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Sensitivity Analysis for the Configuration of a Multi-Purpose Machines Workshop

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Abstract: In this paper, the multi-purpose machines problem denoted $MPM|split|C_{max}$ is considered. In such a problem, each machine is qualified to process a subset of product types but not necessarily all the product types. The set of the qualifications machine/product represents the *configuration* of the multi-purpose machines workshop. Uncertainties on the demand are considered and a question related to sensitivity analysis is answered to find the neighbourhood of a forecasted demand in which the demands are completed by the configuration before a given deadline. The stability radius of a configuration is moreover computed.

Keywords: production systems, scheduling, configuration management, uncertainty, sensitivity analysis

1. INTRODUCTION

Many industrial production systems can be modelled as a Multi-Purpose Machines (MPM) problem (Brucker et al., 1997). In such a model each machine is able to process a subset of products but not necessarily all the products. To be able to process one type of products, the machine must undergo a specific adjustment. When a machine is set up to process one type of products, the machine is said to be qualified for this product and all the qualifications machine/product represent the *configuration* of the workshop.

A basic problem (considering that all the data are perfectly known and constant) is to find the configuration which minimises the setup costs while ensuring that a load-balanced production plan exists. This basic problem has already been studied in (Aubry et al., 2008). However, the authors have proved that enforcing a strict load-balancing is only relevant for uniform machines.

That is the reason why we propose to consider in this paper a more classical criterion that remains relevant for unrelated machines: the makespan. Moreover, in actual industrial context, considering that all the data are perfectly known and constant is becoming hard to justify. In this context the demand is variable and the completion-time is naturally sensitive to these perturbations.

This paper precisely focuses on the sensitivity analysis (Hall and Posner, 2004) of the configuration of a MPM workshop under demand uncertainties to evaluate which magnitude of perturbations can be dealt with by the configuration such that the completion-time remains less than a given deadline.

In section 2, we present the problem with the proposed model and the type of disturbances that are considered. Then, in section 3, sensitivity analysis is presented and the studied sensitivity analysis problem is defined. The solving approach is highlighted in section 4. A stability radius is defined and calculated in section 5. Finally, the section 6 aims at illustrating all the theoretical results through an academic example.

2. A MODEL FOR THE MPM WORKSHOP AND THE ATTACHED SCHEDULING PROBLEM

2.1 The MPM workshop

In a multi-purpose machines (MPM) workshop, m machines and n types of products are considered, but all the machines cannot necessarily process all the product types: e.g., any machine can only process a subset of product types. The set $\{1, \ldots, m\}$ of the machines is denoted J. The set $\{1, \ldots, n\}$ of the product types is denoted I.

The machines. Multi-purpose machines are classically modelled as follows (Brucker et al., 1997): every product type i can be processed by a set O_i of machines, where O_i is a subset of J. In this paper, we use a slightly different but equivalent model. The workshop machines are regarded as parallel machines and some *technological constraints* are used to prevent assigning products to a machine that cannot process them for technological reasons. All the technological constraints are modelled by a n-by-m binary technological matrix T. Moreover, to be able to process one type of products, the machine must be qualified for the product type by undergoing a specific adjustment. The set of qualifications is modelled by a n-by-m binary *configuration* matrix Q. Q is defined as follows: $Q(i, j) = 1$ if machine j is qualified to process product type i (there is no qualification constraint between machine j and product type i), otherwise $Q(i, j) = 0$. A non-qualification may be the result of:

(1) Technology: for some technological or physical reasons, machine j cannot process any product of type i. This information is given by T when $T(i,j) = 0$ ($T(i,j) = 0 \Rightarrow Q(i,j) = 0$).
(2) Cost control: for financial reasons, all the machines may not be qualified for all the product types because this would be too expensive, and often useless (Aubry et al., 2008).

Note that the multi-purpose machines problem generalises classical parallel machine problems. Parallel machines are multi-purpose machines with no technological constraint and no non-qualification: e.g., $T(i,j) = Q(i,j) = 1, \; \forall (i,j) \in I \times J$.

To process the different product types, machines may have different speeds. A n-by-m real *speed matrix* V is defined. $V(i,j)$ is the number of products of type i that machine j is able to process during one unit of time. Machines are assumed to be unrelated.

The demand. The product types are the jobs to be processed by the machines. They are also referred to as the *demand*. The actual demand is modelled as a real n-column vector N. $N(i)$ is the total amount of products of type i to be processed by the workshop for all i in I. It is also assumed that a forecasted demand N_{ref} is available. The actual demand N is regarded as the result of a variation ΔN around N_{ref}: $N = N_{ref} + \Delta N$. We define a neighbourhood $P_{a_1,\dots,a_n}(N_{ref})$ around N_{ref} by:

$$P_{a_1,\dots,a_n}(N_{ref}) = \left\{ N | N = N_{ref} + \sum_{k\in I} \alpha_k . a_k . e_k \wedge \sum_{k\in I} \alpha_k \leq 1 \wedge \alpha_k \geq 0, \forall k \in I \right\} \quad (1)$$

with $(e_k)_{k\in I}$ the canonical basis of $\mathbb{R}_n^+$ and a_k the positive variation magnitude on the product type k.

We define the demand $N_{ref}^{a_k}$ by:

$$N_{ref}^{a_k} = N_{ref} + a_k . e_k \quad (2)$$

Production plan. Scheduling the production in the workshop requires assigning the products to the machines. The corresponding scheduling problem is dealt with in the next section. The result of the scheduling problem defines a *production plan* that can be modelled by a n-by-m real matrix R. $R(i,j)$ is the total amount of time that machine j spends processing products of type i. Furthermore, preemption and splitting are assumed: several machines may be working on the same product type at the same time. The order of the jobs on the machines has no importance, only the affectation matters.

2.2 The scheduling problem

Given a demand N, the speed data V, and the configuration Q, the scheduling problem consists in finding a production plan R such that the makespan C_{max} is minimised. Using the three fields notation introduced by (Graham et al., 1979) and extended by (Blazewicz et al., 1996; T'Kindt and Billaut, 2002), this problem can be stated as $RMPM|split|C_{max}$. $RMPM$ stands for unrelated multi-purpose machines. *split* means that splitting is allowed. C_{max} means that the makespan (or the maximum completion-time of the machines) must be minimised. This problem is an easy problem (in the sense of the $\mathcal{NP}$-completeness theory) as it can be solved by the following linear program denoted LP which is inspired from the linear program of Lawler and Labetoulle (1978) used to solve $R|pmtn|C_{max}$:

$$LP \begin{cases} \min(C_{max}) & & \\ \sum_{j\in J} V(i,j) \times R(i,j) = N(i) & \forall i \in I & (a) \\ \sum_{i\in I} R(i,j) \leq C_{max} & \forall j \in J & (b) \\ (1 - Q(i,j)) \times R(i,j) & \forall (i,j) \in I \times J & (c) \\ R(i,j) \geq 0 & \forall (i,j) \in I \times J & (d) \end{cases} \quad (3)$$

The speed $V(i,j)$, the configuration matrix $Q(i,j)$ and the demand $N(i)$ are given data whereas $R(i,j)$ and C_{max} are decision variables. The set of constraints ($3a$) enforces that the demand is exactly met for any type of products whereas the set of constraints ($3b$) enforces that C_{max} is greater than or equal to the completion-time of any machine in the shop. The set of constraints ($3c$) ensures that product type i is assigned to machine j (e.g., $R(i,j) > 0$) if and only if the machine is effectively qualified for this product type (e.g., $Q(i,j) = 1$). The last constraints define the variation domains of the variables $R(i,j)$.

In the following, we will denote the optimal value of the objective function of $LP(V,Q,N)$ by $C_{max}^{Q,V}(N)$. If $\left(R, C_{max}^{Q,V}(N)\right)$ is an optimal solution for $LP(V,Q,N)$ then the following equality holds:

$$C_{max}^{Q,V}(N) = \max_{j\in J} \left(\sum_{i\in I} R(i,j) \right) \quad (4)$$

3. SENSITIVITY ANALYSIS

3.1 Definition of sensitivity analysis

In (Mahjoub et al., 2005), sensitivity analysis is defined as follows.

Definition 1. (Mahjoub et al., 2005). Given an optimisation problem $\mathcal{P}$ and an instance $\mathcal{I}$ of $\mathcal{P}$, let $\mathcal{S}_{\mathcal{I}}^*$ be an optimal solution for $\mathcal{I}$ and let $z_{\mathcal{I}}^*$ be the optimal value of $\mathcal{S}_{\mathcal{I}}^*$. A sensitivity analysis on $\mathcal{P}$ and $\mathcal{I}$, consists in answering at least one of the following questions:

(1) In what neighbourhood of $\mathcal{I}$, $\mathcal{S}_{\mathcal{I}}^*$ (resp. $z_{\mathcal{I}}^*$) remains optimal?
(2) In what neighbourhood of $\mathcal{I}$, $\mathcal{S}_{\mathcal{I}}^*$ remains admissible with an acceptable performance?
(3) Given $\mathcal{I}'$ a neighbour of $\mathcal{I}$, is $\mathcal{S}_{\mathcal{I}}^*$ still admissible for $\mathcal{I}'$? If it is, what is the performance degradation?
(4) Given $\mathcal{I}'$ a neighbour of $\mathcal{I}$, what is the new optimal solution (resp. value)?

Moreover, replacing $\mathcal{S}_{\mathcal{I}}^*$ by any solution in the questions (2) and (3) permits to obtain two other questions related to sensitivity analysis.

3.2 Definition of the sensitivity analysis problem

The sensitivity analysis problem presented in this paper consists in answering the question (2) of the sensitivity analysis defined in definition 1. In our context, a solution $\mathcal{S}$ is a configuration Q, a particular instance $\mathcal{I}$ is the forecasted demand N_{ref} and the performance is measured by $C_{max}^{Q,V}(N_{ref})$. So, our sensitivity analysis problem consists in answering the following question: "given a forecasted demand N_{ref}, a configuration Q and the speed data V, in what neighbourhood of N_{ref} does Q remain admissible with an acceptable performance?"

Here we have to define the so-called acceptable performance. The configuration Q leads to an acceptable performance if it let the shop completing the actual demand by a given deadline $\tilde{d}$. More formally, that means that the sensitivity analysis question can be formulated as follows: "Given a forecasted demand N_{ref}, a configuration Q and the speed data V, what is the neighbourhood $P^Q_{a_1,\ldots,a_n}(N_{ref})$ such that $\forall N \in P^Q_{a_1,\ldots,a_n}(N_{ref})$, $C_{max}^{Q,V}(N) \leq \tilde{d}$?" **(SA)**

To ensure that $P^Q_{a_1,\ldots,a_n}(N_{ref})$ is not empty, we assume that the following inequality holds:

$$C_{max}^{Q,V}(N_{ref}) \leq \tilde{d} \qquad (5)$$

4. RESOLUTION OF THE SENSITIVITY ANALYSIS PROBLEM

4.1 Preliminary results

Lemma 2. For any couple of demands (N_1, N_2), the following property holds:

$$C_{max}^{Q,V}(N_1 + N_2) \leq C_{max}^{Q,V}(N_1) + C_{max}^{Q,V}(N_2) \qquad (6)$$

Proof. Let R_1 and R_2 be two optimal production plans for N_1 and N_2 respectively. An admissible production plan (but not necessarily optimal) for the demand defined by $N_1 + N_2$ can be built as follows: $R_3 = R_1 + R_2$. As R_3 is not necessarily optimal, its maximal completion-time defined by $\max_{j\in J}\left(\sum_{i\in I} R_3(i,j)\right)$ is an upper bound for $C_{max}^{Q,V}(N_1 + N_2)$.

Then the following inequality holds:

$$\begin{aligned}
&C_{max}^{Q,V}(N_1+N_2) \leq \max_{j\in J}\left(\sum_{i\in I} R_3(i,j)\right)\\
&\Rightarrow C_{max}^{Q,V}(N_1+N_2) \leq \max_{j\in J}\left(\sum_{i\in I} (R_1(i,j)+R_2(i,j))\right)\\
&\Rightarrow C_{max}^{Q,V}(N_1+N_2) \leq \max_{j\in J}\left(\sum_{i\in I} R_1(i,j)\right) + \max_{j\in J}\left(\sum_{i\in I} R_2(i,j)\right)\\
&\overset{(4)}{\Rightarrow} C_{max}^{Q,V}(N_1+N_2) \leq C_{max}^{Q,V}(N_1) + C_{max}^{Q,V}(N_2)
\end{aligned} \qquad (7)$$

What is exactly the inequality (6). □

Lemma 3. For any demand N and any positive real number α, the following property holds:

$$C_{max}^{Q,V}(\alpha.N) = \alpha.C_{max}^{Q,V}(N) \qquad (8)$$

Proof. First let us show that if $(R, C_{max}^{Q,V}(N))$ is an optimal solution for $LP(V,Q,N)$ then $(\alpha.R, \alpha.C_{max}^{Q,V}(N))$ is an admissible solution for $LP(V,Q,\alpha.N)$.

The demand $\alpha.N$ can be processed by the production plan $\alpha.R$. Indeed, for all i in I the following equalities hold:

$$\sum_{j\in J}(\alpha.R(i,j)\times V(i,j)) = \alpha.\sum_{j\in J}(R(i,j)\times V(i,j)) \overset{(3a)}{=} \alpha.N(i) \qquad (9)$$

Moreover, the makespan associated to the production plan $\alpha.R$ can be written as:

$$\max_{j\in J}\left\{\sum_{i\in I}\alpha.R(i,j)\right\} = \alpha.\max_{j\in J}\left\{\sum_{i\in I}R(i,j)\right\} \overset{(4)}{=} \alpha.C_{max}^{Q,V}(N) \qquad (10)$$

Then let us show by a *reductio ad absurdum* that this solution is optimal for $LP(V,Q,\alpha.N)$. Let us assume that there exists a couple (R^*, C^*_{max}), solution for $LP(V,Q,\alpha.N)$, such that :

$$C^*_{max} < \alpha.C_{max}^{Q,V}(N) \qquad (11)$$

$\frac{R^*}{\alpha}$ is an admissible production plan for N. In fact, for all i in I, $N(i)$ is processed according to this production plan as follows:

$$\begin{aligned}
&\sum_{j\in J}\left(\frac{R^*(i,j)}{\alpha}\times V(i,j)\right) = \frac{1}{\alpha}.\sum_{j\in J}(R^*(i,j)\times V(i,j))\\
&\overset{(3a)}{\Rightarrow}\sum_{j\in J}\left(\frac{R^*(i,j)}{\alpha}\times V(i,j)\right) = \frac{1}{\alpha}(\alpha.N(i))\\
&\Rightarrow\sum_{j\in J}\left(\frac{R^*(i,j)}{\alpha}\times V(i,j)\right) = N(i)
\end{aligned} \qquad (12)$$

Let τ be the makespan associated with $\frac{R^*}{\alpha}$. Then:

$$\begin{aligned}
&\tau \overset{(4)}{=} \max_{j\in J}\left\{\sum_{i\in I}\frac{R^*(i,j)}{\alpha}\right\}\\
&\Rightarrow \tau = \frac{1}{\alpha}\max_{j\in J}\left\{\sum_{i\in I}R^*(i,j)\right\}\\
&\overset{(4)}{\Rightarrow} \tau = \frac{1}{\alpha}C^*_{max}\\
&\overset{(11)}{\Rightarrow} \tau < \frac{1}{\alpha}\left(\alpha.C_{max}^{Q,V}(N)\right)
\end{aligned} \qquad (13)$$

Finally, the following inequality holds:

$$\tau < C_{max}^{Q,V}(N) \qquad (14)$$

Thus, there exists a production plan for N such that the makespan is strictly lower than the optimal makespan, what is contradiction. So the assumption (11) is false and $(\alpha.R, \alpha.C_{max}^{Q,V}(N))$ is an optimal solution for $LP(V,Q,\alpha.N)$. □

Definition 4. Let $P_{\tilde{d}}(Q)$ be the set of the demands that can be completed by the configuration Q in less than $\tilde{d}$ units of time. More formally $P_{\tilde{d}}(Q)$ can be defined as follows:

$$P_{\tilde{d}}(Q) = \left\{N \,\middle|\, C_{max}^{Q,V}(N) \leq \tilde{d}\right\} \qquad (15)$$

Theorem 5. $P_{\tilde{d}}(Q)$ is a convex set.

Proof. Let N_1 and N_2 be two demands belonging to $P_{\tilde{d}}(Q)$, and let α and β be two non-nil real positive numbers such that:

$$\alpha + \beta = 1 \qquad (16)$$

We have to prove that the demand defined by $\alpha.N_1 + \beta.N_2$ belongs to $P_{\tilde{d}}(Q)$ and thus that $C_{max}^{Q,V}(\alpha.N_1 + \beta.N_2) \leq \tilde{d}$.

As N_1 and N_2 belong to $P_{\tilde{d}}(Q)$, the following inequalities hold:

$$\begin{array}{ll} C_{max}^{Q,V}(N_1) \leq \tilde{d} & (a) \\ C_{max}^{Q,V}(N_2) \leq \tilde{d} & (b) \end{array} \tag{17}$$

By using lemma 2 and 3 the following inequality holds:

$$\begin{array}{l} C_{max}^{Q,V}(\alpha.N_1+\beta.N_2) \overset{(6)}{\leq} C_{max}^{Q,V}(\alpha.N_1) + C_{max}^{Q,V}(\beta.N_2) \\ \overset{(8)}{\Rightarrow} C_{max}^{Q,V}(\alpha.N_1+\beta.N_2) \leq \alpha.C_{max}^{Q,V}(N_1) + \beta.C_{max}^{Q,V}(N_2) \\ \overset{(17)}{\Rightarrow} C_{max}^{Q,V}(\alpha.N_1+\beta.N_2) \leq \alpha.\tilde{d} + \beta.\tilde{d} \\ \overset{(16)}{\Rightarrow} C_{max}^{Q,V}(\alpha.N_1+\beta.N_2) \leq \tilde{d} \quad \square \end{array} \tag{18}$$

4.2 Building a neighbourhood $P^Q_{a_1,\ldots,a_n}(N_{ref})$ included in $P_{\tilde{d}}(Q)$

Definition 6. For all k in I, the linear program LP_k is defined as follows:

$$LP_k \left\{ \begin{array}{lll} \max(a_k) & & \\ \sum_{j \in J} V(i,j) \times R(i,j) = N_{ref}(i) & \forall i \neq k \in I & (a) \\ \sum_{j \in J} (V(k,j) \times R(k,j)) - a_k = N_{ref}(k) & & (b) \\ \sum_{i \in I} R(i,j) \leq \tilde{d} & \forall j \in J & (c) \\ (1 - Q(i,j)) \times R(i,j) = 0 & \forall (i,j) \in I \times J & (d) \\ R(i,j) \geq 0 & \forall (i,j) \in I \times J & (e) \end{array} \right. \tag{19}$$

$V(i,j)$, $N_{ref}(i)$, $Q(i,j)$ and $\tilde{d}$ are given data whereas a_k and $R(i,j)$ are decision variables. The sets of constraints (19a) and (19b) enforce that the demand $N_{ref}^{a_k}$ is satisfied. The set of constraints (19c) enforces that $\tilde{d}$ is greater than or equal to the completion-time of any machine in the shop. The set of constraints (19d) ensures that product type i is assigned to machine j (e.g., $R(i,j) > 0$) if and only if the machine is effectively qualified for this product type (e.g., $Q(i,j) = 1$). The last constraints define the variation domains of the variables $R(i,j)$.

Solving LP_k to optimality is equivalent to answer the following question: "given a configuration Q, a forecasted demand N_{ref} and a deadline $\tilde{d}$, what is the maximal additional number a_k of products of type k that can be completed by the configuration Q without missing the deadline $\tilde{d}$?"

In the following of the paper, we will denote the optimal value of the objective function of $LP_k\left(Q,V,N_{ref},\tilde{d}\right)$ by $a_k\left(Q,V,N_{ref},\tilde{d}\right)$.

Lemma 7. The neighbourhood $P^Q_{a_1,\ldots,a_n}(N_{ref})$, defined by the family $\left(a_k\left(Q,V,N_{ref},\tilde{d}\right)\right)_{k\in I}$ is included in $P_{\tilde{d}}(Q)$.

Proof. Let N be in $P^Q_{a_1,\ldots,a_n}(N_{ref})$. There is a family $(\alpha_k)_{k\in I}$ of positive real numbers such that $N = N_{ref} + \sum_{k\in I} \alpha_k.a_k\left(Q,V,N_{ref},\tilde{d}\right).e_k$ and $\sum_{k\in I}\alpha_k \leq 1$.

Let β be the real positive number such that $\beta = 1 - \sum_{k\in I}\alpha_k$. N can be written as follows:

$$\begin{array}{l} N = \left(\beta + \sum_{k\in I}\alpha_k\right).N_{ref} + \sum_{k\in I}\alpha_k.a_k\left(Q,V,N_{ref},\tilde{d}\right).e_k \\ \Rightarrow N = \beta.N_{ref} + \sum_{k\in I}\alpha_k.\left(N_{ref} + a_k\left(Q,V,N_{ref},\tilde{d}\right).e_k\right) \\ \Rightarrow N = \beta.N_{ref} + \sum_{k\in I}\alpha_k.N_{ref}^{a_k\left(Q,V,N_{ref},\tilde{d}\right)} \end{array} \tag{20}$$

By definition of the family $\left(a_k\left(Q,V,N_{ref},\tilde{d}\right)\right)_{k\in I}$, each demand $N_{ref}^{a_k\left(Q,V,N_{ref},\tilde{d}\right)}$ belongs to $P_{\tilde{d}}(Q)$. Moreover, according to inequality (5), N_{ref} belongs to $P_{\tilde{d}}(Q)$ too. N is thus a convex combination of elements belonging to $P_{\tilde{d}}(Q)$. According to the convex property of $P_{\tilde{d}}(Q)$ (see theorem 5), N belongs to $P_{\tilde{d}}(Q)$ and so $P^Q_{a_1,\ldots,a_n}(N_{ref}) \subseteq P_{\tilde{d}}(Q)$. □

The neighbourhood $P^Q_{a_1,\ldots,a_n}(N_{ref})$, defined by the family $\left(a_k\left(Q,V,N_{ref},\tilde{d}\right)\right)_{k\in I}$, answers the sensitivity analysis question **(SA)**.

5. DEFINITION AND CALCULATION OF A STABILITY RADIUS

5.1 Definition of the stability radius of an ϵ-approximate solution

Definition 8. An ϵ-approximate solution $\mathcal{S}$ for an instance $\mathcal{I}$ relatively to a performance criterion z to be minimised is a solution that satisfies the following inequality:

$$z_{\mathcal{I}}(\mathcal{S}) \leq (1+\epsilon)z^*_{\mathcal{I}} \tag{21}$$

with $z^*_{\mathcal{I}}$ the optimal value of the objective function z on $\mathcal{I}$.

Definition 9. (Sotskov et al., 1998). Given an optimisation problem $\mathcal{P}$, an instance $\mathcal{I}$ of $\mathcal{P}$, the optimal value $z^*_{\mathcal{I}}$ and an ϵ-approximate solution $\mathcal{S}$ for $\mathcal{I}$, the stability radius $\rho_{\mathcal{S}}(\mathcal{I})$ of the ϵ-approximate solution $\mathcal{S}$ is the maximal radius of a ball with centre $\mathcal{I}$ in which the solution $\mathcal{S}$ does not go away from more than ϵ percents of $z^*_{\mathcal{I}}$.

5.2 Stability radius of an ϵ-approximate configuration Q

For any demand N, the configuration $Q = T$ is the one that leads to the minimum completion-time. $C^{T,V}_{max}(N)$ is thus the minimal makespan for any demand N. An ϵ-approximate configuration for a forecasted demand N_{ref} is a configuration Q such that the following inequality holds:

$$C^{Q,V}_{max}(N_{ref}) \leq (1+\epsilon)C^{T,V}_{max}(N_{ref}) \tag{22}$$

Finding the stability radius of an ϵ-approximate configuration Q is finding the maximal radius of a ball B with centre N_{ref} such that:

$$C^{Q,V}_{max}(N) \leq (1+\epsilon)C^{T,V}_{max}(N_{ref}), \quad \forall N \in B \tag{23}$$

Let $\tilde{d}$ be equal to $(1+\epsilon)C^{T,V}_{max}(N_{ref})$.

Theorem 10. The stability radius of an ϵ-approximate configuration Q is given by:

$$\rho_Q(N_{ref}) = \min_{k \in I} \left\{ a_k \left(Q, V, N_{ref}, \tilde{d} \right) \right\} \quad (24)$$

Proof. First we prove that the ball B with centre N_{ref} and with radius $\rho_Q(N_{ref})$ is such that the inequality (23) holds.

We define a_0 as $a_0 = \min_{k \in I} \left\{ a_k \left(Q, V, N_{ref}, \tilde{d} \right) \right\}$. Moreover, k_0 is the integer of I such that $a_{k_0} = a_0$.

B can be formally defined as:

$$B = \left\{ N | N = N_{ref} + a_0 . \sum_{k \in I} \alpha_k . e_k \wedge \|\alpha_k\|_1 \leq 1 \right\} \quad (25)$$

Let us show that B is a subset of $P_{\tilde{d}}(Q)$.

Let N be in B. There is a family $(\alpha_k)_{k \in I}$ of positive real numbers such that $N = N_{ref} + a_0 . \sum_{k \in I} \alpha_k . e_k$ and $\|\alpha_k\|_1 \leq 1$.

Let β be the real positive number such that:

$$\beta = 1 - \|\alpha_k\|_1 \quad (26)$$

For all k in I, the following inequality holds:

$$\begin{aligned} N(k) = N_{ref}(k) + a_0 . \alpha_k \\ \Rightarrow N(k) \leq N_{ref}(k) + a_0 . |\alpha_k| \\ \Rightarrow N(k) \leq N_{ref}(k) + a_k \left(Q, V, N_{ref}, \tilde{d} \right) . |\alpha_k| \end{aligned} \quad (27)$$

Let N^+ be the demand defined by:

$$N^+ = N_{ref} + \sum_{k \in I} \left(a_k \left(Q, V, N_{ref}, \tilde{d} \right) . |\alpha_k| . e_k \right) \quad (28)$$

According to equation (26), N^+ can be written as follows:

$$\begin{aligned} N^+ = \beta . N_{ref} + \sum_{k \in I} |\alpha_k| . \left(N_{ref} + a_k \left(Q, V, N_{ref}, \tilde{d} \right) . e_k \right) \\ \Rightarrow N^+ = \beta . N_{ref} + \sum_{k \in I} |\alpha_k| . N_{ref}^{a_k (Q, V, N_{ref}, \tilde{d})} \end{aligned} \quad (29)$$

By definition of the family $\left(a_k \left(Q, V, N_{ref}, \tilde{d} \right) \right)_{k \in I}$, each demand $N_{ref}^{a_k (Q, V, N_{ref}, \tilde{d})}$ belongs to $P_{\tilde{d}}(Q)$. Moreover, according to inequality (5), N_{ref} belongs to $P_{\tilde{d}}(Q)$ too. N^+ is thus a convex combination of elements belonging to $P_{\tilde{d}}(Q)$. According to the convex property of $P_{\tilde{d}}(Q)$ (see theorem 5), N^+ belongs to $P_{\tilde{d}}(Q)$. Thus the following inequality holds:

$$C_{max}^{Q,V}(N^+) \overset{(15)}{\leq} \tilde{d} \quad (30)$$

According to equation (27), each element of the demand N^+ is greater than or equal to the one of the demand N, thus the following inequality holds:

$$\begin{aligned} C_{max}^{Q,V}(N) \leq C_{max}^{Q,V}(N^+) \\ \overset{(30)}{\Rightarrow} C_{max}^{Q,V}(N) \leq \tilde{d} \end{aligned} \quad (31)$$

By definition of $P_{\tilde{d}}(Q)$, N belongs to $P_{\tilde{d}}(Q)$. So, B is a subset of $P_{\tilde{d}}(Q)$ and the following inequality holds:

$$C_{max}^{Q,V}(N) \leq \tilde{d}, \ \forall N \in B \quad (32)$$

By definition of $\tilde{d}$, the inequality (32) is equivalent to the inequality (23).

Now, we prove that a_0 is the maximal radius. By definition, a_0 is the maximal additional number of products of type k_0 that can be completed by the configuration Q without exceeding $\tilde{d} = (1+\epsilon) C_{max}^{T,V}(N_{ref})$. $N_{ref}^{a_{k_0}}$ is thus a member of the frontier of B and each demand N defined by $N = N_{ref}^{a_{k_0}} + \delta . e_{k_0}$, with δ any positive real number, is such that $C_{max}^{Q,V}(N) > (1+\epsilon) C_{max}^{T,V}(N_{ref})$. So, a_0 cannot increase anymore. □

6. EXAMPLE

This section aims at illustrating all the theoretical results through a simple example. This example considers two product types and two machines. The technological matrix T, the speed matrix V, the configuration matrix Q and the forecast demand N_{ref} are given below:

$$T = V = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \ Q = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \ N_{ref} = \begin{bmatrix} 5 \\ 4 \end{bmatrix} \quad (33)$$

6.1 Scheduling problem

Solving $LP(V, Q, N_{ref})$ to optimality yields to:

$$R = \begin{bmatrix} 0 & 5 \\ 4 & 0 \end{bmatrix}, \ C_{max}^{Q,V}(N_{ref}) = 5 \quad (34)$$

It can be noted that the machines do not share the same completion-time. This is due to the non-qualification of machine 1 for the products of type 1.

Moreover, solving $LP(V, T, N_{ref})$ to optimality yields to $C_{max}^{T,V}(N_{ref}) = 4.5$.

6.2 Sensitivity analysis

The aim of sensitivity analysis is to determine which magnitude of perturbations can be dealt with by the configuration without passing a deadline $\tilde{d}$.

For our example the deadline is set to $\tilde{d} = 133\% C_{max}^{T,V}(N_{ref}) = 6$. This is equivalent to considering that the completion-time deviation cannot exceed $\epsilon = 33\%$ of $C_{max}^{T,V}(N_{ref})$.

It can be noted that Q is an ϵ-approximate configuration as it verifies $C_{max}^{Q,V}(N_{ref}) \leq (1+\epsilon) C_{max}^{T,V}(N_{ref})$

Calculation of $a_k \left(Q, V, N_{ref}, \tilde{d} \right)$ *for* $k = 1$ *and* $k = 2$. a_k is the maximum amount of additional products of type k that can be completed by the workshop without passing the deadline $\tilde{d}$.

Solving $LP_k \left(Q, V, N_{ref}, \tilde{d} \right)$ for $k = 1$ and $k = 2$ yields to:

$$\begin{cases} a_1 = a_1 \left(Q, V, N_{ref}, \tilde{d} \right) = 1 \\ a_2 = a_2 \left(Q, V, N_{ref}, \tilde{d} \right) = 3 \end{cases} \quad (35)$$

The figure 1 shows the set of demands resulting from an additive amount of products without passing the deadline $\tilde{d}$.

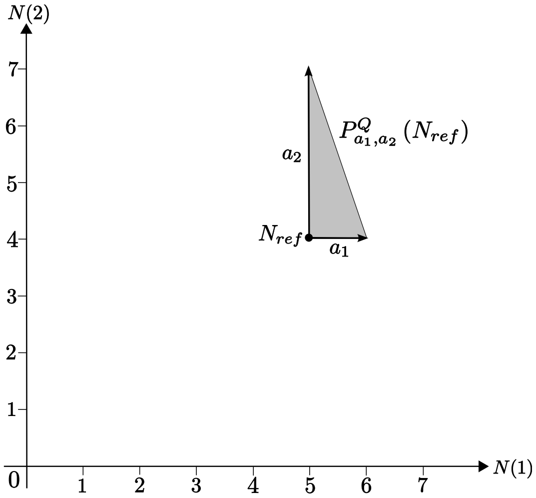

Fig. 1. Graphical representation of $P^Q_{a_1,a_2}(N_{ref})$

Then, any demand N defined by $N = Nref + \alpha.a_1.e_1 + \beta.a_2.e_2$ with $\alpha + \beta \leq 1$ can be completed by the deadline.

It should be stressed that $P^Q_{a_1,a_2}(N_{ref})$ is included in $P_{\tilde{d}}(Q)$. Then, any demand in $P^Q_{a_1,a_2}(N_{ref})$ can be processed in less than $\tilde{d}$ units of time. However, for any demand out of $P^Q_{a_1,a_2}(N_{ref})$, the completion-time must be computed with LP in order to know if the deadline is met or not.

Let us illustrate this point on the following three demands denoted N_1, N_2 and N_3 defined by:

$$N_1 = \begin{bmatrix} 5.5 \\ 5 \end{bmatrix}, \quad N_2 = \begin{bmatrix} 5.5 \\ 6 \end{bmatrix}, \quad N_3 = \begin{bmatrix} 5.5 \\ 7 \end{bmatrix}$$

The optimal production plans and optimal makespans associated with these demands are given below:

$$R_1 = \begin{bmatrix} 0 & 5.5 \\ 5 & 0 \end{bmatrix}, \; R_2 = \begin{bmatrix} 0 & 5.5 \\ 5.75 & 0.25 \end{bmatrix}, \; R_3 = \begin{bmatrix} 0 & 5.5 \\ 6.25 & 0.75 \end{bmatrix}$$

$$C^{Q,V}_{max}(N_1) = 5.5, \;\; C^{Q,V}_{max}(N_2) = 5.75, \;\; C^{Q,V}_{max}(N_3) = 6.25$$

As N_1 is in $P^Q_{a_1,a_2}(N_{ref})$, it can be checked that its completion-time is less than 6 units of time. Although the demand N_2 is out of $P^Q_{a_1,a_2}(N_{ref})$, solving LP allows to compute the completion-time that appears to be less than 6 units of time. Then, N_2 in in $P_{\tilde{d}}(Q)$. The demand N_3 is also out of $P^Q_{a_1,a_2}(N_{ref})$, and has a completion-time that is strictly greater that 6 units of time, this shows that N_3 is out of $P_{\tilde{d}}(Q)$.

6.3 Calculation of $\rho_Q(N_{ref})$

$\rho_Q(N_{ref})$ is the stability radius of the ϵ-approximate configuration Q. It values the maximal magnitude of perturbations on N_{ref} such that the completion-time by the configuration does not go away from more than ϵ percents of $C^{T,V}_{max}(N_{ref})$.

Following the theorem 10, $\rho_Q(N_{ref})$ is valued by:

$$\rho_Q(N_{ref}) = \min_{k \in I} \left\{ a_k \left(Q, V, N_{ref}, \tilde{d} \right) \right\} = a_1 = 1$$

7. CONCLUSION

We present in this paper a sensitivity analysis for the multi-purpose machines problem. The question "in which neighbourhood of a given demand, the configuration of the workshop remains admissible with an acceptable performance degradation" is answered by solving n independent linear programs. Moreover, the stability radius of an ϵ-approximate configuration is calculated.

It would be interesting in the future to measure the degradation of the makespan according to the deviation of the demand and to answer all the questions of the sensitivity analysis.

REFERENCES

A. Aubry, A. Rossi, M.-L. Espinouse, and M. Jacomino. Minimizing setup costs for parallel multi-purpose machines under load-balancing constraint. *European Journal of Operational Research*, 187(3):1115–1125, 2008.

J. Blazewicz, K.H. Ecker, E. Pesch, G. Schmidt, and J. Weglarz. *Scheduling Computer and Manufacturing Processes*. Springer-Verlag, Berlin, Germany, 1996.

P. Brucker, B. Jurisch, and A. Krämer. Complexity of scheduling problems with multi-purpose machines. *Annals of Operational Research*, 70:57–73, 1997.

R.L. Graham, E.L. Lawler, J.K. Lenstra, and A.H.G. Rinnooy Kan. Optimization and approximation in deterministic sequencing and scheduling: a survey. *Annals of Discrete Mathematics*, 5:287–326, 1979.

N.G. Hall and M.E. Posner. Sensitivity analysis for scheduling problems. *Journal of Scheduling*, 7:49–83, 2004.

E. Lawler and J. Labetoulle. On preemptive scheduling of unrelated parallel processors by linear programming. *Journal of the Association for Computing Machinery*, 25:612–619, 1978.

A. Mahjoub, A. Moukrim, and E. Sanlaville. *Flexibilité et Robustesse en Ordonnancement*, chapter 4. Hermès Science, Paris, France, 2005.

Y.N. Sotskov, A.P.M. Wagelmans, and F. Werner. On the calculation of the stability radius of an optimal or an approximate schedule. *Annals of Operations Research*, 83:213–252, 1998.

V. T'Kindt and J.-C. Billaut. *Multicriteria Scheduling: Theory, Models and Algorithms*. Springer-Verlag, Berlin, Germany, 2002. ISBN 3540436170.