Proceedings of the 17th World Congress
The International Federation of Automatic Control
Seoul, Korea, July 6-11, 2008

Solutions in nonantagonistic positional differential games with vector payoff functionals ⋆

A.F.Kleimenov *

* *Institute of Mathematics and Mechanics*
Ural Branch of Russian Academy of Sciences,
S.Kovalevskaya str. 16, Ekaterinburg 620066, Russia,
e-mail: kleimenov@imm.uran.ru

Abstract: Nonantagonistic two-person differential game with fixed final time and with vector terminal payoff functionals for players is considered. Both the players act in the class of strategies defined as arbitrary functions of a position of the game and a precision parameter. A formalization of the game is given. A concept of a solution of the game is introduced. It is shown that the problem of finding the solutions of the game is reduced to a problem of constructing program controls of players satisfying some special conditions. Two examples are considered. The first one is devoted to control motions of a material point on the plane. The second one deals with repeated prisoners' dilemma.

1. INTRODUCTION

At present the theory of differential games with scalar payoff functionals for players is developed sufficiently well (see, for example, the monographs: Krasovskii [1985], Krasovskii and Subbotin [1988], Basar and Olsder [1999]). Considerably less papers deal with differential games with vector payoff functionals. Note the monograph Zhukovskiy and Salukvadze [1994] which is devoted to vector-valued problems under uncertainty.

This paper deals with nonantagonistic two-person positional differential games with vector payoff functionals. Formalization of players' strategies and motions generated by them is similar to the formalization introduced in Krasovskii [1985], Krasovskii and Subbotin [1988] with the exception of technical details (see Kleimenov [1993]).

Binary preference relations on the set of collections of sets in a finite-dimensional Euclidean space such as $P(i)$- and $S(i)$- dominations are introduced. They are generalizations of well-known concepts of Pareto and Sleiter optimality. On this base a concept of a solution is defined.

The paper is organized as follows. Section 2 contains formalization of strategies and motions. Admissible pairs of strategies are defined. Definition of a solution of the game is given in Section 3. It is proved that the problem of finding the solutions of the game is reduced to a problem of constructing program controls of players satisfying some special conditions. Section 4 contains two examples. The first one deals with control motions of a material point on the plane. The second one deals with repeated prisoners' dilemma.

⋆ This work was supported by the Russian Foundation for Basic Researches, project 06-01-00436.

2. FORMALIZATION OF TWO-PERSON NONANTAGONISTIC POSITIONAL DIFFERENTIAL GAMES WITH VECTOR PAYOFF FUNCTIONALS

2.1 Dynamics and payoff functionals

Let dynamics of two-person nonantagonistic positional differential game be described by the equation

$$\dot{x} = f(t, x, u, v), \quad t \in [t_0, \theta], \quad x(t_0) = x_0 \qquad (1)$$

where $x \in \mathbb{R}^n$ is a phase vector, the controls $u \in P \in comp\,\mathbb{R}^p$ and $v \in Q \in comp\,\mathbb{R}^q$ are handled by Player 1 (P1) and Player 2 (P2), respectively, and θ is a fixed final time.

Let G be a compact set in $\mathbb{R}^1 \times \mathbb{R}^n$ whose projection on the time axis is equal to the given interval $[t_0, \theta]$. We assume, that all the trajectories of system (1), beginning at an arbitrary position $(t_*, x_*) \in G$, remain within G for all $t \in [t_*, \theta]$.

Let the following assumptions be fulfilled.

1^0. The function $f : G \times P \times Q \mapsto \mathbb{R}^n$ is continuous over the set of arguments, and satisfies the Lipschitz condition with respect to x.

2^0. There exists a constant $\lambda > 0$ such that

$$\|f(t, x, u, v)\| \le \lambda(1 + \|x\|)$$

for all $(t, x) \in G,\ u \in P,\ v \in Q$.

3^0. The function $f(t, x, u, v)$ satisfies the condition

$$\max_{u\in P}\min_{v\in Q} s^T f(t, x, u, v) = \min_{v\in Q}\max_{u\in P} s^T f(t, x, u, v)$$

for any $s \in \mathbb{R}^n$ and $(t, x) \in G$.

Here and below the upper symbol ^{T} denotes the operation of transposition.

Player i chooses his control to maximize the vector payoff functional

978-1-1234-7890-2/08/$20.00 © 2008 IFAC

10.3182/20080706-5-KR-1001.2793

$$I_i = \sigma_i(x(\theta)) = (\sigma_{i1}(x(\theta)), ..., \sigma_{il_i}(x(\theta))), \quad i = 1, 2, \quad (2)$$

where $\sigma_{ij} : R^n \to R^1$, $i = 1, 2$, $j = 1, ..., l_i$ are given continuous functions.

2.2 Formalization of strategies

Suppose that both players have complete information about the current position $(t, x(t))$ of the game. The formalization of players' strategies and of motions generated by them in nonantagonistic positional differential games is similar to the formalization introduced for antagonistic positional differential games in Krasovskii [1985], Krasovskii and Subbotin [1988] with the exception of technical details (see Kleimenov [1993]).

A pure strategy (or strategy for short) of P1 is identified with a pair $U \div \{u(t, x, \varepsilon), \ \beta_1(\varepsilon)\}$, where $u(\cdot)$ is an arbitrary function depending on the position (t, x) and on a positive precision parameter ε and having values in the set P. The function $\beta_1 : (0, \infty) \mapsto (0, \infty)$ is a continuous monotone one and satisfies the condition $\beta_1(\varepsilon) \to 0$ if $\varepsilon \to 0$.

The function $\beta_1(\cdot)$ has the following sense. For a fixed ε the value $\beta_1(\varepsilon)$ is the upper bound for the step of a subdivision of the interval $[t_0, \theta]$ which P1 uses for forming step-by-step motion.

A strategy $V \div \{v(t, x, \varepsilon), \beta_2(\varepsilon)\}$ of P2 is defined analogously.

Remind that in antagonistic positional differential games theory a strategy of P1 is identified with a function $U^a \div u(t, x, \varepsilon)$ and a strategy of P2 is identified with a function $V^a \div v(t, x, \varepsilon)$ (see Krasovskii [1985]).

2.3 Formalization of motions

Motions of two types: approximated (step-by-step) ones and ideal (limit) ones are considered as motions generated by a pair of strategies of players.

Approximated motion $x[\cdot, t_0, x_0, U, \varepsilon_1, \Delta_1, V, \varepsilon_2, \Delta_2]$ is introduced for fixed values of players' precision parameters ε_1 and ε_2 and for fixed subdivisions $\Delta_1 = \{t_i^{(1)}\}$ and $\Delta_2 = \{t_j^{(2)}\}$ of the interval $[t_0, \theta]$ chosen by P1 and P2, respectively, under the conditions

$$\delta(\Delta_i) \leq \beta_i(\varepsilon_i), \ i = 1, 2. \quad (3)$$

Here step of subdivision Δ_i is $\delta(\Delta_i) = \max_k (t_{k+1}^{(i)} - t_k^{(i)})$.

A limit motion generated by the pair of strategies (U, V) from the initial position (t_0, x_0) is a continuous function $x[t] = x[t, t_0, x_0, U, V]$ for which there exists a sequence of approximated motions

$$\{x[t, t_0^k, x_0^k, U, \varepsilon_1^k, \Delta_1^k, V, \varepsilon_2^k, \Delta_2^k]\}$$

uniformly converging to $x[t]$ on $[t_0, \theta]$ as

$$k \to \infty, \quad \varepsilon_1^k \to 0, \quad \varepsilon_2^k \to 0, \quad t_0^k \to t_0,$$
$$x_0^k \to x_0, \quad \delta(\Delta_i^k) \leq \beta_i(\varepsilon_i^k).$$

A pair of strategies (U, V) generates a nonempty compact (in the metric of the space $C[t_0, \theta]$) set $X(t_0, x_0, U, V)$ consisting of limit motions $x[\cdot, t_0, x_0, U, V]$.

In Subsection 3.2 we use the sets of motions $X(t_0, x_0, U^a)$ and $X(t_0, x_0, V^a)$ generated by the strategy of P1 $U^a \div u(t, x, \varepsilon)$ and by the strategy of P2 $V^a \div v(t, x, \varepsilon)$, respectively. Refine the determination of these sets. For instance, a motion of the set $X(t_0, x_0, U^a)$ is determined as an uniform limit for some sequence of approximated motions

$$\left\{x\left[t, t_o^k, x_0^k, U^a, \varepsilon^k, \Delta_1^k, v^k[\cdot]\right]\right\}$$

as $k \to \infty$, $\varepsilon^k \to 0$, $t_0^k \to t_0$, $x_0^k \to x$ and $v^k[\cdot]$ are measurable realizations of control of P2. Note that the condition (3), $i = 1$ is lacking here.

2.4 Guaranteed payoffs for players

Let a pair strategies (U, V) and a position $(t_*, x_*) \in G$ be fixed. Define the sets

$$\Gamma_i(t_*, x_*, U, V) = P^{\min}(\sigma_i(X(t_*, x_*, U, V)), \quad i = 1, 2$$

where

$$\sigma_i(X(t_*, x_*, U, V)) = \{z \in R^{l_i}, \quad z = \sigma_i(x[\theta]),$$
$$x[\cdot] \in X(t_*, x_*, U, V)\}$$

and the symbol $P^{\min}(A)$ denotes the set of Pareto-minimal elements of the set $A \in R^k$.

The sets $\Gamma_i(t_*, x_*, U, V)$ are nonempty compacts. Elements of set $\Gamma_i(t_*, x_*, U, V)$ can be interpreted as "guaranteed payoffs" of i-th player when both players use strategies (U, V). The guarantee is understood in the following sense: for any motion $x[\cdot] \in X(t_*, x_*, U, V)$ there exists a vector $\xi \in \Gamma_i(t_*, x_*, U, V)$ such that the inequality $\sigma_i(x(\theta) \geq \xi$ holds.

Here and below the vector inequality $\geq$ denotes the same inequality for corresponding coordinates.

2.5 Binary preference relations

Introduce the following binary preference relations.

Let A, B, A_i and B_i be nonempty sets in the space R^k.

Definition 1. A set A ρ - dominates a set B (use the notation $A\rho B$) if for any $a \in A$ there exists $b \in B$ such that $a \geq b$, and moreover, for at least one such pair $a \neq b$.

Definition 2. A pair of sets (A_1, A_2) $S(i)$ - dominates a pair of sets (B_1, B_2), $i = 1, 2$, if the condition $A_i \rho B_i$ is fulfilled.

Definition 3. A pair of sets (A_1, A_2) $P(i)$ - dominates a pair of sets (B_1, B_2), $i = 1, 2$, if the condition $A_i \rho B_i$ is fulfilled, and moreover, the set B_{3-i} does not ρ - dominate the set A_{3-i}.

Definition 4. A pair of sets (A_1, A_2) $P(1, 2)$ - dominates a pair of sets (B_1, B_2), $i = 1, 2$, if the pair (A_1, A_2) $P(i)$ - dominates the pair (B_1, B_2) for at least one $i \in \{1, 2\}$.

2.6 Admissible pairs of strategies

The next definition determines admissible pair of strategies in the considered game.

Definition 5. A pair of strategies $\left(\tilde{U}, \tilde{V}\right)$ is called an admissible one in the game, if for any trajectory $\tilde{x}[\cdot] \in X(t_0, x_0, \tilde{U}, \tilde{V})$, any $\tau \in [t_0, \theta)$, and any strategies U and V the pair of sets $(\Gamma_1(\tau, \tilde{x}(\tau), \tilde{U}, \tilde{V}),\ \Gamma_2(\tau, \tilde{x}(\tau), \tilde{U}, \tilde{V}))$ is not $S(1)$ - dominated by the pair of sets $(\Gamma_1(\tau, \tilde{x}(\tau), U, \tilde{V}),\ \Gamma_2(\tau, \tilde{x}(\tau), U, \tilde{V}))$ and is not $S(2)$ - dominated by the pair of sets $(\Gamma_1(\tau, \tilde{x}(\tau), \tilde{U}, V),\ \Gamma_2(\tau, \tilde{x}(\tau), \tilde{U}, V))$.

It follows from this definition that it does not pay for each player to deviate from admissible pair of strategies at any instant of time $\tau \in [t_0, \theta)$. The utility of deviation is understood as regards $S(i)$-domination of the deviator's set of guaranteed payoffs.

Denote by D^* the set of admissible pairs of strategies. Obviously, a pair of strategies which is not admissible can not be a solution of the game.

Remark. The set D^* is, in general, nonclosed in the sense of convergence understood as the convergence of families of sets $\{\Gamma_i(t_*, x_*, U, V)\}$ in the Hausdorf metric.

3. DEFINITION OF SOLUTIONS OF THE GAME. STRUCTURE OF SOLUTIONS

3.1 A concept of a solution of the game

Formulate the following problem.

Problem 1. It is required to find a subset $D^0 \subseteq D^*$ of admissible pair of strategies (U, V) such that for any pairs $(U^{(1)}, V^{(1)}) \in D^0$ and $(U^{(2)}, V^{(2)}) \in D^0$ the pair $(\Gamma_1(t_0, x_0, U^{(1)}, V^{(1)}),\ (\Gamma_2(t_0, x_0, U^{(1)}, V^{(1)}))$ and the pair $(\Gamma_1(t_0, x_0, U^{(2)}, V^{(2)}),\ (\Gamma_2(t_0, x_0, U^{(2)}, V^{(2)}))$ do not $P(1,2)-$dominate each other.

Definition 6. Any element of the set D^0 is called a solution of the game.

Remark. Because of nonclosure of the set D^* Problem 1 may have no solutions. In this case one can solve Problem 1 for the closure of the set D^* and then any element of the obtained set D^0 can be approximated by a sequence of elements belonging to the set D^*.

Proposition 1. It is sufficiently to solve Problem 1 in the class of pair of strategies (U, V) which generate a single limit motion.

Proof. Indeed, if the set $X(t_0, x_0, U, V)$, $(U, V) \in D^0$ is not singleton, then one can choose an element $x^*[\cdot] \in X(t_0, x_0, U, V)$ such that vector $\sigma_i(x^*[\theta])$ ρ - dominates the set $\Gamma_i(t_0, x_0, U, V)$ for at least one number $i \in \{1, 2\}$. Let a pair of strategies (U^*, V^*) generate a single trajectory $x^*[\cdot]$. Then the pair $(\Gamma_1(t_0, x_0, U^*, V^*),\ \Gamma_2(t_0, x_0, U^*, V^*))$ $P(1,2)$- dominates the pair $(\Gamma_1(t_0, x_0, U, V),\ \Gamma_2(t_0, x_0, U, V))$. This implies that $(U, V) \notin D^0$. The contradiction proofs the proposition.

3.2 Auxiliary approach- evasion games

Now consider trajectories $x(t)$, $t_0 \le t \le \theta$ of the system (1) generated by all possible pairs of measurable controls $(u(t), v(t),\ t_0 \le t \le \theta)$. Denote the set of these trajectories by L. The endpoints of these trajectories form the attainability set $G(\theta)$ of the system (1).

Let a trajectory $x^*(\cdot) \in L$ and a value of parameter $\varepsilon > 0$ be given. Consider the sets

$$M^\varepsilon(x^*(\cdot), i, j) = \{x \in G(\theta) : \sigma_{ij}(x) \le \sigma_{ij}(x^*(\theta)) - \varepsilon\},\quad i = 1, 2;\ j = 1, ..., l_i,$$

$$N_i^\varepsilon(x^*(\cdot)) = \bigcup_{j \in \overline{1, l_i}} M^\varepsilon(x^*(\cdot), i, j),\ i = 1, 2.$$

Consider the following family of antagonistic positional differential games of approach- evasion $\Gamma_1(\tau, x^*(\cdot), \varepsilon)$ and $\Gamma_2(\tau, x^*(\cdot), \varepsilon)$, $\tau \in [t_0, \theta)$, $\varepsilon > 0$ (Krasovskii [1985], Krasovskii and Subbotin [1988]). Dynamics of the game is described by the equation (1). The position $(\tau, x^*(\tau))$ is an initial one. The players act in the class of positional strategies $U^a \div u(t, x, \varepsilon)$ and $V^a \div v(t, x, \varepsilon)$.

In the game $\Gamma_1(\tau, x^*(\cdot), \varepsilon)$ P1 chooses his strategy U^a such that for any motion $x(\cdot) \in X(\tau, x^*(\tau), U^a)$ the state $x(\theta)$ contacts the set $N_1^\varepsilon(x^*(\cdot))$. On the other hand, P2 chooses his strategy V^a such that for any motion $x(\cdot) \in X(\tau, x^*(\tau), V^a)$ the condition $x(\theta) \notin N_1^\varepsilon(x^*(\cdot))$ holds.

Analogously, in the game $\Gamma_2(\tau, x^*(\cdot), \varepsilon)$ P2 chooses his strategy V^a such that for any motion $x(\cdot) \in X(\tau, x^*(\tau), V^a)$ the state $x(\theta)$ contacts the set $N_2^\varepsilon(x^*(\cdot))$. On the other hand, P1 chooses his strategy U^a such that for any motion $x(\cdot) \in X(\tau, x^*(\tau), U^a)$ the condition $x(\theta) \notin N_2^\varepsilon(x^*(\cdot))$ holds.

It is well known (see Krasovskii and Subbotin [1988]) that under the assumptions $1^0 - 3^0$ the games $\Gamma_1(\tau, x^*(\cdot), \varepsilon)$ and $\Gamma_2(\tau, x^*(\cdot), \varepsilon)$, $\tau \in [t_0, \theta)$, $\varepsilon > 0$ have saddle points and solving strategies for both players.

Denote a solving strategy of P2 in the game $\Gamma_1(\tau, x^*(\cdot), \varepsilon)$ by $v(t, x, \varepsilon | \tau, x^*(\cdot))$ and a solving strategy of P1 by $u(t, x, \varepsilon | \tau, x^*(\cdot))$.

3.3 Acceptable trajectories

Introduce the following definitions.

Definition 7. A trajectory $x(\cdot) \in L$ is called an acceptable one for P1, if for any position $(\tau, x^*(\tau))$, $\tau \in [t_0, \theta)$ there exists a strategy of P2 such that P1 can not guarantee payoff which ρ- dominates his payoff on the trajectory $x^*(\cdot)$. Analogously, a trajectory $x(\cdot) \in L$ is called an acceptable one for P2, if for any position $(\tau, x^*(\tau))$, $\tau \in [t_0, \theta)$ there exists a strategy of P1 such that P2 can not guarantee payoff which ρ- dominates his payoff on the trajectory $x^*(\cdot)$.

Definition 8. A trajectory $x(\cdot) \in L$ is called an acceptable one, if it is an acceptable one for both players simultaneously.

Denote the set of acceptable trajectories by L^0.

3.4 Structure of solutions

Let a pair of measurable controls $(u^*(t), v^*(t),\ t_0 \le t \le \theta)$ generate a trajectory $x^*(\cdot) \in L^0$ of the system (1). Consider the following strategies of P1 and P2.

$$U^0 \div \{u^0(t, x, \varepsilon), \beta_1(\varepsilon)),\ V^0 \div \{v^0(t, x, \varepsilon), \beta_2(\varepsilon)) \quad (4)$$

where

$$u^0(t,x,\varepsilon) = \tag{5}$$
$$\begin{cases} u^*(t), \ if \ \|x - x^*(t)\| < \varepsilon\varphi(t) \\ u(t,x,\varepsilon|\tau,x^*(\cdot)), \ if \ \|x - x^*(t)\| \geq \varepsilon\varphi(t) \end{cases},$$

$$v^0(t,x,\varepsilon) = \tag{6}$$
$$\begin{cases} v^*(t), if \ \|x - x^*(t)\| < \varepsilon\varphi(t) \\ v(t,x,\varepsilon|\tau,x^*(\cdot)), \ if \ \|x - x^*(t)\| \geq \varepsilon\varphi(t) \end{cases}$$

for all $(t,x) \in G$, $\varepsilon > 0$.

The functions $\beta_i(\cdot)$ and positive increasing function $\varphi(\cdot)$ are chosen so that the following ineguality

$$\left\|x(t,t_0,x_0,U^0,\varepsilon,\Delta_1,V^0,\varepsilon,\Delta_2) - x^*(t)\right\| < \varepsilon\varphi(t)$$

holds for $t \in [t_0,\theta)$, if $\delta(\Delta_i) \leq \beta_i(\varepsilon)$.

The strategies $u(t,x,\varepsilon|\tau,x^*(\cdot))$ and $v(t,x,\varepsilon|\tau,x^*(\cdot))$ are defined in Subsection 3.2.

These strategies can be interpreted as universal penalty strategies used in the case when the partner refuses to follow the trajectory $x^*(\cdot)$ at some moment of time $\tau \in [t_0,\theta)$.

Penalty strategies were considered in Kononenko [1976], Kleimenov [1982] and Tolwinskii et al. [1986].

So, for any trajectory $x^*(\cdot) \in L^0$ one can find the pair of strategies (U^0,V^0) (4) - (6) which generates a single limit motion $x^*(\cdot)$. Let the set of such pair of strategies be denoted by W^*. It follows from acceptability of the trajectory $x^*(\cdot)$ that it does not pay for any player to deviate from the pair (U^0,V^0) moving along the trajectory.

Denote by W^0 the subset of the set W^* such that for any pairs of strategies $(U^{(1)},V^{(1)}) \in W^0$ and $(U^{(2)},V^{(2)}) \in W^0$ the pair $(\Gamma_1(t_0,x_0,U^{(1)},V^{(1)})$, $(\Gamma_2(t_0,x_0,U^{(1)},V^{(1)}))$ and the pair $(\Gamma_1(t_0,x_0,U^{(2)},V^{(2)})$, $(\Gamma_2(t_0,x_0,U^{(2)},V^{(2)}))$ do not $P(1,2)-$dominate each other.

Proposition 2. The sets D^0 and W^0 coincide.

Indeed, the inclusion $W^0 \subseteq D^0$ is obvious. The inverse inclusion follows from the definions of the sets.

Proposition 2 gives a way for finding solutions of the game which is based on determination of acceptable trajectories.

In conclusion note that, in general, Problem 1 is difficult. We give its solution for two special examples in next Section.

4. EXAMPLES

4.1 Example 1. Motion of material point on the plane

The vector equation

$$\ddot{\xi} = u + v, \ \ \xi, u, v \in \mathbb{R}^2, \ \ \|u\| \leq 1, \|v\| \leq 1, \tag{7}$$
$$\xi[t_0] = \xi_0, \ \dot{\xi}[t_0] = \xi_0$$

describes the motion of a material point of unit mass on the plane (ξ_1,ξ_2) under the action of a force $F = u + v$. P1 (P2) which governs the control $u(v)$ tends to maximize the cost functional $\sigma_1(\xi[\theta])$ $(\sigma_2(\xi[\theta]))$ where

$$\sigma_i(\xi[\theta]) = (< \xi[\theta]), \ a^{(i)} >, \ \ < \xi[\theta]), \ b^{(i)} >), \ i = 1,2 \tag{8}$$
$$\xi = (\xi_1,\xi_2)^T, \ a^{(i)} = (a_1^{(i)},a_2^{(i)})^T, \ b^{(i)} = (b_1^{(i)},b_2^{(i)})^T.$$

Here $a^{(i)}, b^{(i)}$, $i = 1,2$, are given vectors on the plane (ξ_1,ξ_2); θ is the given final time. The symbol $< \cdot \, , \, \cdot >$ denotes the scalar product.

By setting

$$y_1 = \xi_1, \ \ y_2 = \xi_2, \ \ y_3 = \dot{\xi}_1, \ \ y_4 = \dot{\xi}_2$$

and making the following change of variables

$$x_1 = y_1 + (\theta - t)y_3, \ \ x_2 = y_2 + (\theta - t)y_4, \ \ x_3 = y_3, \ \ x_4 = y_4$$

we shall get a system whose first and second equations are

$$\begin{aligned} \dot{x}_1 &= (\theta - t)(u_1 + v_1), \\ \dot{x}_2 &= (\theta - t)(u_2 + v_2). \end{aligned} \tag{9}$$

Further, (7) can be written

$$\sigma_i(x[\theta]) = (< x[\theta]), \ a^{(i)} >, \ \ < x[\theta]), \ b^{(i)} >), \tag{10}$$
$$x = (x_1,x_2)^T, \quad i = 1,2.$$

Since the cost functional (10) depends on variables x_1 and x_2 only and the right-hand side of (9) does not depend on other variables, one can conclude that it is sufficiently to consider only the shortened system (9) with vector cost functionals (10).

Then initial conditions for the system (9) are given by formulas

$$x_i[t_0] = x_{0i} = \xi_{0i} - (\theta - t_0)\dot{\xi}_{0i}, \quad i = 1,2.$$

Let the following initial conditions be given:

$t_0 = 0$, $\xi_{01} = -1.6$, $\xi_{02} = 1,2$, $\dot{\xi}_{01} = 0.8$, $\dot{\xi}_{02} = -0.6$, $\theta = 2$.

Then we have $x_{01} = 0$, $x_{02} = 0$.

Consider two variants of vectors $a^{(i)}$ and $b^{(i)}$:

1) $a^{(1)} = (-5, \ 2)^T$, $b^{(1)} = (-4, \ 4)^T$, $a^{(2)} = (4, \ 4)^T$, $b^{(2)} = (5, \ 2)^T$ (Fig. 1).

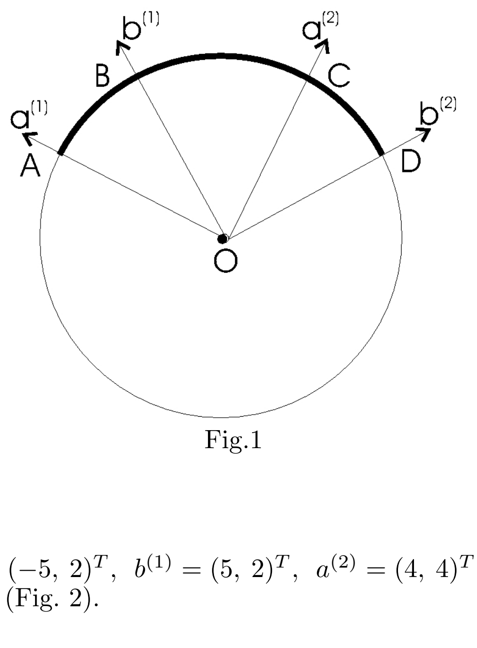

Fig.1

2) $a^{(1)} = (-5, \ 2)^T$, $b^{(1)} = (5, \ 2)^T$, $a^{(2)} = (4, \ 4)^T$, $b^{(2)} = (-4,4)^T$ (Fig. 2).

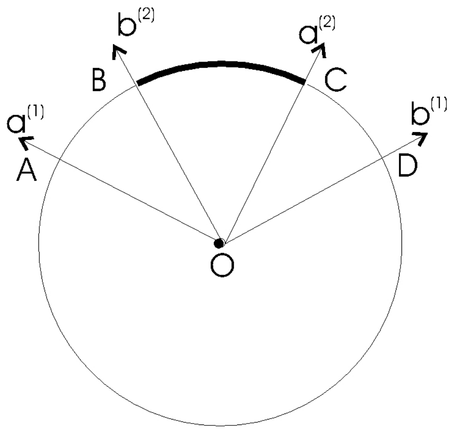

Fig.2

Sets of solutions of the game were calculated. Describe these sets of solution by means of sets of endpoints of trajectories generated by considered solutions.

In Fig.1 and Fig.2, the circle of radius 4 with center at the initial point $(0,0)$ represents the attainability set of the system (9) at the moment of time $\theta = 2$. In Fig.1 the bold-faced arc ABCD is a set of endpoints of the trajectories for the variant 1). In Fig.2 the arc BC is the set of endpoints of the trajectories for the variant 2).

For the solution (U^0, V^0) (4) - (6) with $u^*(t) \equiv (0,1)^T$, $v^*(t) \equiv (0,1)^T$, $0 \leq t \leq 2$ the generated trajectory in the plane (ξ_1, ξ_2) was calculated. This trajectory is represented in Fig.3 by the curve AB.

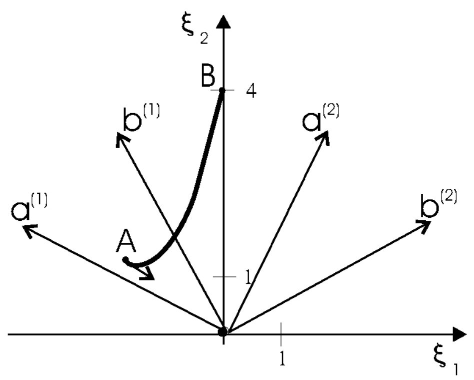

Fig.3

4.2 Example 2. Repeated prisoner's dilemma

In this Subsection some results of the paper (Kleimenov [2000]) are interpreted from point of view of theory of dynamics game with vector payoff functionals. The talk is about repeated prisoner's dilemma (see, for example, Smale [1980], Kleimenov [1998]).

Remind that in the bimatrix "prisoner's dilemma" game, the payoff matrices of P1 and P2 are

$$A = \left\| \begin{matrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{matrix} \right\|, \; B = \left\| \begin{matrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{matrix} \right\|, \qquad (11)$$

where the inequalities

$$a_{21} > a_{12} > a_{22} > a_{12}, \quad 2a_{11} > a_{21} + a_{22} \qquad (12)$$

hold. Each player has two strategies. The first one is C (cooperate) and the second one is D (defect). It follows from (12) that the pair (D,D) is an unique Nash equilibrium. In this pair, each player gets payoff equals to a_{22}. At the same time, both players get payoff $a_{11} > a_{22}$ on the pair (C,C) which is not a Nash equilibrium.

Let the players act in the class of mixed strategies. A pair $(p, 1-p)$, where $p \in [0,1]$, is a mixed strategy of P1 and a pair $(q, 1-q)$, where $q \in [0,1]$, is a mixed strategy of P2. Then payoffs of P1 and P2 are defined by

$$f_1(p,q) = cpq - c_1 p - c_2 q + a_{22} \qquad (13)$$
$$f_2(p,q) = cpq - c_2 p - c_1 q + a_{22} \qquad (14)$$

where

$$c = a_{11} - a_{12} - a_{21} + a_{22}, \quad c_1 = a_{22} - a_{12}, \quad c_2 = a_{22} - a_{21}$$

A point (p,q) in the unit square

$$E = \{(p,q) : 0 \leq p \leq 1, \quad 0 \leq q \leq 1\}$$

characterizes the state of the game.

Now let this bimatrix game be repeated. Then P1 and P2 can govern dynamics of the process. It is natural to require that a state (p,q) arrives at the cooperation state (C,C).

Under this additional requirement the repeated prisoner's dilemma was investigated in Kleimenov [2000]. Furtermore, it was assumed in this paper, that dynamics of the repeated prisoner's dilemma is constructed according to some special procedure given in Kleimenov [1997]. The following points are characteristic in this procedure. First, using the principle of non-decrease of players'payoffs, and, secondly, using of Nash equalibria in some auxiliary "local" bimatrix games.

Generated trajectories were constructed for various values of parameters a_{ij} of the game. For one of considered cases these trajectories are represented in Fig.4.

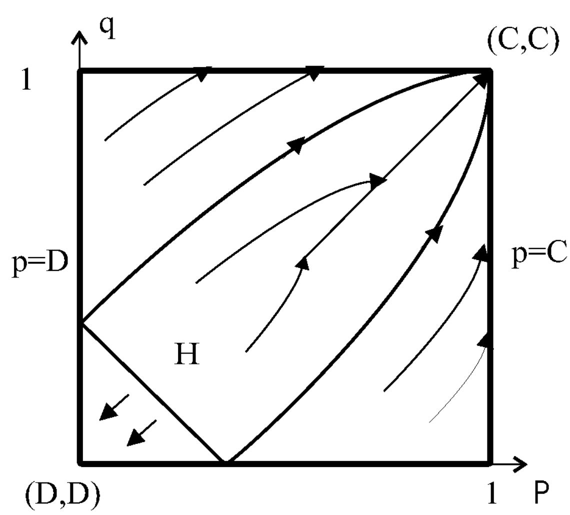

Fig.4

Arrows in Fig.4 show directions of motions for a current state (p, q). Let the set bounded by bold line be denoted by H. Analyzing the trajectories one obtain that:

(i) For initial states $(p_0, q_0) \in H$ dynamics leads a state (p, q) to the state (C, C) in a finite number of rounds; moreover, both functions (13), (14) reach the maximum along the trajectory in the state (C, C).

(ii) For initial states $(p_0, q_0) \in E \backslash H$ dynamics does not lead to the state (C, C) but the functions (13), (14) reach the maximum along the trajectory in the state (C, C) again.

This process can be interpreted as a two-person dynamic game with vector payoff functionals. Indeed, each player has his own payoff (13) or (14) and, in addition, the another payoff which consists in leading a current state (p, q) to the state (C, C). So, for initial states of the set H P1 and P2 maximize their functionals (13), (14) and lead the system to the state (C, C). For initial states of the set $E \backslash H$ P1 and P2 maximize payoffs (13), (14) but they do not lead a state (p, q) to the state (C, C).

5. CONCLUSION

This approach can be generalized for the case of m players ($m > 2$). For this case one can use the formalization of the game given in Kleimenov [1993].

It will be interesting to develop general methods of finding solutions in nonantagonistic differential game with vector payoff functionals.

REFERENCES

T. Basar and G.J. Olsder. *Dynamic Noncooperative Game Theory.* Academic Press, New Jork, 2nd edition, 1999.

A.F. Kleimenov. Equilibrium coalitional mixed strategies in differential games of m players. *Problems of Control and Inform. Theory,* v.11, No.2, pp. 85-95, 1982.

A.F. Kleimenov. *Nonantagonistic Positional Differential Games.* Nauka, Ekaterinburg (in Russian), 1993.

A.F. Kleimenov. Solutions in a nonantagonistic positional differential game. *J. Appl. Math Mechs*, v.61,No.5, pp.717–723, 1997.

A.F. Kleimenov. An approach to building dynamics for repeated bimatrix 2x2 games involving various behavior types. In: *Dynamic and Control,*. pp. 195–204, Gordon & Breach Sci. Publ., London, 1998.

A.F. Kleimenov. Problem of constructing dynamics for nonantagonistic positional differential games. In: *Proceedings of the Steklov Institute of Mathematics.* Vol. Suppl. 2. pp. S67–S78, 2000.

A.F. Kononenko. On equilibrium positional strategies in nonantagonistic differential games. *Dokl. Akad. Nauk SSSR*, 231(2), pp.285-288, 1976 (in Russian).

N.N. Krasovskii. *Control of a Dynamical System.* Nauka, Moscow, 1985 (in Russian).

N.N. Krasovskii and A.I. Subbotin. *Game-Theoretical Control Problems.* Springer, NewYork, Berlin, 1988.

S.Smale. The prisoner's dilemma and dynamical system associated to noncooperative games. *Econometrica*, 48, pp. 1617-1634, 1980.

B. Tolwinski, A. Haurie, and G. Leitman. Cooperative equilibria in differential games. *J.Math. Anal. Appl.*, v.119, No.1-2, pp. 182-202, 1986.

V.I. Zhukovskiy, and M.E. Salukvadze. *The Vector-Valued Maximum.* Academic Press, New Jork, 1994.