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Stability Analysis for a Class of Networked/Embedded Control Systems: Output Feedback Case

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Abstract: Motivated by the widespread use of networked and embedded control systems, an algorithm for stability analysis is proposed for sampled-data feedback control systems with uncertainly time-varying sampling intervals. The output feedback case with a dynamic compensator is considered. The algorithm is based on the robustness of related discrete-time systems against perturbation caused by the variation of sampling intervals. The validity of the algorithm is demonstrated by numerical examples.

1. INTRODUCTION

The sampled-data control theory (See Chen and Francis [1995] and references therein) has been well-developed in the last two decades, where the crucial properties is the periodicity of the closed-loop systems which comes from the periodic sampling. It is reasonable to consider the periodic sampling in the conventional implementation of sampled-data systems. We, however, recently encounter applications where the periodic sampling is almost impossible. In particular, resources for measurement and control are restricted in networked and/or embedded control systems (See Hristu-Varsakelis and Levine [2005], Hespanha et al. [2007] and references therein) and hence the sampling operation results to be aperiodic and uncertainly time-varying. In view of the widespread use of networked and/or embedded control systems, it is theoretically and practically important to study the robustness of such systems against variation of sampling intervals. One can find pioneering work for the issue in the literature including Walsh et al. [1999], Zhang et al. [2001], Zhang and Branicky [2001].

Recently the so-called input delay approach was proposed in Fridman et al. [2004], Yue et al. [2004] to treat the systems with aperiodic sampling, and a significant reduction of the conservatism is achieved. The basic idea of the approach is modeling the aperiodic sample and hold operations by a time-varying uncertain time delay at control input, and hence one can apply methodologies developed for delay systems to the aperiodic sampled-data systems. One can find applications of the input delay approach to several analysis and synthesis problems: Fridman et al. [2004], Yue et al. [2004], Fridman et al. [2005], Naghshtabrizi and Hespanha [2005], Yue et al. [2005], Suplin et al. [2007]. This approach also has inspired the discussion of the problem from the viewpoints of hybrid systems in Naghshtabrizi et al. [2006, 2007] and robust control in Mirkin [2007], Fujioka [2007b].

The existing results listed above verify the stability by showing the existence of a continuous-time Lyapunov function, although it might be implicit. As a different approach, one can check the stability by showing the existence of a discrete-time quadratic Lyapunov function as shown in Zhang and Branicky [2001]. They constructed a randomized algorithm for the search the Lyapunov function. Their algorithm, however, checks the quadratic stability of a set of discrete-time systems corresponding to finite number of prespecified sampling intervals between bounds of sampling intervals. In other words, the algorithm in Zhang and Branicky [2001] determines if a necessary condition (for a sufficient condition for the stability) holds or not, and hence cannot conclude the stability. In order to solve the issue, an algorithm of checking the quadratic stability for all sampling intervals uncertainly varying between given lower and upper bounds has been recently derived in Fujioka [2007a], by exploiting the robustness against perturbation caused by the variation of sampling intervals based on the small-gain condition.

The purpose of this paper is to extend the results in Fujioka [2007a], where the state feedback with a constant gain is considered, to a more practical setup. To be more concrete, we will consider the output feedback case where the controller is dynamic. The dynamic compensators have also been considered in the continuous-time approach, e.g., Fridman et al. [2005], Naghshtabrizi and Hespanha [2005], Suplin et al. [2007], mainly for the synthesis. They assume that the dynamics of the controller is given by differential equations, and the output of the controller generates the control input to the plant through sample-and-hold devices. In contrast the controller is given in terms of difference equations in this paper, and the output of the controller generates the control input to the plant through a zeroth-order hold device, as in the standard sampled-data control setup. The existing setups and that considered in this paper will be compared later in detail.

This paper is organized as follows: The problem is formulated in Section 2. Section 3 provides a stability criteria and an algorithm to verify the stability based on the criteria. The validity of the algorithm is demonstrated in Section 4.

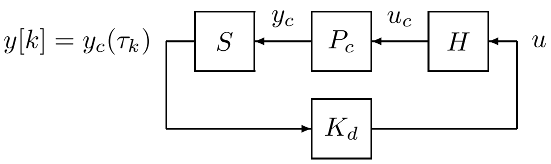

Fig. 1. Feedback control with aperiodic sample and hold actions

2. PROBLEM FORMULATION

Let the continuous-time plant P_c be given in the state-space form:

$$P_c : \begin{bmatrix} \dot{x}_c(t) \\ y_c(t) \end{bmatrix} = \begin{bmatrix} A_c & B_{c2} \\ C_2 & 0 \end{bmatrix} \begin{bmatrix} x_c(t) \\ u_c(t) \end{bmatrix} \tag{1}$$

where x_c, u_c, and y_c respectively denote the state, the control input, and the measurement output taking values in $\mathbb{R}^{n_c}$, $\mathbb{R}^{m_2}$, and $\mathbb{R}^{p_2}$. A_c, B_{c2}, and C_{c2} are matrices of compatible dimensions. The feedback compensator K_d, which is a discrete-time system, is also given in the state-space form:

$$K_d : \begin{bmatrix} x_K[k+1] \\ u[k] \end{bmatrix} = \begin{bmatrix} A_K & B_K \\ C_K & D_K \end{bmatrix} \begin{bmatrix} x_K[k] \\ y[k] \end{bmatrix} \tag{2}$$

where x_K, y, and u respectively denote the state, the input, and the output of the controller taking values in $\mathbb{R}^{n_K}$, $\mathbb{R}^{p_2}$, and $\mathbb{R}^{m_2}$. A_K, B_K, C_K, and D_K are matrices of compatible dimensions.

Finally we suppose that P_c and K_d are connected in feedback to compose the closed-loop system T, as depicted in Fig. 1, through the following sample and hold operations:

- The k-th input to the controller is the sampled version of the measured output of the plant at $t = \tau_k$ ($k = 0, 1, \ldots$)
$$y[k] = y_c(\tau_k)$$
where $\{\tau_k\}$ is an uncertain set of discrete time instances satisfying
$$\tau_0 = 0$$
and
$$0 < h_\ell \le \tau_{k+1} - \tau_k \le h_u < \infty \tag{3}$$
for given h_ℓ and h_u.
- The control input u_c is determined by the output of the controller in the zero-th order hold fashion:
$$u_c(t) = u[k], \quad {}^\forall t \in [\tau_k,\, \tau_{k+1}). \tag{4}$$

Applications of this scenario can be found in networked and/or embedded control systems, where resources for measurement and control are restricted. Readers are referred to, e.g., Hristu-Varsakelis and Levine [2005], Hespanha et al. [2007]. Note that (3) implies

$$\lim_{k \to \infty} \tau_k = \infty$$

since $h_\ell > 0$.

The purpose of this paper is to provide stability criteria for T. If τ_k's satisfy

$$\tau_{k+1} - \tau_k = \tilde{h}$$

for some $\tilde{h} \in [h_\ell,\, h_u]$, the resultant feedback control system is periodic. This special scenario is the one well-studied in the so-called sampled-data control theory, e.g., Chen and Francis [1995]. Indeed the stability can be easily verified by checking the spectral radius of $\Phi(\tilde{h})$ for the special scenario, where

$$\Phi(h) := \begin{bmatrix} A(h) + B_2(h) D_K C_2 & B_2(h) C_K \\ C_2 B_K & A_K \end{bmatrix}, \tag{5}$$

$$A(h) := \mathrm{e}^{Ah}, \quad B_2(h) := \int_0^h \mathrm{e}^{A(h-\eta)} B \,\mathrm{d}\eta, \quad C_2 := C_{c2}.$$

It is, however, obvious that our general scenario is much more complicated, because of the uncertainly time-varying nature.

The scenario given above can be regarded as a natural generalization of the standard sampled-data control setup to the time-varying sampler. Here we compare it to the existing setups in the literature from the following three viewpoints:

- The sample and the hold operations are synchronized or not.
- The control input u_c is piece-wise constant or not.
- The discrete-time controller is time-invariant or not.

In our setup, the sample and the hold operations are synchronized, u_c is piece-wise constant, and K_d is time-invariant.

The type 2 controller in Suplin et al. [2007] is similar to our controller; the sample and the hold operations are synchronized, and u_c is piece-wise constant, but the controller is time-varying in the discrete-time domain. The type 2 controller generates u_c from y as follows:

$$\begin{cases} \dot{x}_C(t) = A_{C0} x_C(t) + A_{C1} x_C(\tau_k) + B_C y[k], \\ u_c(t) = C_C x_C(\tau_k) \end{cases}$$

where $t \in [\tau_k,\, \tau_{k+1})$ and x_C is the state of the controller. This can be recovered by (4) and

$$\begin{bmatrix} \bar{x}_K[k+1] \\ u[k] \end{bmatrix} = \begin{bmatrix} \bar{A}_K[k] & \bar{B}_K[k] \\ C_C & 0 \end{bmatrix} \begin{bmatrix} \bar{x}_K[k] \\ y[k] \end{bmatrix}$$

instead of (2), where

$$\bar{A}_K[k] := \mathrm{e}^{A_{C0}(\tau_{k+1} - \tau_k)} + \int_{\tau_k}^{\tau_{k+1}} \mathrm{e}^{A_{C0}(\tau_{k+1} - \eta)} A_{C1} \,\mathrm{d}\eta,$$

$$\bar{B}_K[k] := \int_{\tau_k}^{\tau_{k+1}} \mathrm{e}^{A_{C0}(\tau_{k+1} - \eta)} B_C \,\mathrm{d}\eta.$$

One can similarly show that all the discrete-time controllers considered in Fridman et al. [2005], Naghshtabrizi and Hespanha [2005], Suplin et al. [2007] are time-varying, although they are not explicitly given. The time dependence could improve the performance, but it would be practical to consider the time-invariant discrete-time dynamics which results a simple implementation.

The control input u_c is not supposed to be piece-wise constant in Fridman et al. [2005], the type 1 controller in Suplin et al. [2007], and the anticipative controller in Naghshtabrizi and Hespanha [2005].

The asynchronous sample and hold operations are considered in Naghshtabrizi and Hespanha [2005] and the type 3 controller in Suplin et al. [2007] where, if the control input is piece-wise constant, (4) is replaced by

$$u_c(t) = u[k], \quad {}^\forall t \in [\sigma_k,\, \sigma_{k+1})$$

and σ_k's denote the instants updating the control input. It would be natural to consider the case of $\sigma_k \ge \tau_k$ according to time delay at the path from K_d to P_c, as considered in Naghshtabrizi and Hespanha [2005]. The discussion in the sequel can be extended for the case by using the transformation in Hara et al. [1994] if the delay is constant. The method developed in Kao and Rantzer [2007] could be combined for the time-varying delay.

3. MAIN RESULTS

3.1 The Discrete-Time Approach

The purpose of this paper is to extend the results in Fujioka [2007a], where the state feedback with a constant gain is considered, for the more practical setup provided in the previous section. To be more concrete we will verify the stability of T based on the following lemma which is an extension of that in Zhang and Branicky [2001], Hespanha et al. [2007] for the state feedback case:

Lemma 1. The origin of T is exponentially stable, i.e., there exist α, $\beta > 0$ satisfying

$$\left\| \begin{bmatrix} x_c(t) \\ x_K[k] \end{bmatrix} \right\| \le \alpha \mathrm{e}^{-\beta t} \left\| \begin{bmatrix} x_c(0) \\ x_K[0] \end{bmatrix} \right\|$$

for all $t \in [\tau_k, \tau_{k+1})$ and $k = 0, 1, \ldots$, if there exists a matrix $0 < P = P^* \in \mathbb{R}^{n\times n}$ satisfying

$$(\Phi(h))^* P \Phi(h) - P < 0 \tag{6}$$

for all $h \in [h_\ell, h_u]$, where $\Phi(\cdot)$ is defined in (5).

The proof is parallel to that for the original lemma so it is omitted.

Note that Lemma 1 is based on the quadratic stability of the accompanying discrete-time system T_d defined by

$$\xi[k+1] = \Phi(\tau_{k+1} - \tau_k)\xi[k]$$

with the discrete-time Lyapunov function

$$V(\xi[k]) := \xi^*[k] P \xi[k]$$

where $\xi[k] := x(\tau_k)$.

It is hard to find a matrix P in Lemma 1 since the inequality must hold for all values in $[h_\ell, h_u]$, and Φ is a strongly nonlinear function of h.

For the state feedback case, Zhang and Branicky [2001] proposed a randomized algorithm to search a P on a grid between h_ℓ and h_u, although it determines a necessary condition and cannot conclude the stability. Fujioka [2007a] developed an algorithm which can verify the stability based on the robustness of discrete-time systems against the perturbation caused by the variation of sampling interval.

The basic idea of the stability analysis in this paper follows that in Fujioka [2007a], i.e., we will verify the existence of $P > 0$ satisfying (6) for all $h \in [h_\ell, h_u]$ from that of $P > 0$ satisfying (6) for all $h \in \mathcal{G}$ where $\mathcal{G}$ is a grid:

$$\mathcal{G} = \{h_1, h_2, \ldots, h_N\} \subset [h_\ell, h_u].$$

Hence in the sequel we will first discuss the robustness of systems with uniform sampling interval against the perturbation caused by the variation of sampling interval. After that we will derive a method to combine robustness results for each point of the grid.

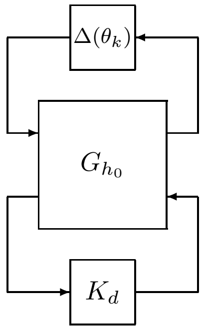

Fig. 2. Alternative representation of T_d

3.2 Stability Criteria

In order to discuss the robustness against the variation of sampling interval, we consider the following manipulation of Φ: Fix $h_0 \in (h_\ell, h_u)$ and then one can define θ_k so that

$$\tau_{k+1} - \tau_k = h_0 + \theta_k.$$

One has the following property, which is simple but plays a key role in this paper:

Proposition 2. The function $\Phi(\cdot)$ defined in (5) satisfies

$$\Phi(\tau_{k+1} - \tau_k) = \Phi(h_0) + \Gamma \Delta(\theta_k) \Psi(h_0), \tag{7}$$

where

$$\Gamma := \begin{bmatrix} B_1 \\ 0 \end{bmatrix}, \quad B_1 := I,$$

$$\Psi(h) := \left[C_1(h) + D_{12}(h) D_K C_2 \;\; D_{12}(h) C_K \right], \tag{8}$$

$$C_1(h) := A_c A(h), \quad D_{12}(h) := A_c B_2(h) + B_{c2},$$

$$\Delta(\theta) := \int_0^\theta \mathrm{e}^{A\eta}\, \mathrm{d}\eta.$$

Proof. By definition one has

$$\begin{aligned} A(\tau_{k+1} - \tau_k) &= \mathrm{e}^{A_c(h_0+\theta_k)} = \mathrm{e}^{A_c\theta_k} A(h_0) \\ &= (I + \Delta(\theta_k) A_c)\, A(h_0) \\ &= A(h_0) + \Delta(\theta_k) C_1(h_0), \end{aligned}$$

and

$$\begin{aligned} &B_2(\tau_{k+1} - \tau_k) \\ &= \int_0^{h_0+\theta_k} \mathrm{e}^{A_c(h_0+\theta_k-\eta)} B_{c2}\, \mathrm{d}\eta \\ &= \int_0^{h_0} \mathrm{e}^{A_c(h_0+\theta_k-\eta)} B_{c2}\, \mathrm{d}\eta + \int_{h_0}^{h_0+\theta_k} \mathrm{e}^{A_c(h_0+\theta_k-\eta)} B_{c2}\, \mathrm{d}\eta \\ &= \mathrm{e}^{A\theta_k} B_2(h_0) + \Delta(\theta_k) B_{c2} \\ &= (I + \Delta(\theta_k) A_c) B_2(h_0) + \Delta(\theta_k) B_{c2} \\ &= B_2(h_0) + \Delta(\theta_k) D_{12}(h_0). \end{aligned}$$

Then it is straightforward to derive (7) by substituting the above results. This completes the proof.

Now one can regard T_d as a feedback connection of an LTI discrete-time system Σ_{h_0}:

$$\Sigma_{h_0}[z] := \Psi(h_0)(zI - \Phi(h_0))^{-1}\Gamma$$

and a time-varying matrix $\Delta(\theta_k)$. See Fig. 2, where we in addition use the identity

$$\Sigma_{h_0}[z] = \mathcal{F}_\ell\left(G_{h_0}, K\right)[z],$$

$$G_h[z] := \begin{bmatrix} C_1(h) \\ C_2 \end{bmatrix} (zI - A(h))^{-1} \left[B_1 \;\; B_2(h)\right] + \begin{bmatrix} 0 & D_{12}(h) \\ 0 & 0 \end{bmatrix}.$$

Thus we obtain the following lemma as a simple application of the small-gain theorem [1]:

Lemma 3. Let an interval $\mathcal{H} \subseteq (0, \infty)$ be given. There exists a matrix $0 < P = P^* \in \mathbb{R}^{n\times n}$ satisfying (6) for all $h \in \mathcal{H}$ if $\rho(\Phi(h_0)) < 1$ and

$$\gamma \left\|\Delta(\theta)\right\| \leq 1 \tag{9}$$

for all $\theta \in \mathcal{H} - h_0$, where γ is an upper bound of $\|\Sigma\|_\infty$:

$$\gamma > \|\Sigma\|_\infty . \tag{10}$$

Since minimization of γ in (10) is routine, one can verify the stability from (9) by bounding $\|\Delta(\theta)\|$. For the purpose, as in Fujioka [2007a], we invoke the following property found in, e.g., Loan [1977]:

Lemma 4. For given $A \in \mathbb{R}^{n\times n}$ and $t \geq 0$ one has

$$\left\|\mathrm{e}^{At}\right\| \leq \mathrm{e}^{\mu(A)t} \tag{11}$$

where $\mu(A)$ denotes the log norm of A:

$$\mu(A) = \lambda_{\max}\left(\frac{A + A^*}{2}\right).$$

Remark 1. One can continue the following discussion by replacing the bound in (11) by other bounds found in, e.g., Loan [1977], Kågström [1977].

In order to state the main results of this paper we need the following interval defined with given $h > 0$ and $\gamma > 0$:

$$\mathcal{H}(h, \gamma) := [h_L, h_U] \cap (0, \infty) \tag{12}$$

where h_L and h_U are given as follows:

L1) If $\mu(-A_c) = 0$, $h_L = h - \gamma^{-1}$,
L2) else if $\mu(-A_c) \leq -\gamma$, $h_L = -\infty$,
L3) else

$$h_L = h - \frac{1}{\mu(-A_c)} \log\left(1 + \gamma^{-1}\mu(-A_c)\right).$$

U1) If $\mu(A_c) = 0$, $h_U = h + \gamma^{-1}$,
U2) else if $\mu(A_c) \leq -\gamma$, $h_U = \infty$,
U3) else

$$h_U = h + \frac{1}{\mu(A_c)} \log\left(1 + \gamma^{-1}\mu(A_c)\right).$$

Now we are ready to state the basic robustness results, which is a natural generalization of that in Fujioka [2007a]:

Theorem 5. Let $h_0 > 0$ be given so that $\rho(\Phi(h_0)) < 1$. For $\gamma > 0$ satisfying (10), there exists a matrix $0 < P = P^* \in \mathbb{R}^{n\times n}$ satisfying (6) for all $h \in \mathcal{H}(h_0, \gamma)$.

Proof. See Appendix.

3.3 Algorithm for Stability Analysis

Theorem 5 provides a robustness condition for T based on the property of the nominal system determined by the fixed sampling period h_0. A direct use of Theorem 5, however, can be conservative in the sense that there might not exist an $h_0 > 0$ such that $[h_\ell, h_u] \subseteq \mathcal{H}(h_0, \gamma)$ even though there exists a matrix P satisfying (6) for all $h \in [h_\ell, h_u]$, mainly because of the small-gain type modeling of $\Delta(\theta_k)$.

In order to reduce the conservatism we introduce the multi-model to obtain the following theorem which is a natural generalization of that in Fujioka [2007a]:

Theorem 6. Let $h_i > 0$ $(i = 1, 2, \ldots, N)$ be given. If there exist a matrix $0 < X = X^* \in \mathbb{R}^{n\times n}$ and $\alpha_i > 0$ $(i = 1, 2, \ldots, N)$ satisfying N matrix inequalities

$$\begin{bmatrix} \Phi(h_i) & \Gamma \\ \Psi(h_i) & 0 \end{bmatrix} \begin{bmatrix} X & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \Phi(h_i) & \Gamma \\ \Psi(h_i) & 0 \end{bmatrix}^* - \begin{bmatrix} X & 0 \\ 0 & \alpha_i I \end{bmatrix} < 0 \tag{13}$$

then (6) is satisfied with $P = X^{-1}$ for all

$$h \in \bigcup_{i=1}^{N} \mathcal{H}(h_i, \sqrt{\alpha_i})$$

where $\Phi(\cdot)$, $\Psi(\cdot)$, $\mathcal{H}(\cdot, \cdot)$ are defined in (5), (8), and (12), respectively.

Proof. The proof is parallel to that in Fujioka [2007a]. Consider the case $i = 1$. The condition (13) with $i = 1$ is an equivalent representation of

$$\left\|\Psi(h_1)(zI - \Phi(h_1))^{-1}\Gamma\right\|_\infty < \sqrt{\alpha_i}.$$

Hence, by invoking Theorem 5, there exists a matrix $0 < P = P^* \in \mathbb{R}^{n\times n}$ satisfying (6) for all $h \in \mathcal{H}(h_1, \sqrt{\alpha_1})$. Moreover we can verify that one of such P is given by X^{-1} from the standard procedure. With similar discussion, we can conclude that there exists a matrix $0 < P = P^* = X^{-1} \in \mathbb{R}^{n\times n}$ satisfying (6) for all $h \in \mathcal{H}(h_i, \sqrt{\alpha_i})$, $i = 2, \ldots, N$. This concludes the proof.

Once we find a matrix $P > 0$ satisfying (6) on a grid by any methods, e.g., one proposed in Zhang and Branicky [2001], we can verify the robustness by invoking Theorem 6. In this paper we propose the following concrete algorithm instead for stability analysis which is again a natural generalization of that in Fujioka [2007a] and generates a grid effectively based on Theorem 6:

Algorithm 1. Given $0 < h_\ell < h_u < \infty$, and a large positive integer N_0.

0. Initialization: $\mathcal{G} \leftarrow \{(h_\ell + h_u)/2\}$
1. If there exists an $h \in \mathcal{G}$ satisfying $\rho(\Phi(h)) \geq 1$, the origin of T is unstable. Stop.
2. If $\#(\mathcal{G}) \geq N_0$, stop without deciding the stability of the origin of T. Here $\#(\mathcal{G})$ denotes the number of elements in $\mathcal{G}$.
3. Minimize

$$\sum_{i=1}^{\#(\mathcal{G})} \frac{\beta_i}{\|\Sigma_i\|_\infty^2}$$

subject to

$$\begin{bmatrix} \Phi(h_i) & \Gamma \\ \Psi(h_i) & 0 \end{bmatrix} \begin{bmatrix} X & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \Phi(h_i) & \Gamma \\ \Psi(h_i) & 0 \end{bmatrix}^* - \begin{bmatrix} X & 0 \\ 0 & \beta_i I \end{bmatrix} < 0$$

for all h_i's and $X > 0$, where

$$\Sigma_i[z] := \Psi(h_i)(zI - \Phi(h_i))^{-1}\Gamma,$$

and h_i is the i-th smallest element in $\mathcal{G}$.
4. If

$$[h_\ell, h_u] \subseteq \bigcup_{i=1}^{\#(\mathcal{G})} \mathcal{H}(h_i, \sqrt{\alpha_i}),$$

the origin of T is exponentially stable. Stop. Here

$$\alpha_i := \lambda_{\max}(R_i - S_i^*(Q_i - X_i)^{-1}S_i) + \varepsilon$$

[1] Readers are referred to, e.g., Khargonekar et al. [1990] on the relationship between the quadratic stability and the small-gain condition.

where ε is a small positive number and

$$\begin{bmatrix} Q_i & S_i \\ S_i^* & R_i \end{bmatrix} := \begin{bmatrix} \Phi(h_i) & \Gamma \\ \Psi(h_i) & 0 \end{bmatrix} \begin{bmatrix} X & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \Phi(h_i) & \Gamma \\ \Psi(h_i) & 0 \end{bmatrix}^* .$$

5. Update $\mathcal{G}$ by

$$\mathcal{G} \leftarrow \mathcal{G} \cup \{(L_j + U_j)/2\}$$

for all $j \in \{1, 2, \ldots, M\}$ where L_j, U_j and M are determined so that

$$L_1 < U_1 < L_2 < U_2 < \cdots < L_M < U_M,$$

$$\bigoplus_{j=1}^{M} (L_j, U_j) = (h_\ell, h_u) \backslash \left((h_\ell, h_u) \cap \bigcup_{i=1}^{\#(\mathcal{G})} \mathcal{H}(h_i, \sqrt{\alpha_i}) \right)$$

are satisfied. Go to Step 1.

Algorithm 5 has similar features to that in Fujioka [2007a]: Step 2 is introduced to avoid numerical issues which could happen when $\#(\mathcal{G})$ is too large. The performance of the algorithm can be tuned by modifying the objective function in Step 3. Note that α_i satisfies (13) with X determined in Step 3 and $\alpha_i \leq \beta_i$ with sufficiently small ε. The integer M in Step 4 is $\#(\mathcal{G}) + 1$ at most.

4. ILLUSTRATIVE EXAMPLE

Consider the following parameters found in Naghshtabrizi and Hespanha [2005]:

$$A_c = \begin{bmatrix} 0 & 1 \\ 0 & -0.1 \end{bmatrix}, \quad B_{c2} = \begin{bmatrix} 0 \\ 0.1 \end{bmatrix}, \quad C_{c2} = [1 \ 0] .$$

They have reported that their results guarantee the stability of closed-loop system by the following observer-based controller:

$$\dot{x}_C(t) = A_c x_C(t) + B_{c2} u_c(t) + L(y[k] - C_{c2} x_C(\tau_k))$$
$$u_c(t) = -F x_C(\sigma_k)$$

if $\tau_k = kh^s$, $h^s = 0.5$ and $\sigma_k = kh^a$ as long as $h^a \leq 0.733$, where F and L are given by

$$F = [3.3348 \ 9.9103] , \quad L = \begin{bmatrix} 0.6772 \\ 0.1875 \end{bmatrix} .$$

By using the parameters we determine K_d by

$$A_K = \mathrm{e}^{A_c h^s} - \int_0^{h^s} \mathrm{e}^{A_c(h^s - \eta)} (BF + LC) \, \mathrm{d}\eta,$$

$$B_K = \int_0^{h^s} \mathrm{e}^{A_c(h^s - \eta)} L \, \mathrm{d}\eta, \quad C_K = -F, \quad D_K = 0$$

and check the stability of the closed-loop system.

We have implemented Algorithm 1 on MATLAB 7.4 with Robust Control Toolbox as an LMI solver and YALMIP (R20070810) in Löfberg [2004] as an LMI parser. The stability of the closed-loop system is verified by Algorithm 1 for $h_\ell = 0.17$ and $h_u = 0.88$, with

$$P = X^{-1} = \begin{bmatrix} 1.0540 & -0.5882 & -1.1648 & 0.1586 \\ -0.5882 & 2.8879 & 0.3702 & -2.5441 \\ -1.1648 & 0.3702 & 1.7945 & 1.2804 \\ 0.1586 & -2.5441 & 1.2804 & 7.1118 \end{bmatrix} \times 10^{-3}.$$

We cannot compare the results to that in Naghshtabrizi and Hespanha [2005] since the formulation is different, however, h_u is larger than 0.733, which is guaranteed for the time-varying discrete-time component, while K_d is time-invariant. Hence we could conclude the effectiveness of the proposed analysis method.

The search took 29.16 [s] on a laptop with Intel Core Solo (1.20GHz) running Linux 2.6.22, and the maximal $\#(\mathcal{G})$ in the search was 48.

5. CONCLUDING REMARKS

We have considered the stability of sampled-data feedback control systems where the measurement output is sampled aperiodically, motivated by the widespread use of networked and embedded control systems.

We have proposed a stability analysis algorithm by showing robustness of sampled-data systems against perturbation caused by variation of sampling intervals based on the small-gain framework, as a generalization of Fujioka [2007a] for the state feedback case. The effectiveness of the proposed algorithm has been shown by numerical examples.

In this paper we have considered an analysis problem, however, application to more practical analysis and synthesis problems are not hard and under development.

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Proof of Theorem 5

We here prove that (9) holds for all $h \in [h_0, h_U]$. The proof for the interval $[h_L, h_0]$ is similar so it is omitted. Although the proof is almost the same to that in Fujioka [2007a] we show it for the paper completeness. Note that $H(h_0, \gamma) \subseteq [h_L, h_U]$.

Invoking Lemma 4 we have

$$\|\Delta(\theta)\| \le \int_0^\theta \left\|\mathrm{e}^{A_c t}\right\| \mathrm{d}t \le \int_0^\theta \mathrm{e}^{\mu(A_c)t}\,\mathrm{d}t$$

when $\theta \ge 0$. If $\mu(A_c) = 0$

$$\|\Delta(\theta)\| \le \theta.$$

Hence (9) holds as long as $\gamma\theta \le 1$. This completes the proof for the case U1.

Let us next consider the case of $\mu(A_c) \ne 0$. In this case

$$\|\Delta(\theta)\| \le \frac{\mathrm{e}^{\mu(A_c)\theta} - 1}{\mu(A_c)}. \qquad (.1)$$

Suppose that $\mu(A_c) < 0$. Noting that the right hand side goes to $-1/\mu(A_c)$ when θ tends to ∞. Hence (9) holds for all $\theta > 0$ if

$$-\frac{\gamma}{\mu(A_c)} \le 1.$$

This completes the proof for the case U2.

Finally let us consider the case of $\mu(A_c) \ne 0$ and

$$-\frac{\gamma}{\mu(A_c)} > 1.$$

The small gain condition (9) holds for all $\theta > 0$ if

$$\gamma \frac{\mathrm{e}^{\mu(A_c)\theta} - 1}{\mu(A_c)} \le 1.$$

Noting that $1 + \gamma^{-1}\mu(A_c) > 0$ in this case, this condition turns to

Case A) If $\mu(A_c) > 0$

$$\mu(A_c)\theta \le \log(1 + \gamma^{-1}\mu(A_c)).$$

Case B) If $\mu(A_c) < 0$

$$\mu(A_c)\theta \ge \log(1 + \gamma^{-1}\mu(A_c)).$$

Hence we have

$$\theta \ge \frac{1}{\mu(A_c)} \log(1 + \gamma^{-1}\mu(A_c)).$$

for both cases. This completes the proof for the case U3.