}^*)^{\mathrm{T}} R (K_{LQ} x_{N|t}^*) - x_{N-1|t+1}^{\mathrm{T}} Q x_{N-1|t+1} \end{aligned} \quad (48)$$

holds, (46) yields the following inequality.

$$V(x_{t+1}) - V^*(x_t) = -u_{0|t}^{*\ \mathrm{T}} R u_{0|t}^* - x_{0|t}^{*\ \mathrm{T}} Q x_{0|t}^* < 0 \quad (49)$$

Finally, the relation

$$\begin{aligned} \Delta V &= V^*(x_{t+1}) - V^*(x_t) \\ &\leq V(x_{t+1}) - V^*(x_t) < 0 (x_t \neq 0) \end{aligned} \quad (50)$$

is obtained from the fact that the cost $V(x_{t+1})$(not necessarily optimal) is an upper bound of the optimal cost $V^*(x_{t+1})$. Thus V^* is a Lyapunov function of the system (1) and the reduced order MPC law $u_t = u_{0|t}^*$ asymptotically stabilizes the system. □

Remark 2. For the traditional MPC based on the control problem $\mathcal{P}_{LQ}$ (4),(5), imposing the terminal constraint set $\mathbb{X}_f$ is sufficient to ensure the closed-loop stability and feasibility (Mayne et al. [2000]). If the terminal constraint is imposed for the reduced order MPC (Hara and Kojima [2007a,b]) without the auxiliary input sequence, the similar Lyapunov stability argument is not applicable as we cannot ensure that the value function becomes Lyapunov function. In the case of the reduced order MPC, in addition to the terminal set constraint, the auxiliary input sequence $\bar{u}_k$ in (28), which provides a feasible solution for the optimization problem, is essential to guarantee the stability and feasibility of the reduced order MPC law. This difference arises from the fact that the reduced order MPC constructs the input obtained from the decomposed input space whereas the standard MPC constructs the input from the input moves over the time-discretized prediction horizon. □

Next theorem provides a set of initial conditions which ensures the stability of the reduced order MPC.

Theorem 3. Define a polytope in $\mathbb{R}^{n+N_s}$ as follows:

$$\mathbb{Y} = \left\{ y \in \mathbb{R}^{n+N_s} \mid \begin{bmatrix} C_U \\ C_X \\ C_{X_f} \end{bmatrix} y \leq \begin{bmatrix} d_U \\ d_X \\ d_{X_f} \end{bmatrix} \right\}, \tag{51}$$

$$C_U := \begin{bmatrix} \mathbf{0}_{\bar{m}\times n} & G_U R^{-\frac{1}{2}} W_0^1 \\ \mathbf{0}_{\bar{m}\times n} & G_U R^{-\frac{1}{2}} W_1^1 \\ \vdots & \vdots \\ \mathbf{0}_{\bar{m}\times n} & G_U R^{-\frac{1}{2}} W_{N-1}^1 \end{bmatrix},$$

$$C_X := \begin{bmatrix} G_X & G_X(W_0^2 - W_0) \\ G_X A & G_X(W_1^2 - AW_0) \\ \vdots & \vdots \\ G_X A^{N-1} & G_X(W_{N-1}^2 - A^{N-1}W_0) \end{bmatrix},$$

$$C_{X_f} := \begin{bmatrix} MA^N & M(W_N^2 - A^N W_0) \end{bmatrix},$$

$$d_U := \begin{bmatrix} E_U \\ \vdots \\ E_U \end{bmatrix}, \; d_X := \begin{bmatrix} E_X \\ \vdots \\ E_X \end{bmatrix}, \; d_{X_f} := m. \tag{52}$$

Then, for any initial condition in the polytope defined by the projection

$$\mathbb{X}_0 = \left\{ x \in \mathbb{R}^n \mid \exists \bar{v} \in \mathbb{R}^{N_s} : \begin{bmatrix} x \\ \bar{v} \end{bmatrix} \in \mathbb{Y} \right\}, \tag{53}$$

the constrained system (1) can be stabilized by the reduced order MPC law $u_t = u^*_{0|t}$.

Proof. Consider the optimization problem $\mathcal{P}$ at $t = 0$. By rewriting the constraints (39),(40),(41) in terms of the initial condition x_0 and the coefficients $[\alpha_1, \cdots, \alpha_{N_s}]^{\mathrm{T}}$, the following expression is obtained

$$\begin{bmatrix} C_U \\ C_X \\ C_{X_f} \end{bmatrix} \begin{bmatrix} x_0 \\ \bar{v} \end{bmatrix} \leq \begin{bmatrix} d_U \\ d_X \\ d_{X_f} \end{bmatrix}, \tag{54}$$

$$\bar{v} := [\alpha_1, \cdots, \alpha_{N_s}]^{\mathrm{T}}.$$

Note that we only consider the coefficients $\alpha_i (i = 1, \cdots, N_s)$ as $\alpha_{N_s+1} = 0$ at $t = 0$. If there exists a coefficient vector $\bar{v}$ which satisfies the inequality (54) for a given initial state x_0, the optimization problem is feasible at $t = 0$. Then the stability of the reduced order MPC follows from Theorem 2. Thus the orthogonal projection of the polytope described by (54) onto $\mathbb{R}^n$ characterizes the set of all initial states for which the closed-loop stability of the reduced order MPC law is guaranteed. □

Remark 3. As we mentioned in Remark 1(a), the zero eigenvalues and corresponding eigenvectors characterize the LQ control. Thus, by including n zero eigenvalues and eigenvectors in the N_s bases, the stabilizing reduced order MPC law $u_t = u^*_{0|t}$ becomes the LQ control after the state is once steered to the invariant set $\mathbb{X}_f$. □

Remark 4. In Rojas et al. [2004], an approximate MPC for input constrained systems has been proposed based on SVD of the Hessian of a quadratic cost functional. In this approach, the input is constructed by the linear combination of the variable basis vectors in the input space at each sample time while our approach employs the fixed number of the basis vectors obtained from the system decomposition method. □

Table 1. Eigenvalues of Δ

λ_1	0
λ_2	0
λ_3	6419.1396
λ_4	124.277
λ_5	13.5067
λ_6	3.3997
λ_7	1.7541
λ_8	1.3266
$\vdots$	$\vdots$

5. NUMERICAL EXAMPLE

Consider the discrete-time system

$$x_{k+1} = \begin{bmatrix} 1 & 0.2 \\ 0 & 1 \end{bmatrix} x_k + \begin{bmatrix} 0.02 \\ 0.2 \end{bmatrix} u_k, \tag{55}$$

which is obtained by discretizing the double integrator $Y(s) = \frac{1}{s^2}U(s)$ with sample time 0.2 [s]. The control objective is to regulate the system to the origin while fulfilling the input and state constraints:

$$-1 \leq u_k \leq 1, \; \begin{bmatrix} -3 \\ -3 \end{bmatrix} \leq x_k \leq \begin{bmatrix} 3 \\ 3 \end{bmatrix}. \tag{56}$$

A standard MPC law with $Q = \mathrm{diag}(15, 1)$, $R = 1$, $N = 25$ in (6) is obtained by solving the QP-problem where the number of the optimization variable is 25. For the design of the standard MPC formulation, the invariant set (45) is imposed for the terminal constraint. Fig.1(MPC) shows the system responses of the standard MPC law.

We next design a reduced order MPC law. By Lemma 1, the eigenvalues of the matrix Δ are calculated as Table 1. By employing three dominant eigenvalues $\lambda_3 \cdots \lambda_5$ in addition to zero eigenvalues λ_1, λ_2 which correspond to LQ control (Remark 1(a)), the stabilizing reduced order MPC law is obtained by Theorem 2. The control input and state response are depicted in Fig.1(reduced order MPC) and it is observed that the responses are quite similar to those of the standard MPC. In highlight with the standard MPC, it should be pointed out that reduced order MPC law is obtained from the optimization problem where the dimension of the optimization variable is 6 ($= N_s + 1$) (25 for the standard MPC). Fig.3 shows the computation time required to solve the optimization problems at each time step for the initial condition $x_0 = [1, -2.5]^{\mathrm{T}}$. The computation is performed on Sun Microsystems Ultrasparc IIi 550MHz using Matlab with CPLEX QP solver. We investigate the relation between the parameter N_s and the feasible initial state by Theorem 3. The sets of feasible initial states $\mathbb{X}_0$ for $N_s = 5, 10$ are depicted by Fig.2. The computation of the projection of polytopes is performed using MPT (Kvasnica et al. [2004]). It is observed that the set $\mathbb{X}_0$ for $N_s = 10$, which guarantees the stability of the reduced order MPC for $\forall x_0 \in \mathbb{X}_0$, is larger than that for $N_s = 5$. In this numerical example, the set $\mathbb{X}_0$ for $N_s = 10$ almost coincides with the set of initial conditions for which the stability of the standard MPC is guaranteed. From the trajectories for some initial conditions, we see that the reduced order MPC law inherits the control strategy of the standard MPC. We also show the standard MPC for the prediction horizon $N = 6$ (depicted by +). In this case, no terminal constraint is imposed. It is observed that the response has deteriorated for the initial condition

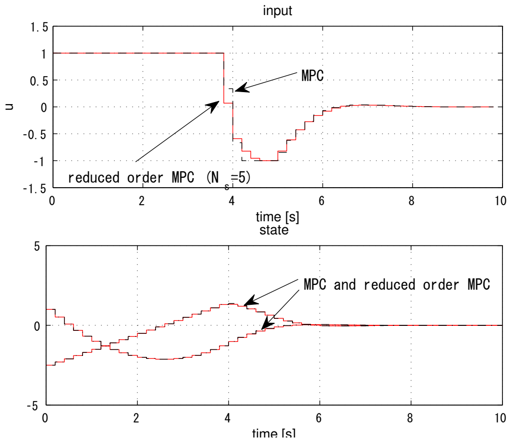

Fig. 1. Input (upper) and state (lower) responses

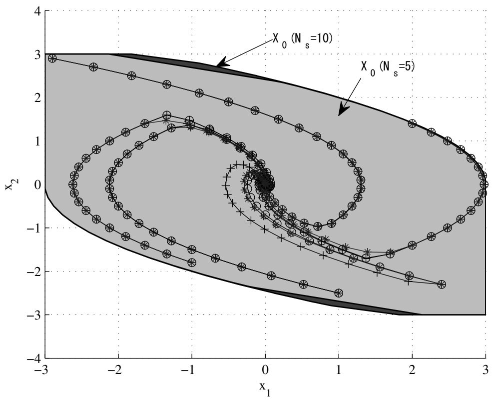

Fig. 2. Set of feasible initial states $\mathbb{X}_0$ and trajectories of the closed-loop system($*$:reduced order MPC, o:MPC, +:MPC($N = 6$))

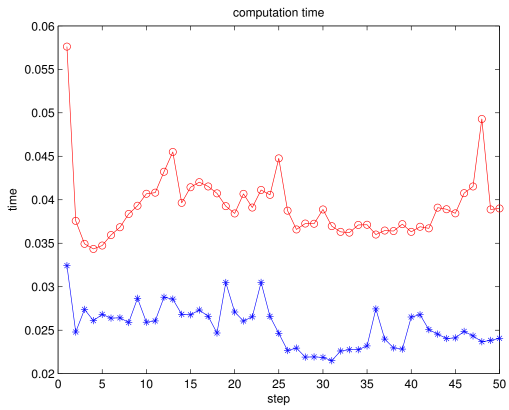

Fig. 3. Computation time at each time step for $x_0 = [1, -2.5]^{\mathrm{T}}$($*$:reduced order MPC,o:MPC)

$x_0 = [2.4, -2.3]^{\mathrm{T}}$ while the responses for other initial conditions are similar to those of the standard MPC for $N = 25$.

6. CONCLUSION

A reduced order MPC law, which guarantees stability of the closed-loop system, is derived by employing the system decomposition method. The MPC law is obtained from a low dimensional on-line optimization problem and a set of initial states for which the control law ensures stability of the closed-loop system is clarified. A numerical example is provided to illustrate the features of the proposed method.

The system decomposition technique (Lemma 1) requires the nonsingular state transition matrix A, which restricts the application of the method to systems with I/O delays. The generalization for the singular case is a direction of future research.

REFERENCES

J. M. Maciejowski, *Predictive control with constraints*, Addison-Wesley, 2002.

M. Morari and J. H. Lee, Model predictive control: past, present and future,*Computers and Chemical Engineering*, Vol.23, No.4–5, pp.667–682, 1999.

W. H. Kwon and S. Han, *Receding horizon control*, Springer, 2005.

A. Bemporad, M. Morari, V. Dua and E. N. Pistikopoulos, The explicit linear quadratic regulator for constrained systems, *Automatica*, Vol.38, No.1, pp.3–20, 2002a.

A. Bemporad, F. Borrelli and M. Morari, Model predictive control based on linear programming - the explicit solution, *IEEE Trans. Automatic Control*, Vol.47, No.12, pp.1974–1985, 2002b.

O. J. Rojas, G. C. Goodwin, M. M. Serón and A.Feuer, An SVD based strategy for receding horizon control of input constrained linear systems, *Int. J. Robust and Nonlinear Control*, Vol.14, No.13–14, pp.1207–1226, 2004.

A. Kojima and M. Morari, LQ control for constrained continuous time systems, *Automatica*, Vol.40, No.7, pp.1143–1155, 2004.

R. Cagienard, P. Grieder, E. C. Kerrigan and M. Morari, Move blocking strategies in receding horizon control, *Journal of Process Control*, Vol.17, No.6, pp.563–570, 2007.

N. Hara and A. Kojima, Reduced order model predictive control method based on decomposition of discrete-time linear systems, *Proc. of the 46th IEEE CDC*, WePI19.6, pp.1269–1274, New Orleans, USA, 2007a.

N. Hara and A. Kojima, Reduced order model predictive control – An approach based on system decomposition – , *SICE Annual Conference 2007*, 3A12-4, pp.2226–2229, Kagawa, Japan, 2007b.

E.G. Gilbert and K.T. Tan: Linear systems with state and control constraints: The theory and application of maximal output admissible sets, *IEEE Trans. on Automatic Control*, Vol.36, No.9, pp.1008–1020, 1991.

D. Q. Mayne, J. B. Rawlings, C. V. Rao and P. O. M. Scokaert, Constrained model predictive control: Stability and optimality, *Automatica*, Vol.36, No.6, pp.789–814, 2000.

M. Kvasnica, P. Grieder and M. Baotić, Multi-Parametric Toolbox (MPT), http://control.ee.ethz.ch/˜mpt/, 2004.

J. Richalet, A. Rault, J. L. Testud and J. Papon, Model predictive heuristic control: applications to industrial processes, *Automatica*, Vol.14, No.5, pp.413–428, 1978.

N. Petit, Y. Creff, L. Lemaire and P. Rouchon, Minimum time constrained control of acid strength on a sulfuric acid alkylation unit, *Chemical Engineering Science*, Vol.56, Issue 8, pp.2767–2774, 2001.