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Synchronizing Chaotic Systems Based on Tridiagonal Structure

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Abstract: The design approach based on tridiagonal structure combines the structure analysis with the design of stabilizing controller. During the design procedure, the original nonlinear affine systems is transformed into a stable system with special tridiagonal structure. In this study, the method is proposed for synchronizing chaotic systems. There are several advantages in this method for synchronizing chaotic systems: (a) it presents a systematic procedure for construct a proper controller in chaos synchronization; (b) it can be applied to a variety of chaotic systems with lower triangular structure. Examples of Lorenz system, Chua's circuit and Duffing system are presented.

1. INTRODUCTION

Synchronizing chaotic systems and circuits has received great interest in recent years. Generally the two chaotic systems in synchronization are called drive system and response system respectively. The idea of synchronization is to use the output of the drive system to control the response system, and make the output of the response system follow the output of the drive system. Many approaches have been proposed for chaos synchronization Ott [1990],Carroll [1991],Bai [1997] and Liao [2000]. Backstepping method Sepulchre [1997] has become one of the most important approaches for synchronizing chaotic systems Li [2006],Zhang [2004] in recent years. The main advantage of this method is the systematic construction of a Lyapunov function for the nonlinear systems, and control goal can be achieved with reduced control effort.

Notice the original systems is transformed into the system with special tridiagonal structure using backstepping. There are a class of design methods proposed in Liu [2007] including direct design method and recursive design method based on tridiagonal structurem, which is to transform the original systems into the system with stable tridiagonal structure by inputs and coordinate change. The recursive design method is similar to backstepping and can design the controllers with more parameters than backstepping for nonlinear systems.

This paper focuses on showing the effectiveness of the recursive design method based on tridiagonal structure to a wider variety of chaotic systems. The paper is organized as follows: In Section 2 the class of chaotic systems considered in this work and the problem formulation are presented. In Section 3, two theorems about systems with tridiagonal structure are given and the recursive design method based on special tridiagonal structure is given. In Section 4, the method is utilized for several systems such as Lorenz system, Chua's circuit and Duffing system. Numerical simulations are carried out to confirm the validity of the proposed theoretical approach. In Section 5 conclusion is presented.

2. PROBLEM FORMULATION

In general, typical dynamics of chaotic systems such as Lorenz system, Chua's circuit and Duffing system all belong to the system as following:

$$\dot{\mathbf{x}} = f(\mathbf{x}) \tag{1}$$

where $\mathbf{x} = [x_1, \cdots, x_n]^T \in \mathbb{R}^n$ are state variables.

Assume that drive system is expressed as Eq. (1). Then response system which is coupled with system (1) by $\mathbf{u}$ is as following:

$$\dot{\mathbf{y}} = f_1(\mathbf{y}) + g_1(\mathbf{y})\mathbf{u} \tag{2}$$

where $\mathbf{y} = [y_1, \cdots, y_n]^T \in \mathbb{R}^n$ are state variables and $\mathbf{u} \in \mathbb{R}^m$ are inputs.

Let us define the state errors between the response system and the drive system as

$$e_1 = y_1 - x_1, e_2 = y_2 - x_2 \cdots e_n = y_n - x_n \tag{3}$$

$$\mathbf{e} = [e_1, \cdots, e_n]^T \tag{4}$$

Subtract (1) from (2). Notice Eqs.(3) and (4), finally error system can be derived as

$$\dot{\mathbf{e}} = f_1(\mathbf{y}) - f(\mathbf{x}) + g_1(\mathbf{y})\mathbf{u} \tag{5}$$

The problem to realize the synchronization between two chaotic systems now is transformed into another problem on how to choose control law $\mathbf{u}$ to make $\mathbf{e}$ converge to zero with time increasing.

3. CONTROL BASED ON TRIDIAGONAL STRUCTURE

In this section, nonlinear control methods based on tridiagonal structure are introduced. First of all, two theorems about systems with specially tridiagonal structure are introduced.

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Theorem 1. Consider the systems with state-dependent coefficients as follows:

$$\dot{\mathbf{x}} = A(\mathbf{x})\mathbf{x} \tag{6}$$

where $\mathbf{x} = [x_1, \cdots, x_n]^T$, $A(\mathbf{x})$ is called coefficient matrix and has a class of tridialognal structure. If the $A(\mathbf{x}) = [a_{ij}] \in \mathbb{R}^{n\times n}, i, j = 1, \cdots, n$ satisfies the following conditions:

(1) $|i-j| > 1, a_{ij} = 0;$
(2) $|i-j| = 1, a_{ij}/a_{ji} = const \in \mathbb{R}^-;$
(3) $|i-j| = 0, a_{ij} < 0;$

the system (6) is asymptotically stable.

Proof.

Let $k_i = -\frac{a_{ij}}{a_{ji}}, (j-i=1)$, and take the matrix Λ as

$$\Lambda = diag\{d_1, d_2, \cdots, d_n\} \tag{7}$$

where $d_1 = 1, d_2 = k_1, \cdots, d_i = d_{i-1}\sqrt{k_i}$. Choose the Lyapunov function candidate as

$$V = \frac{1}{2}\mathbf{x}^T \Lambda^{-2}\mathbf{x} \tag{8}$$

The derivative of $V$is given by

$$\dot{V} = -\mathbf{x}^T P \mathbf{x} \tag{9}$$

where matrix $P = diag\{-a_{11}, \cdots, -a_{nn}\}$is a positive matrix. Therefore, the equilibrium $x = 0$ is globally stable.

We know the stable systems with special structure. Then, the stabilization of nonlinear affine systems can be transformed into the construction of the system from the original nonlinear affine systems. There exist a method based on tridiagonal structure for the systems with lower triangular structure. The method is suitable for the systems with lower triangular structure. Consider the system with lower triangular

$$\begin{aligned} \dot{x}_1 &= f_1(x_1) + g_1(x_1)x_2 \\ \dot{x}_2 &= f_2(x_1, x_2) + g_2(x_1, x_2)u \end{aligned} \tag{10}$$

where $\mathbf{x} = [x_1, x_2]^T \in \mathbb{R}^n, \mathbf{u} \in \mathbb{R}$. The objective of control based on tridiagonal structure is to transform the systems into a system that has a special tridiagonal structure and satisfies the conditions of theorem 1. The design procedure is as follows:

(1) Take the variable x_2 as virtual input, and design the controller of the systems as follows:

$$\dot{x}_1 = f_1(x_1) + g_1(x_1)x_2 \tag{11}$$

The virtual controller can be got as

$$x_2 = \alpha_1(x_1) = \frac{1}{g_1(x_1)}(-f_1(x_1) - k_1x_1)$$

(2) The variable x_2 is not the practical controller, and the error between x_2 and $\alpha_1(x_1)$ is $z_2 = x_2 - \alpha_1(x_1)$. The derivative of z_2 is as follows:

$$\dot{z}_2 = f_2(x_2) + g_2(x_2)u - \dot{\alpha}(x_1) \tag{12}$$

Take the practical controller as

$$u = \frac{1}{g_2(x_1, x_2)}(-l_1g_1(x_1) - f_2(x_1) - k_2z_2 + \dot{\alpha}(x_1)) \tag{13}$$

We can get the new systems as follows:

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} -k_1 & g_1(z_1) \\ -l_1g_1(z_1) & -k_2 \end{bmatrix} \begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} \tag{14}$$

where $z_1 = x_1$. The asymptotical stability of (14) can be guaranteed by Theorem 1.

The similar procedure can be used for the system with lower triangular structure as follows:

$$\begin{aligned} \dot{x}_1 &= f_1(x_1) + g_1(x_1)x_2 \\ \dot{x}_2 &= f_2(x_1, x_2) + g_2(x_1, x_2)x_3 \\ &\vdots \\ \dot{x}_n &= f_n(x_1, \cdots, x_n) + g_n(x_1, \cdots, x_n)u \end{aligned} \tag{15}$$

where $g_i(\cdot) \neq 0$. The above system can be transformed into the system as follows:

$$\dot{z} = \begin{bmatrix} -k_1 & g_1(z) & & \\ -l_1g_1(z) & -k_2 & \ddots & \\ & \ddots & \ddots & g_{n-1}(z) \\ & & -l_{n-1}g_{n-1}(z) & -k_n \end{bmatrix} z \tag{16}$$

where $k_i > 0, z \in \mathbb{R}^n$. The objective of the design method is to transform the system (15) into the systems (16) by using coordinate changes and inputs. In the next section, the design method is used to synchronize chaotic systems.

4. SYNCHRONIZATION VIA THE DESIGN BASED ON TRIDIAGONAL STRUCTURE

In this section, Lorenz system, Chua's circuit and Duffing system are presented for synchronizing by the design method based on tridialognal structure.

4.1 Lorenz system

In Lorenz system, external excitation does not exit. Drive Lorenz system and response Lorenz system can be described respectively as (17) and (18)

$$\begin{aligned} \dot{x}_1 &= \sigma(y_1 - x_1) \\ \dot{y}_1 &= \rho x_1 - y_1 - x_1z_1 \\ \dot{z}_1 &= -\beta z_1 + x_1y_1 \end{aligned} \tag{17}$$

$$\begin{aligned} \dot{x}_2 &= \sigma(y_2 - x_2) \\ \dot{y}_2 &= \rho x_2 - y_2 - x_2z_2 \\ \dot{z}_2 &= -\beta z_2 + x_2y_2 + u \end{aligned} \tag{18}$$

where $\sigma, \rho, \beta > 0$. Let

$$e_x = x_2 - x_1, e_y = y_2 - y_1, e_z = z_2 - z_1 \tag{19}$$

Subtract Eq. (17) from Eq. (18), consider Eq. (19), and obtain

$$\begin{aligned} \dot{e}_x &= \sigma(e_y - e_x) \\ \dot{e}_y &= \rho e_x - e_y - e_xe_z - e_xz_1 - x_1e_z \\ \dot{e}_z &= -\beta e_z + e_xe_y + e_xy_1 + x_1e_y + u \end{aligned} \tag{20}$$

The problem of synchronization between drive and response system can be transformed into a problem on how to realize the asymptotical stabilization of system (20). Now the objective is to find a control law u for transforming the system (20) into a system with tridiagonal structure.

First we consider the system (21)

$$\dot{e}_x = \sigma(e_y - e_x) \tag{21}$$

Take e_y as a virtual control, we can get

$$e_y = \alpha_1(e_x) = 0, k_1 > 0 \tag{22}$$

Define a new variable $e_2 = e_y - \alpha_1(e_x)$, and obtain the system

$$\begin{aligned} \dot{e}_x &= -\sigma e_x + \sigma z_2 \\ \dot{e}_2 &= f_2(e_x, e_y, z_1, \dot{e}_x) + g_2(e_x, e_y, z_1)e_z \end{aligned} \tag{23}$$

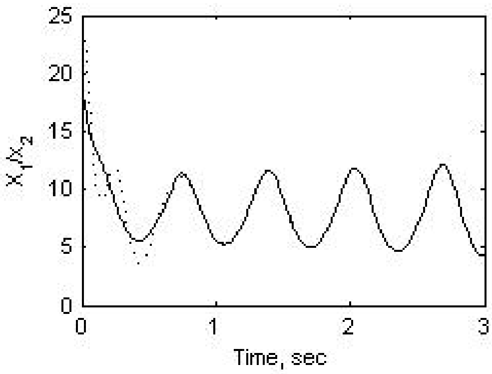

Fig. 1. Synchronized states x_1, x_2 of modified Lorenz system

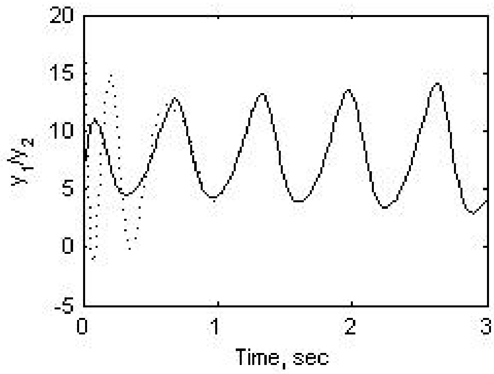

Fig. 2. Synchronized states y_1, y_2 of modified Lorenz system

where

$$\begin{array}{l} f_2\left(e_x, e_y, z_1, \dot{e}_x\right) = \rho e_x - e_y - e_x z_1 \\ g_2\left(e_x, e_y, z_1\right) = -e_x - x_1 \end{array} \tag{24}$$

Suppose $g_2(\cdot) \neq 0$, and take e_y as a virtual control, we can get

$$\begin{aligned} e_z &= \alpha_2\left(e_x, e_y, z_1\right) \\ &= \frac{1}{g_2\left(e_x, e_y, z_1\right)}\left(-f_2\left(e_x, e_y, z_1, \dot{e}_x\right) - \right. \\ &\quad \left. l_1 \sigma e_x - k_2 e_2\right) \end{aligned} \tag{25}$$

Define a new variable $e_3 = e_z - \alpha_2(e_x, e_y, z_1)$, and obtain the system

$$\begin{array}{l} \dot{e}_x = -\sigma\left(k_1 - 1\right) e_x - \sigma z_2 \\ \dot{e}_2 = l_1 \sigma e_x - k_2 e_2 + g_2\left(e_x, e_y, z_1\right) e_3 \\ \dot{e}_3 = f_3\left(e_x, e_y, e_z, z_1\right) + g_3\left(e_x, e_y, e_z, z_1\right) u \end{array} \tag{26}$$

where

$$\begin{aligned} f_3\left(e_x, e_y, e_z, z_1, \dot{\alpha}_2\right) = &-\beta e_z + e_x e_y + e_x y_1 + x_1 e_y \\ &-\dot{\alpha}_2\left(e_x, e_y, z_1\right) \end{aligned} \tag{27}$$

$$g_3\left(e_x, e_y, e_z, z_1\right) = 1 \tag{28}$$

Let the controller as

$$\begin{aligned} u = &\frac{1}{g_3\left(e_x, e_y, e_z, z_1\right)}\left(-f_3\left(e_x, e_y, e_z, z_1, \dot{\alpha}_2\right)\right. \\ &\left. -k_3 e_3 - l_2 g_2 e_2\right) \end{aligned} \tag{29}$$

Substitute (29) into (26), and obtain

$$\begin{bmatrix} \dot{e}_x \\ \dot{e}_2 \\ \dot{e}_3 \end{bmatrix} = \begin{bmatrix} -k_1 & \sigma & 0 \\ -l_1 \sigma & -k_2 & g_2(\cdot) \\ 0 & -l_2 g_2(\cdot) & -k_3 \end{bmatrix} \begin{bmatrix} e_x \\ e_2 \\ e_3 \end{bmatrix} \tag{30}$$

The globally asymptotical stability of (30) can be assured according to theorem 1, that is $e_x, e_2, e_3 \to 0$. From the definition of e_2, e_3, we can get e_x, e_y, e_z coverage to zero with time increasing.

The parameters are selected as $\sigma = 10, \beta = 8/3, \rho = 28, l_1 = l_2 = k_1 = k_2 = k_3 = 2$ and initial condition is taken as $x_1(0) = 20, y_1(0) = 5, z_1(0) = 20$, $x_2(0) = 24$, $y_2(0) = 20, z_2(0) = 28$. With the control law (29), as we can see from Fig. 1-Fig. 3, the slave system is driven to chaotic state gradually, which synchronizes with the master system.

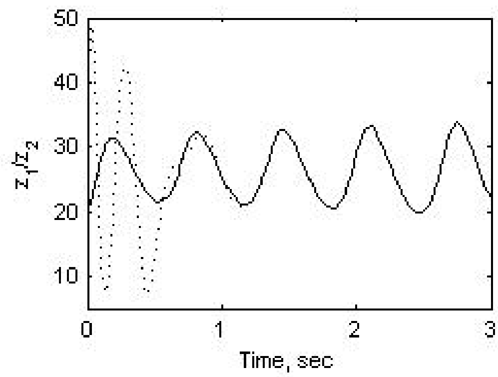

Fig. 3. Synchronized states z_1, z_2 of modified Lorenz system

4.2 Chua's circuit

In order to further test the effectiveness of the method, Chua's circuit, which was the first physical dynamical system capable of generating chaotic phenomena in the laboratory, is proposed for synchronizing. The circuit considered here contains a cubic nonlinearity and the drive system (31) and response system (32) are described by the following set of differential equations:

$$\begin{array}{l} \dot{x}_1 = \alpha\left(y_1 - x_1^3 - c x_1\right) \\ \dot{y}_1 = x_1 - y_1 + z_1 \\ \dot{z}_1 = -\beta y_1 \end{array} \tag{31}$$

$$\begin{array}{l} \dot{x}_2 = \alpha\left(y_2 - x_2^3 - c x_2\right) \\ \dot{y}_2 = x_2 - y_2 + z_2 \\ \dot{z}_2 = -\beta y_2 \end{array} \tag{32}$$

Subtract (32) from (33), and obtain the error systems as follows:

$$\begin{array}{l} \dot{e}_x = -\alpha e_y - \alpha e_x\left(e_x^2 + 3 x_1 e_x + 3 x_1^2\right) - \alpha c e_x + u \\ \dot{e}_y = e_x - e_y + e_z \\ \dot{e}_z = -\beta e_y \end{array} \tag{33}$$

where $e_x = x_2 - x_1, e_y = y_2 - y_1, e_z = z_2 - z_1$. Now the objective is to find a control law u for transforming the system (33) into a system with tridiagonal structure.

First we consider the system (34)

$$\dot{e}_z = -\beta e_y \tag{34}$$

Take e_y as a virtual control, we can get

$$e_y = \alpha_1\left(e_z\right) = k_1 e_z, k_1 > 0 \tag{35}$$

Take a new variable $e_2 = e_y - \alpha_1(e_z)$, and obtain the system as follows:

$$\begin{aligned} \dot{e}_z &= -k_1\beta e_z - \beta z_2 \\ \dot{e}_2 &= f_2(e_z, e_y) + g_2(e_z, e_y)\, e_x \end{aligned} \tag{36}$$

where

$$\begin{aligned} f_2(e_z, e_y) &= -e_y + e_z - \dot{\alpha}_1(e_z) \\ g_2(e_z, e_y) &= -1 \end{aligned} \tag{37}$$

Suppose $g_2(\cdot) \neq 0$, and take e_y as a virtual control, we can get

$$\begin{aligned} e_x &= \alpha_2(e_x, e_y, z_1) \\ &= \frac{1}{g_2(e_x, e_y, z_1)}\left(-f_2(e_x, e_y, z_1, \dot{e}_x)\right. \\ &\quad \left. + l_1\beta e_z - k_2 e_2\right) \end{aligned} \tag{38}$$

Take a new variable $e_3 = e_x - \alpha_2(e_z, e_y, z_1)$, and obtain the system

$$\begin{aligned} \dot{e}_z &= -k_1\beta e_z - \beta e_2 \\ \dot{e}_2 &= l_1\beta e_z - k_2 e_2 + g_2(e_z, e_y)\, e_3 \\ \dot{e}_3 &= f_3(e_x, e_y, e_z, z_1) + g_3(e_x, e_y, e_z, z_1)\, u \end{aligned} \tag{39}$$

where

$$\begin{aligned} f_3(e_x, e_y, e_z, x_1) &= -\alpha e_y - \alpha e_x\left(e_x^2 + 3x_1 e_x + 3x_1^2\right) - \alpha c e_x \\ &\quad - \dot{\alpha}_2(e_x, e_y, z_1) \\ g_3(e_x, e_y, e_z, z_1) &= 1 \end{aligned} \tag{40}$$

Let the controller as

$$u = \frac{1}{g_3(e_x, e_y, e_z, z_1)}\left(-f_3(e_x, e_y, e_z, z_1, \dot{\alpha}_2) - k_3 e_3 - l_2 g_2\right) \tag{41}$$

Substitute (41) into (41), we can obtain

$$\begin{bmatrix} \dot{e}_z \\ \dot{e}_2 \\ \dot{e}_3 \end{bmatrix} = \begin{bmatrix} -k_1\beta & -\beta & 0 \\ l_1\beta & -k_2 & g_2(\cdot) \\ 0 & -l_2 g_2(\cdot) & -k_3 \end{bmatrix} \begin{bmatrix} e_x \\ e_2 \\ e_3 \end{bmatrix} \tag{42}$$

The globally asymptotical stability of (42) can be assured according to theorem 1, that is $e_z, e_2, e_3 \to 0$. From the definition of e_2, e_3, we can get e_x, e_y, e_z coverage to zero with time increasing.

Choose $\alpha = 10, \beta = 16, c = -0.143, l_1 = l_2 = k_1 = k_2 = k_3 = 2$ and take initial condition as $x_1(0) = 1, y_1(0) = 2, z_1(0) = 1, x_2(0) = 10, y_2(0) = 5, z_2(0) = 5$. With the control law (41), as we can see from Fig. 4-Fig. 6, the slave system is driven to chaotic state gradually, which synchronizes with the master system.

4.3 Duffing system

Lorenz system and Chua's circuit discussed above can generate chaotic phenomena under no external excitation condition while Duffing system can generate chaotic phenomena only under external excitation. So here we classify Duffing system to another chaotic system and make Duffing system an example to illustrate how to use this method to synchronize chaotic systems with external excitation. The following set of differential equations formulates two Duffing systems. The first is drive system and the second response system

$$\begin{aligned} \dot{x}_1 &= y_1 \\ \dot{y}_1 &= \alpha x_1 + b y_1 - x_1^3 + c\cos(0.4t) \end{aligned} \tag{43}$$

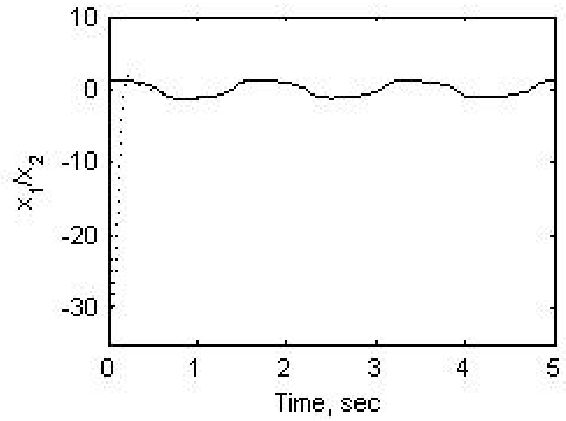

Fig. 4. Synchronized states x_1, x_2 of modified Chua 's circuit

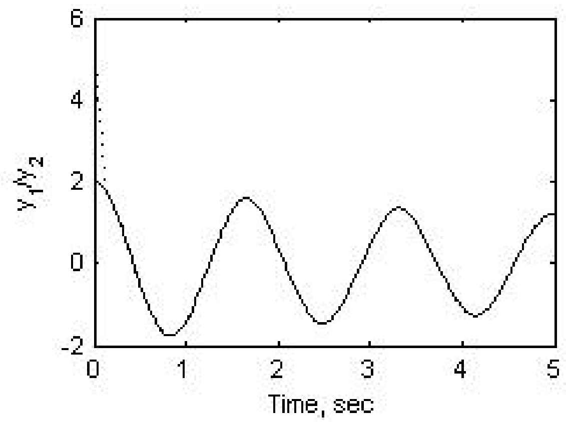

Fig. 5. Synchronized states y_1, y_2 of modified Chua 's circuit

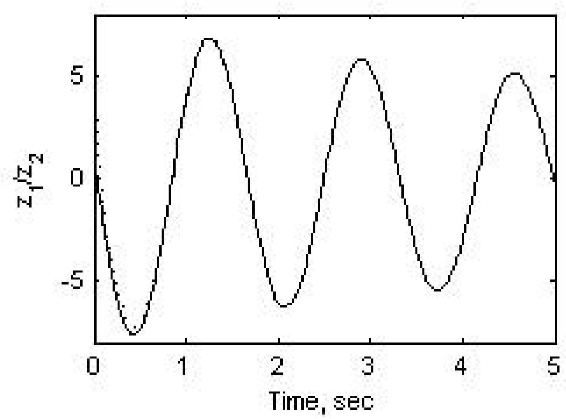

Fig. 6. Synchronized states z_1, z_2 of modified Chua 's circuit

$$\begin{aligned} \dot{x}_2 &= y_2 \\ \dot{y}_2 &= \alpha x_2 + b y_2 - x_2^3 + c\cos(t) \end{aligned} \tag{44}$$

where $a > 0, b < 0, c$ are known parameters.

Subtract (43) from (44), and obtain the error system as follows:

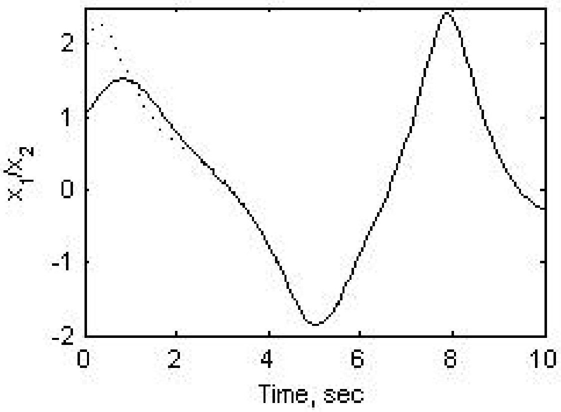

Fig. 7. Synchronized states x_1, x_2 of modified Duffing system

$$\begin{aligned}\dot{e}_x &= e_y \\ \dot{e}_y &= ae_x + be_y - e_x\left(e_x^2 + 3x_1e_x + 3x_1^2\right) \\ &\quad + c\left[\cos t - \cos(0.4t)\right] + u\end{aligned} \qquad (45)$$

where $e_x = x_2 - x_1, e_y = y_2 - y_1$. Now the objective is to find a control law u for transforming the system (33) into a system with tridiagonal structure.

First we consider the system (46)

$$\dot{e}_x = e_y \qquad (46)$$

Take e_y as a virtual control, we can get

$$\alpha_1(e_x) = -k_1e_x, k_1 > 0 \qquad (47)$$

Take a new variable $e_2 = e_y - \alpha_1(e_x)$, and obtain the system

$$\begin{aligned}\dot{e}_x &= -k_1e_x + z_2 \\ \dot{e}_2 &= f_2(e_x, e_y, t) + g_2(e_x, e_y)\,u\end{aligned} \qquad (48)$$

where

$$\begin{aligned}f_2(e_x, e_y, t) &= ae_x + be_y - e_x\left(e_x^2 + 3x_1e_x + 3x_1^2\right) \\ &\quad + c\left[\cos t - \cos(0.4t)\right] - \dot{\alpha}_1\end{aligned} \qquad (49)$$

$$g_2 = 1 \qquad (50)$$

Take the controller as

$$u = \frac{1}{g_2(e_x, e_y, t)}\left(-f_2(e_x, e_y, t) - k_2e_2 - l_1g_1e_x\right) \qquad (51)$$

Substitute (41) into (41), we can obtain

$$\begin{bmatrix}\dot{e}_x \\ \dot{e}_2\end{bmatrix} = \begin{bmatrix}-k_1 & 1 \\ -l_1 & -k_2\end{bmatrix}\begin{bmatrix}e_x \\ e_2\end{bmatrix} \qquad (52)$$

The globally asymptotical stability of (42) can be assured according to theorem 1, that is $e_x, e_2 \to 0$. From the definition of e_2, we can get e_x, e_y coverage to zero with time increasing.

Take the parameters as $a = 1.8, \beta = -0.1, c = -1.1, l_1 = k_1 = k_2 = 2$, and select initial condition as $x_1(0) = 1, \quad y_1(0) = 1, \quad x_2(0) = 2, y_2(0) = 2$. With the control law (51), as we can see from Fig. 7 and Fig. 8, the slave system is driven to chaotic state gradually, which synchronizes with the master system.

5. CONCLUSION

In this paper, the design method based on tridiagonal structure has been proposed and used to synchronize

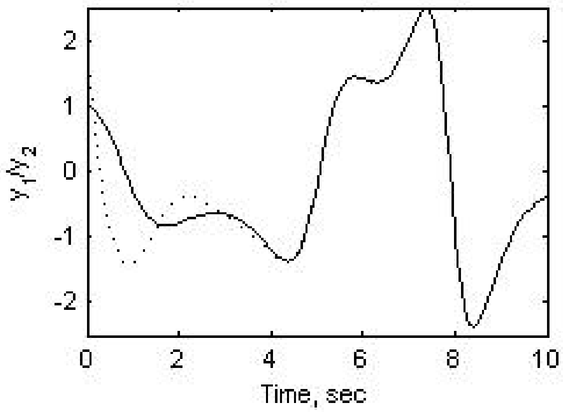

Fig. 8. Synchronized states y_1, y_2 of modified Duffing system

chaotic systems. The advantages of this method can be summarized as follows: (a) it presents a systematic procedure for selecting proper controllers in chaos synchronization; (b) it can be applied to a variety of chaotic systems with lower triangular structure. The technique has been successfully applied to the Lorenz system, Chua's circuit and Duffing system. Numerical simulations have verified the effectiveness of the method.

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