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Averaging of zero dynamics for systems controlled by the vertically transverse function approach

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Abstract: In this paper, a method is explored to introduce dissipation in the closed-loop, zero-dynamics systems that arise in the vertically transverse function approach (VTFA) recently proposed by the authors. The VTFA is an attempt to extend the transverse function approach (TFA) of Morin and Samson to deal with practical point-stabilization of a class of critical simple mechanical systems on Lie groups. This class comprises systems that are not kinematic reductions and hence fall beyond the scope of application of the TFA as originally formulated. The VTFA gives rise to a nontrivial zero dynamics that ultimately determines the qualitative nature of the trajectories and, in order to constrain the velocity (or "fiber") coordinates to vanish asymptotically, as required in typical applications, dissipation must be injected into the zero dynamics. Reported below is a possible way to reach that goal based on the adjunction of additional auxiliary inputs, via generalized vertically transverse functions, and the use of nonlinear, high-order averaging theory. Also included is an illustrative example as well as numerical simulations suggesting that the feedback laws herein designed yield promising results.

Keywords: Mechanical system, generalized vertically transverse function, high-order averaging

1. INTRODUCTION.

Recently, a methodology was introduced in Lizárraga and Sosa (2005) for the control of a class of second-order systems on tangent Lie groups by means of *vertically transverse functions* (VTF). The ultimate application of that methodology is to practically stabilize points—and more general trajectories whenever possible—for simple mechanical systems (SMS), whose dynamic equations are derived from the Euler-Lagrange principle with Lagrangians equal to kinetic minus potential energies, cf. e.g. Bullo and Lewis (2005). Within this class one finds systems that are fully or partially actuated, holonomically or nonholonomically constrained and, in particular, systems (referred to as "critical") that at some states do not satisfy Brockett's necessary condition for point-stabilization by continuous state feedback.

The idea of using VTF arises as an attempt to generalize, to the more ample case of SMS, the methodology proposed by Morin and Samson (2003) for critical controllable driftless systems. In essence, the term *practical stabilization* refers to the property that the trajectories of the controlled system ultimately converge to a predefined neighborhood of the desired equilibrium. The procedure in Lizárraga and Sosa (2005) may be roughly described as follows. One starts with a second-order system, the *target* system, defined on a tangent Lie group TG. More specifically, it is assumed that the target is determined by the specification of a *second-order, drift* vector field on TG, and a *control distribution* spanned by vector fields that are *vertical lifts* of vector fields on G. It is also assumed that the original, *unlifted* vector fields define a driftless system which is locally accessible at some point. As proved in Morin and Samson (2001), the latter condition is equivalent to the existence of a mapping (a transverse function) which, in a sense that differs slightly from the usual one, is *transverse* to the associated control distribution. In turn, the *tangent* mapping of a transverse function is *vertically transverse* for the corresponding lifted distribution, as defined and stated in Lizárraga and Sosa (2005). The image of the VTF is an immersed submanifold of TG, submanifold on which one defines an additional *auxiliary* second-order system. Together, the target and the auxiliary define what is called the *compound* system, the state of which is referred to as the compound state. An *error* signal is then defined, with the aid of the Lie group operation on TG, to quantify the difference between the states of the target and auxiliary systems; under appropriate assumptions, the dynamics of the error signal can be assigned arbitrarily by smooth feedback defined in terms of the compound state. If the error dynamics is assigned to be positive-complete and to admit the zero-velocity state over the identity element as an exponentially stable equilibrium, the configuration coordinates ultimately approach a prescribed neighborhood of the desired configuration. In that case, however, the evolution of the fiber coordinates—the *velocities* when the target is a SMS—is determined by a *nontrivial*, autonomous zero dynamics, with the structure

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of a (perturbed) affine connection control system, cf. e.g. (Bullo and Lewis, 2005, Chap. 4).

Much in the spirit of the approach of Morin and Samson (2004), the methodology we propose in this paper is based on the notion of *generalized transverse function* (GTF), and it aims at providing the closed-loop zero dynamics with *new, independent control inputs*. In principle, these inputs appear as an additional degree of freedom which allows one to influence the behavior of the zero dynamics, and we estimate that by using appropriately defined feedback laws, dissipation could be injected into the system to have the velocities asymptotically vanish. Although the zero dynamics that arises in our approach is an affine-connection control system, the associated affine-connection may not be the Levi-Civita connection of any Riemannian metric, in which case it is not clear how to assign the extra inputs to introduce dissipation. As an alternative, we analyze the possibility of designing such control inputs by techniques of high-order averaging, especially as recently developed in Sarychev (2001) and Vela (2003), both of whom make use of tools from the chronological calculus developed in Agrachev and Gramkrelidze (1979).

The paper is organized as follows. In Section 2, the basic notations are fixed and some mathematical concepts, including that of vertically transverse function, are recalled. Generalized vertically transverse functions are introduced in Section 3, along with a description of its applications to the control of SMS and the general structure of the resulting zero dynamics. In Section 4, a brief description is made of the main ideas of averaging theory and the way we plan to use it to introduce dissipation. An example is developed in Section 5, where the construction of time-varying feedback laws by means of averaging theory is explored. Finally, Section 6 contains closing remarks and an outline of our current research efforts.

2. PRELIMINARY CONCEPTS AND NOTATION.

In this section we recall basic concepts and fix the notations used throughout the paper. The reader may consult Warner (1983); Abraham and Marsden (1985); Bullo and Lewis (2005) for further reference. All manifolds, mappings, vector fields and related constructs involved in the sequel are assumed to be smooth (of class C^∞), unless otherwise stated.

2.1 Differential-geometric notions and Lie groups.

Given a (finite-dimensional, paracompact) manifold Q, T_qQ is the tangent space at $q \in Q$, and $\pi_Q : TQ \longrightarrow Q$ is the associated tangent bundle. Given $f : Q \longrightarrow P$, $T_qf : T_qQ \longrightarrow T_{f(q)}P$ is the tangent mapping of f at $q \in Q$. The tangent bundle map covering f is denoted by Tf. The set of (smooth) vector fields on Q (resp. TQ) is denoted by $\Gamma(TQ)$ (resp. $\Gamma(TTQ)$). Given a vector field X, we write either X_q or $X(q)$ to denote its value at a point q. A vector field $X \in \Gamma(TTQ)$ is said to be ***second-order*** if $T\pi_Q \circ X = \mathrm{id}_{TQ}$, and ***vertical-valued*** (or simply vertical) if $T\pi_Q \circ X = 0$. Given $v \in TQ$, the ***vertical space over*** v is the subset of T_vTQ given by $T_vTQ^{\mathrm{vert}} = \{\alpha \in T_vTQ : T_v\pi(\alpha) = 0\}$. The disjoint union of the spaces T_vTQ^{vert}, $v \in TQ$, with the differentiable structure naturally induced by TTQ, is called the ***vertical subbundle*** of TTQ and is denoted by TTQ^{vert}. Given $v, w \in TQ$ with $\pi_Q(v) = \pi_Q(w)$, the vector in T_vTQ defined by $\mathrm{lift}(v, w) = d/dt|_0 (v + tw)$ is the ***vertical lift of w by v***. The ***vertical lift of a vector field*** X on Q is a vector field on TQ given by $X^{\mathrm{lift}}(v) = \mathrm{lift}(v, X_v)$. The ***zero-section*** of a tangent bundle TQ is the subbundle $Z(TQ)$ of zero-vectors in TQ; as an embedded submanifold, it is diffeomorphic to Q. The ***Liouville vector field*** C on Q is defined by $C(v) = \mathrm{lift}(v, v)$. A ***spray*** S is a second-order vector field satisfying $[C, S] = S$. $\mathbb{T}^\kappa$ denotes the torus of dimension κ. Let $h : T\mathbb{T}^{\kappa_1} \times T\mathbb{T}^{\kappa_2} \longrightarrow T(\mathbb{T}^{\kappa_1} \times \mathbb{T}^{\kappa_2})$ and $H : TT\mathbb{T}^{\kappa_1} \times TT\mathbb{T}^{\kappa_2} \longrightarrow TT(\mathbb{T}^{\kappa_1} \times \mathbb{T}^{\kappa_2})$ denote natural identifications.

Assume that G is a Lie group with group composition denoted by $\hat{\mu}$, and let $\widehat{L}_a, \widehat{R}_a : G \longrightarrow G$ denote the left and right translations by a on G, respectively. The tangent bundle TG, with composition given by $\mu(x, y) = T\widehat{L}_{\pi_G(x)}(y) + T\widehat{R}_{\pi_G(y)}(x)$, is also a Lie group, usually referred to as the ***tangent Lie group*** of G. We let $L_v, R_v : TG \longrightarrow TG$ denote the left and right translations by v on TG, respectively. Sometimes we use $x \cdot y$ or xy in place of $\mu(x, y)$. We write e for the identity element in G, and 0_e (the zero vector in T_eG) for the corresponding identity in TG. A vector field X on G is said to be left-invariant if $X_{gh} = T\widehat{L}_g(X_h)$ for all $g, h \in G$.

2.2 Vertically transverse functions.

In this section we recall some of the results presented in Lizárraga and Sosa (2005). Consider a set $\{X_1, \ldots, X_m\}$ of left-invariant vector fields on a n-dimensional Lie group G ($m \leq n$) satisfying the LARC (*Local Accessibility Rank Condition*) at a point which, without loss of generality, we assume to be $e \in G$. Consider also a second-order vector field S on TG. The class of control systems we consider are of the form

$$\dot{x} = S_x + \sum_{i=1}^{m} u^i X_{i,x}^{\mathrm{lift}}. \tag{1}$$

When dealing with SMS, we require S to be equal to the geodesic spray of some Riemannian metric minus the vertical lift of the gradient of a potential energy function.

As shown in Morin and Samson (2003), given a neighborhood $\mathcal{U}$ of e, the local accessibility of $\{X_1, \ldots, X_m\}$ at e is equivalent to the existence of a ***transverse function*** near e, i.e., a mapping $f : \mathbb{T}^\kappa \longrightarrow G$, with $\kappa = n - m$, such that $f(\mathbb{T}^\kappa) \subset \mathcal{U}$ and, for every $\theta \in \mathbb{T}^\kappa$,

$$T_{f(\theta)}G = Tf(T_\theta \mathbb{T}^\kappa) \oplus \mathrm{span}_{\mathbb{R}}\{X_{1,f(\theta)}, \ldots, X_{m,f(\theta)}\}. \tag{2}$$

In this paper we focus on systems evolving on Lie groups; nevertheless, this notion of transversality also makes sense for systems on more general manifolds, in which case $\kappa \geq n - m$ and the sum in (2) need not be direct.

In Lizárraga and Sosa (2005), it is proved that the tangent of a transverse function for a set $\{X_1, \ldots, X_m\}$ at $e \in G$ is ***vertically transverse*** to the set of lifted vector fields $\{X_1^{\mathrm{lift}}, \ldots, X_m^{\mathrm{lift}}\}$ in the sense that

$$T_{Tf(\omega)}TG^{\mathrm{vert}} = TTf((T_\omega T\mathbb{T}^\kappa)^{\mathrm{vert}}) \oplus \mathrm{span}_{\mathbb{R}}\{X_{1,Tf(\omega)}^{\mathrm{lift}}, \ldots, X_{m,Tf(\omega)}^{\mathrm{lift}}\},$$

for every $\omega \in T\mathbb{T}^{\kappa}$. The procedure in Lizárraga and Sosa (2005), which appeals to VTF for practical point-stabilization of second-order systems, may be seen as a particular case of the procedure described in the next section, which makes use of vertically transverse functions derived from generalized transverse functions.

3. GENERALIZED VERTICALLY TRANSVERSE FUNCTIONS FOR CONTROL.

In this section we recall a slightly weakened version of generalized transverse function introduced in Morin and Samson (2004), and then define the related notion of generalized vertically transverse function (GVTF). A natural way to construct a GVTF is by taking the tangent map associated to a generalized transverse function.

Definition 1. A ***generalized transverse function*** (GTF) for a given set of vector fields $\{X_1, \ldots, X_m\} \subset \Gamma(TG)$, $n = \dim(G)$, near $e \in G$, for a given neighborhood $\mathcal{U}$ of e, is a map $f : \mathbb{T}^{\kappa_1} \times \mathbb{T}^{\kappa_2} \longrightarrow G$, $\kappa_1 = n - m$, $\kappa_2 \geq 1$, such that **(a)** $f(\mathbb{T}^{\kappa_1} \times \mathbb{T}^{\kappa_2}) \subset \mathcal{U}$; and **(b)** for every $\sigma = (\theta_1, \theta_2) \in \mathbb{T}^{\kappa_1} \times \mathbb{T}^{\kappa_2}$,

$$T_{f(\sigma)}G = \text{span}_{\mathbb{R}}\{X_{1,f(\sigma)}, \ldots, X_{m,f(\sigma)}\} \oplus Tf_{\theta_2}(T_{\theta_1}\mathbb{T}^{\kappa_1}), \tag{3}$$

with $(f_{\theta_2})_{\theta_2 \in \mathbb{T}^{\kappa_2}}$ being the associated family of maps defined by $f_{\theta_2}(\theta_1) = f(\theta_1, \theta_2)$.

In a coordinate chart $\theta = (\theta_1, \theta_2) = (\theta_1^i, \theta_2^j)$, for $\mathbb{T}^{\kappa_1} \times \mathbb{T}^{\kappa_2}$, $(i = 1, \ldots, \kappa_1; j = 1, \ldots, \kappa_2)$, condition **(b)** in Definition 1, reduces to

$$\begin{aligned} \mathbb{R}^n \; = \; & \text{span}_{\mathbb{R}}\{X_{1_{f(\theta)}}, \ldots, X_{m,f(\theta)}\} \\ & \oplus \text{span}_{\mathbb{R}} \left\{ \frac{\partial f_{\theta_2}}{\partial \theta_1^1}(\theta), \ldots, \frac{\partial f_{\theta_2}}{\partial \theta_1^{\kappa 1}}(\theta) \right\}. \end{aligned}$$

The following proposition, a consequence of (3), is rather straightforward to prove.

Proposition 2. Let $f : \mathbb{T}^{\kappa_1} \times \mathbb{T}^{\kappa_2} \longrightarrow G$ with $\kappa_1 = n - m$, $\kappa_2 \geq 1$, be a GTF for $\{X_1, \ldots, X_m\} \subset \Gamma(TG)$ around e. Then $Tf : T(\mathbb{T}^{\kappa_1} \times \mathbb{T}^{\kappa_2}) \longrightarrow TG$ is a ***generalized vertically transverse function*** for $\{X_1^{\text{lift}}, \ldots, X_m^{\text{lift}}\} \subset \Gamma(TTG^{\text{vert}})$ in the sense that, for every $\nu \in T(\mathbb{T}^{\kappa_1} \times \mathbb{T}^{\kappa_2})$,

$$\begin{aligned} T_{Tf(\nu)}TG^{\text{vert}} \; = \; & TTf \circ H((T_\omega T\mathbb{T}^{\kappa_1})^{\text{vert}} \times \{0\}) \\ & \oplus \text{span}_{\mathbb{R}}\{X_{1,Tf(\nu)}^{\text{lift}}, \ldots, X_{m,Tf(\nu)}^{\text{lift}}\}, \end{aligned} \tag{4}$$

with H the natural diffeomorphism $TT\mathbb{T}^{\kappa_1} \times TT\mathbb{T}^{\kappa_2} \longrightarrow TT(\mathbb{T}^{\kappa_1} \times \mathbb{T}^{\kappa_2})$.

3.1 Application of generalized VTF to control.

In this subsection we explore the use of GVTF to practically stabilize points for systems of the form (1), as an extension of the methodology presented in Lizárraga and Sosa (2005).

Consider an auxiliary control system, with state $(\omega(t), \eta(t))$ evolving on $T\mathbb{T}^{\kappa_1} \times T\mathbb{T}^{\kappa_2}$, given by

$$\begin{aligned} \dot{\omega} \; &= \; \Delta_\omega + \textstyle\sum_{i=1}^{\kappa_1} v^i \Omega_{i,\omega} \\ \dot{\eta} \; &= \; \Lambda_\eta + \textstyle\sum_{i=1}^{\kappa_2} w^i \Upsilon_{i,\eta}, \end{aligned} \tag{5}$$

where Δ and Λ are second-order vector fields on $\Gamma(TT\mathbb{T}^{\kappa_1})$ and $\Gamma(TT\mathbb{T}^{\kappa_2})$ resp. and $\{\Omega_i : i = 1, \ldots, \kappa_1\}$, $\{\Upsilon_i : i = 1, \ldots, \kappa_2\}$ are global frames for $(TT\mathbb{T}^{\kappa_1})^{\text{vert}}$ and $(TT\mathbb{T}^{\kappa_2})^{\text{vert}}$, respectively.

Now define a map to quantify the error between the state of (1) and the image by Tf of the auxiliary control system state:

$$z_e(t) = x(t) \cdot Tf(\nu(t))^{-1},$$

where $\nu = h(\omega, \eta)$ and h is the natural diffeomorphism mapping $T\mathbb{T}^{\kappa_1} \times T\mathbb{T}^{\kappa_2}$ onto $T(\mathbb{T}^{\kappa_1} \times \mathbb{T}^{\kappa_2})$.

Whenever $z_e(t) \equiv e$, the state $x(t)$ of the control system (1) coincides with the image by Tf of the state $\nu(t)$, whose dynamics is given by

$$\dot{\nu} = \Pi_\nu + \sum_{i=1}^{\kappa_1} v^i \Phi_{i,\nu} + \sum_{i=1}^{\kappa_2} w^i \Psi_{i,\nu}, \tag{6}$$

for some vertical vector fields Φ_i, Ψ_j on $T(\mathbb{T}^{\kappa_1} \times \mathbb{T}^{\kappa_2})$ associated with Ω_i and Υ_j respectively, $(i = 1, \ldots, \kappa_1; j = 1, \ldots, \kappa_2)$, and for some second-order vector field Π on $T(\mathbb{T}^{\kappa_1} \times \mathbb{T}^{\kappa_2})$ associated with Δ and Λ.

By differentiating along the trajectories of systems (1) and (6), one checks that the error dynamics is given by

$$\dot{z}_e(t) = TR_{Tf(\nu(t))^{-1}} \left(\dot{x}(t) - TL_{z_e(t)} \circ TTf(\dot{\nu}(t)) \right) \tag{7}$$

for every t in $\mathbb{R}$ for which the trajectories of the compound system (1)-(6) are defined. Hence

$$\begin{aligned} \dot{z}_e \; = \; & TR_{Tf(\nu)^{-1}} \left(\left(S_x + \textstyle\sum_{i=1}^m u^i X_{i,x}^{\text{lift}} \right) \right. \\ & \left. - TL_{z_e} \circ TTf \left(\Pi_\nu + \textstyle\sum_{i=1}^{\kappa_1} v^i \Phi_{i,\nu} + \sum_{i=1}^{\kappa_2} w^i \Psi_{i,\nu} \right) \right). \end{aligned} \tag{8}$$

By using the left-invariance of the X_i^{lift}, and the fiberwise linearity of the tangent maps one obtains

$$\begin{aligned} \dot{z}_e \; = \; & TR_{Tf(\nu)^{-1}} \left(D_{(z_e,\nu,w)} \right) + \\ & TR_{Tf(\nu)^{-1}} \circ TL_{z_e} \left(\textstyle\sum_{i=1}^m u^i X_{i,Tf(\nu)}^{\text{lift}} \right. \\ & \qquad \left. - \textstyle\sum_{i=1}^{\kappa_1} v^i TTf\left(\Phi_{i,\nu}\right) \right), \end{aligned} \tag{9}$$

where $D_{(z_e,\nu,w)}$ is defined by $S_{z_e Tf(\nu)} - TL_{z_e} \circ TTf \left(\Pi_\nu + \sum_{i=1}^{\kappa_2} w^i \Psi_{i,\nu} \right)$.

Proposition 3. Given a second-order vector field $S_d \in \Gamma(TTG)$, there is a unique feedback law $(u, v) = \left(u^1, \ldots, u^m, v^1, \ldots, v^{\kappa_1}\right) : TG \times T\mathbb{T}^{\kappa_1} \times T\mathbb{T}^{\kappa_2} \times \mathbb{R}^{\kappa_2} \longrightarrow \mathbb{R}^n$ such that the error dynamics (9) satisfies $\dot{z}_e = S_{d,z_e}$.

The proof of this result follows from straightforward computations using property (4).

In accordance with Proposition 3, by appropriately selecting the input functions for (1)-(6), 0_e can be made an exponentially stable equilibrium point (in some coordinates) for the error dynamics (9). Whenever the solutions of the compound system in closed loop exist, the state of the control system (1) converges to the image by Tf of the auxiliary system (6). Since $f(\mathbb{T}^{\kappa_1} \times \mathbb{T}^{\kappa_2})$ is contained in a predefined neighborhood $\mathcal{U}$ of e by Definition 1, the methodology ensures that the configuration coordinates are ultimately bounded and close to the desired equilibrium point, independently of how the extra control functions w^i $(i = 1, \ldots, \kappa_2)$ are chosen. However, the dynamics of the fiber coordinates is determined by the zero dynamics, which is obtained by setting $z_e = 0_e$ in the closed-loop system. The zero dynamics is given by

$$\begin{aligned} & \left(S_{Tf(\nu)} + \textstyle\sum_{i=1}^m u^i(0_e, \nu, w) X_{i,Tf(\nu)}^{\text{lift}} \right) = \\ & \quad TTf\left(\Pi_\nu + \textstyle\sum_{i=1}^{\kappa_1} v^i(0_e, \nu, w) \Psi_{i,\nu} \right) \\ & \qquad + \textstyle\sum_{i=1}^{\kappa_2} w^i \, TTf \circ \Upsilon_{i,\nu}. \end{aligned} \tag{10}$$

Note that the evolution in zero dynamics of the target system, the left-hand member of equation (10), is determined by the zero dynamics of the auxiliary control system, the right-hand member of (10). Thus, if one accomplishes to design $w^1, \ldots, w^{\kappa_2}$ as functions of the auxiliary state so that the zero-section of $\mathbb{T}^{\kappa_1} \times \mathbb{T}^{\kappa_2}$ is (locally) attractive, the net effect is that the velocities of the auxiliary and target systems tend to zero. Achieving this goal, which turns out to be a nontrivial task, apparently calls for the use of time-varying feedback techniques, and remains an open research problem. The main aim of this paper is to outline an approach based on averaging techniques for the design of those feedback laws. Section 5 illustrates, via a numerical example, that the asymptotic convergence of the velocities to zero may be achievable by means of time-varying feedback laws.

4. AVERAGING THEORY AS A TOOL TO RENDER THE ZERO-SECTION LOCALLY ATTRACTIVE.

Here we recall the basics of high-order averaging theory with a view toward the design of time-varying feedback to render the zero-section (locally) attractive for the zero dynamics (10). The reason to explore time-varying feedback is that there exist obstructions—in fact, extensions of those embodied in the main result of Brockett (1983)—to the existence of continuous state feedback which achieves that goal. Nonlinear, high-order averaging theory has proved to be useful for the design of time-varying stabilizers for driftless controllable systems and for some specific second-order systems, cf. Vela (2003); Sarychev (2001). In essence, that theory aims at reducing the qualitative study of the flow of a periodic *time-varying* vector field to the study of an *autonomous* (i.e., time invariant) one. This is similar to Floquet theory except that a parameter λ is introduced, as in singular perturbation theory, to obtain series approximations of the autonomous vector field. For more enlightening discussions on this topic, see Sarychev (2001), Vela (2003) and the references therein. We shall only detail some of the expressions for the averaging of systems with drift to be used in the next section. Consider a system on $\mathbb{R}^n$, written in the *standard form for averaging*

$$\dot{x} = \lambda X(x,t), \tag{11}$$

with X a T-periodic time-varying vector field and λ a small positive real. By means of the chronological exponential $\overrightarrow{\exp}$, cf. Agrachev and Gramkrelidze (1979), one obtains a system

$$\dot{z} = Z(z), \tag{12}$$

where Z, viewed as "the average of X," is given by

$$Z = \tfrac{1}{T} \ln \overrightarrow{\exp} \left(\int_0^T \lambda\, X(\,\cdot\,,\tau)\, d\tau \right). \tag{13}$$

Z is such that the trajectories of (11) coincide with the trajectories of (12) up to a *time-periodic* diffeomorphism or flow, i.e., $x(t) = P(t, z(t))$, $P(t+T,\,\cdot\,) = P(t,\cdot)$, where P is called the Floquet mapping, cf. Vela (2003). Typically, however, Z is very difficult to compute explicitly, and the way to circumvent this difficulty is by using an infinite series expansion of the form $Z = \frac{1}{T}\sum_{i=1}^{\infty} \lambda^i \Lambda_i(X_\tau)$, where $\Lambda_i(X_\tau)$ is called the ith variation of the identity flow corresponding to $X_\tau = X(\,\cdot\,,\tau)$. By analyzing the mth-partial sum (or "truncate") $\mathrm{Trunc}_m(Z) = \frac{1}{T}\sum_{i=1}^{m} \lambda^i \Lambda_i(X_\tau)$, one may infer some stability properties of (11) for sufficiently small values of λ.

These ideas can also be used to design time-varying feedback for second-order systems, as proposed in Vela (2003). Consider the class of systems on $T\mathbb{R}^n$ of the form

$$\dot{x} = X_x + \sum_{i=1}^{m} u^i(x,t) Y^{\mathrm{lift}}_{i,x}, \tag{14}$$

where X is a spray and Y_i a vector field on $\mathbb{R}^n$. The continuous mapping $u^i : \mathbb{R}^n \times \mathbb{R} \longrightarrow \mathbb{R}$ is assumed to have the form $u^i(x,t) = f^i(x) + \frac{1}{\lambda} v^i(t/\lambda)$ $(i = 1,\ldots,m)$, with $\lambda > 0$ small. Define $V^{(i)}_{(n)}(t) = \int_{t_0}^{t}\int_{t_0}^{s_{n-1}} \cdots \int_{t_0}^{s_2} v^i(s_1)\, ds_1 \ldots ds_{n-1}$ and $V^{(i_1,\ldots,i_k)}_{(n_1,\ldots,n_k)}(t) = V^{(i_1)}_{(n_1)}(t) \cdots V^{(i_k)}_{(n_k)}(t)$ for $k \geq 1$, $i, i_k = 1,\ldots,m$. In a similar manner, we define integrals and averages of these terms, for instance $V^{\widehat{(i)}}_{\widehat{(n)}}(t) = \int_{t_0}^{t} V^{(i)}_{(n)}(\tau)\, d\tau$, $\overline{V^{(i)}_{(n)}(t)} = \frac{1}{T}\int_{t_0}^{t} V^{(i)}_{(n)}(\tau)\, d\tau$. Following the procedure in Vela (2003), we rescale the time variable and consider a truncate of the "nonlinear variation of constants" for (14), yielding a system in the standard form for averaging (11):

$$\dot{x} \;=\; \lambda \left(X_{S,x} + V^{(i)}_{(1)}(t) \left[Y^{\mathrm{lift}}_i, X_S \right]_x - \tfrac{1}{2} V^{(i,j)}_{(1,1)}(t) \left\langle Y_i : Y_j \right\rangle^{\mathrm{lift}}_x \right). \tag{15}$$

Here, $X_S = X + \sum_{i=1}^{m} f^i Y^{\mathrm{lift}}_i$ and the operator $\langle \cdot : \cdot \rangle$, a generalization of the notion of symmetric product, cf. Bullo and Lewis (2005), is given by $\langle X : Y \rangle^{\mathrm{lift}} = \left[X^{\mathrm{lift}}, \left[X_S, Y^{\mathrm{lift}} \right] \right]$.

Assuming that the first order time averages of the inputs v^i vanish, i.e., $\overline{V^{(i)}_{(0)}(t)} = \overline{v^i(t)} = 0$, the first and second order averaged truncates in terms of Lie brackets and symmetric products of the drift and control vector fields in (15) are

$$\begin{aligned} \dot{z} = \mathrm{Trunc}_1(Z)_z \;=\; & X_{S,z} + \overline{V^{(i)}_{(1)}(t)}\; \left[Y^{\mathrm{lift}}_i, X_S \right]_z \\ & - \tfrac{1}{2} \overline{V^{(i,j)}_{(1,1)}(t)}\; \left\langle Y_i : Y_j \right\rangle^{\mathrm{lift}}_z , \end{aligned} \tag{16}$$

$$\begin{aligned} & \dot{z} = \mathrm{Trunc}_2(Z)_z = \\ & \mathrm{Trunc}_1(Z)_z + \lambda \left(\overline{V^{(i)}_{(2)}(t)} - \tfrac{1}{2} T\, \overline{V^{(i)}_{(1)}(t)} \right) \left[\left[Y^{\mathrm{lift}}_i, X_S \right], X_S \right]_z \\ & - \tfrac{1}{2}\lambda \left(\overline{V^{\widehat{(i,j)}}_{\widehat{(1,1)}}(t)} - \tfrac{1}{2} T\, \overline{V^{(i,j)}_{(1,1)}(t)} \right) \left[\left\langle Y_i : Y_j \right\rangle^{\mathrm{lift}}, X_S \right]_z \\ & + \tfrac{1}{2}\lambda \overline{V^{(i,j)}_{(2,1)}(t)} \left[\left[Y^{\mathrm{lift}}_i, X_S \right], \left[Y^{\mathrm{lift}}_j, X_S \right] \right]_z \\ & + \tfrac{1}{2}\lambda \left(\overline{V^{(i,j,k)}_{(2,1,1)}(t)} - \tfrac{1}{2} T\, \overline{V^{(i)}_{(1)}(t) V^{(j,k)}_{(1,1)}(t)} \right) \left\langle Y_i : \left\langle Y_j : Y_k \right\rangle \right\rangle^{\mathrm{lift}}_z . \end{aligned} \tag{17}$$

The rationale to design $u^i(x,t)$ for system (14) is to set $f^i(x)$ such that it stabilizes the state components that are "directly controlled." The inputs v^i are chosen as zero-average functions of t, e.g. $a \sin(\omega t) + b \cos(2\omega t)$, where a and b are to be designed in terms of z in order to render the zero-section locally attractive, as exemplified in the next section.

5. EXAMPLE: THE ECF.

In this section we illustrate the application of GVTF, along with averaging techniques, to examine the possibility of rendering the zero-section of $T\mathbb{T}^{\kappa_1} \times T\mathbb{T}^{\kappa_2}$ locally attractive under the zero dynamics. Although stabilizing only a subset, e.g. the zero-section, of the state manifold may seem more relaxed an aim than stabilizing a point, it should be kept in mind that the zero dynamics in the form

(10) may not be accessible but at generic points. Consider the 3-dimensional Extended Chained Form (ECF), a SMS with state $(x, \dot{x})$ evolving on $T\mathbb{R}^3 \simeq \mathbb{R}^6$, given by:

$$\begin{aligned} \ddot{x}_1 &= u_1 \\ \ddot{x}_2 &= u_2 \\ \ddot{x}_3 &= x_2 u_1. \end{aligned}$$

After relabeling state variables, it can be written as

$$\dot{x} = S_x + u_1 X_{1,x}^{\text{lift}} + u_2 X_{2,x}^{\text{lift}}, \qquad (18)$$

where $S_x = x_4\, \partial/\partial x^1 + x_5\, \partial/\partial x^2 + x_6\, \partial/\partial x^3$ is the geodesic spray given by the Euclidean metric on $\mathbb{R}^3$ and $X_{1,x}^{\text{lift}} = \partial/\partial x^4 + x_2\, \partial/\partial x^6$, $X_{2,x}^{\text{lift}} = \partial/\partial x^5$ are the control vector fields on $T\mathbb{R}^3 \simeq \mathbb{R}^6$. These vector fields are left-invariant provided $T\mathbb{R}^3$ is endowed with the (tangent) Lie group multiplication given by $x \cdot y = (x_1 + y_1, x_2 + y_2, x_3 + y_3 + x_2 y_1, x_4 + y_4, x_5 + y_5, x_6 + y_6 + x_2 y_4 + x_5 y_1)$. Since the set $\{X_1, X_2\}$ satisfies the LARC at $e = 0 \in \mathbb{R}^3$, there exists a transverse function near e, an instance of which is the map $f : \mathbb{T} \longrightarrow \mathbb{R}^3$ given by

$$f(\theta) = \left(\varepsilon \sin(\theta), \varepsilon \cos(\theta), \tfrac{1}{4}\varepsilon^2 \sin(2\theta)\right), \qquad \varepsilon > 0. \qquad (19)$$

Note that by taking ε sufficiently small, $f(\mathbb{T})$ may be made to lie in an arbitrarily predefined neighborhood $\mathcal{U}$ of e. A GTF $g_1 : \mathbb{T} \times \mathbb{T} \longrightarrow \mathbb{R}$ can be constructed by defining $g_1(\theta) = f(\theta_2) \cdot f(\theta_1 + \theta_2)^{-1}$ for $\theta = (\theta_1, \theta_2) \in \mathbb{T} \times \mathbb{T}$. In the sequel we use 's' and 'c' for 'sin' and 'cos' respectively. Explicitly one has

$$\begin{aligned} g_1(\theta_1, \theta_2) = \Big(&\varepsilon(\mathrm{s}(\theta_1 + \theta_2) - \mathrm{s}(\theta_2)), \varepsilon(\mathrm{c}(\theta_1 + \theta_2) - \mathrm{c}(\theta_2)), \\ &\tfrac{\varepsilon^2}{4}(\mathrm{s}(2\theta_2) + \mathrm{s}(2\theta_1 + 2\theta_2)) \\ &- \tfrac{\varepsilon^2}{2}(\mathrm{s}(2\theta_2 + \theta_1) + \mathrm{s}(\theta_1))\Big). \end{aligned}$$

It is readily verified that g_1 is a GTF for (18). Consider the auxiliary control system $\ddot{\theta}_1 = u_3$, $\ddot{\theta}_2 = u_4$ on $T\mathbb{T} \times T\mathbb{T}$. Define an error signal $z = x \cdot Tg_1(\theta, \omega)^{-1}$, where $\omega = \dot{\theta}$. Given a second-order vector field, say $S_d(z) = (z_4, z_5, z_6, -z_1 - z_4, -z_2 - z_5, -z_3 - z_6)$, by Proposition 3 there exists a unique feedback law $(u_1(x, \theta, \omega, u_4), u_2(x, \theta, \omega, u_4), u_3(x, \theta, \omega, u_4))$ which sets the error dynamics equal to $\dot{z} = S_d(z)$. Thus, in closed-loop the trajectories $z(t)$ approach zero exponentially, which in turn forces the state $x(t)$ to approach $Tg_1(\theta, \omega)$ forcing the configuration trajectories to ultimately enter a neighborhood of e. The zero dynamics, however, must be analyzed to determine the evolution of the fiber-coordinates. Moreover, the trajectories should be made to converge to the zero-section. The resulting *controlled* zero dynamics is given by

$$\begin{pmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\omega}_1 \\ \dot{\omega}_2 \end{pmatrix} = \begin{pmatrix} \omega_1 \\ \omega_2 \\ \Gamma_1(\theta, \omega) \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ -2 + 2\,\mathrm{c}(\theta_1) \\ 1 \end{pmatrix} u_4, \qquad (20)$$

where

$$\begin{aligned} \Gamma_1(\theta, \dot{\theta}) = &-\mathrm{s}(2\theta_1 + 2\theta_2)\, \dot{\theta}_1^2 - 2\,\mathrm{s}(2\theta_1 + 2\theta_2)\dot{\theta}_1\dot{\theta}_2 - 2\,\mathrm{s}(\theta_1)\dot{\theta}_1\dot{\theta}_2 \\ &+ 2\,\mathrm{s}(\theta_1 + 2\theta_2)\dot{\theta}_1\dot{\theta}_2 - \mathrm{s}(2\,\theta_2)\dot{\theta}_2^2 + 2\,\mathrm{s}(\theta_1 + 2\theta_2)\dot{\theta}_2^2 \\ &- 2\,\mathrm{s}(2\theta_1 + 2\theta_2)\dot{\theta}_2^2. \end{aligned}$$

Consider an input of the form

$$\begin{aligned} u_4(\omega, t) = -k_2\omega_2 + \tfrac{1}{\tau}(&\alpha_{1,1}\mathrm{s}(t/\tau) + \alpha_{1,2}\mathrm{c}(t/\tau) + \cdots \\ &+ \alpha_{n,1}\mathrm{s}(n\,t/\tau) + \alpha_{n,2}\mathrm{c}(n\,t/\tau)), \end{aligned} \qquad (21)$$

with $k_2 > 0$. Computing the truncated expressions for averaging, at least for $n \leq 4$, the linearization of the truncated first order average expression is of the form

$$\begin{aligned} \dot{z}_1 &= z_3 \\ \dot{z}_2 &= z_4 \\ \dot{z}_3 &= a_{1,1}(z)\, {\alpha_{1,1}}^2 + a_{1,2}(z)\, {\alpha_{1,2}}^2 + \cdots \\ &\quad + a_{n,1}(z)\, {\alpha_{n,1}}^2 + a_{n,2}(z)\, {\alpha_{n,2}}^2 \\ \dot{z}_4 &= -k_2 z_4, \end{aligned}$$

where the terms $a_{i,j}$ are sums of sines and cosines of functions of z_1 and z_2. Note, however, that it is impossible to solve for $\alpha_{1,1}$ and $\alpha_{1,2}$ in terms of z if one wishes to obtain, say, $\dot{z}_3 = -k_1 z_3$.

By examining next the truncated expression for second order averaging, setting u_4 as in (21), with $n = 1$, we end up with a system that is no longer second-order. To preserve the second-order nature, one may choose $u_4 = -k_2 z_4 + \alpha_{1,2}\mathrm{c}(t/\tau)$. However, the application of such input yields $\dot{z}_3 = a_{1,2}(z)\alpha_{1,2}^2$ leading, in turn, to the impossibility of designing $\alpha_{1,2}$ to make the zero-section locally attractive.

The choice of a GTF may be essential to determine properties of the resulting zero dynamics, such as accessibility. Therefore, alternative GTFs may yield a controlled zero dynamics which allows one to achieve the required goals. Consider, for instance, the GTF defined on $\mathbb{T}^3$ given by

$$g(\theta) = \left(\varepsilon\mathrm{s}(\theta_1/2) + \varepsilon\mathrm{s}(\theta_3), \varepsilon\mathrm{c}(\theta_1/2), \tfrac{1}{4}\varepsilon^2\mathrm{s}(\theta_1) + \varepsilon^2\mathrm{s}(\theta_2)\right). \qquad (22)$$

The resulting controlled zero dynamics is

$$\begin{pmatrix} \dot{\theta} \\ \dot{\omega} \end{pmatrix} = S_{(\theta,\omega)} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 4\mathrm{c}(\theta_2) \\ 1 \\ 0 \end{pmatrix} u_4 + \begin{pmatrix} 0 \\ 0 \\ 0 \\ -4\mathrm{c}(\theta_1/2)\mathrm{c}(\theta_3) \\ 0 \\ 1 \end{pmatrix} u_5, \qquad (23)$$

where $S_{(\theta,\omega)} = (\omega_1, \omega_2, \omega_3, \Gamma(\theta, \omega), 0, 0)$ and $\Gamma(\theta, \omega) = -1/2\,\mathrm{s}(\theta_1)\omega_1^2 - 4\,\mathrm{s}(\theta_2)\omega_2^2 + 4\,\mathrm{c}(\theta_1/2)\mathrm{s}(\theta_3)\omega_3^2$.

The *unlifted* control vector fields of (23), $X_{1,\theta} = 4\,\mathrm{c}(\theta_2)\,\partial/\partial\theta^1 + \partial/\partial\theta^2$ and $X_{2,\theta} = -4\,\mathrm{c}(\theta_1/2)\mathrm{c}(\theta_3)\,\partial/\partial\theta^1 + \partial/\partial\theta^3$ satisfy the LARC at every point in $\mathbb{T}^3$ except, possibly, at points in a subset of measure zero. Hence, depending to a great degree on the drift vector field S, we may expect (23) to be accessible at generic points (θ, ω) in $T\mathbb{T}^3$.

The first order average truncate with control inputs of the form

$$\begin{aligned} u_4 &= -k_2 z_2 - k_5 z_5 + 1/\lambda\, \alpha\, \mathrm{s}(t/\lambda) \\ u_5 &= -k_3 z_3 - k_6 z_6 + 1/\lambda\, \beta\, \mathrm{s}(t/\lambda) \end{aligned} \qquad (24)$$

in (23) is given by

$$\begin{aligned} \dot{z}_1 &= z_4 \\ \dot{z}_2 &= z_5 \\ \dot{z}_3 &= z_6 \\ \dot{z}_4 &= a(z, \alpha, \beta) \\ \dot{z}_5 &= -k_2 z_2 - k_5 z_5 \\ \dot{z}_6 &= -k_3 z_3 - k_6 z_6, \end{aligned}$$

where the term $a(z, \alpha, \beta)$ is given by

$$\begin{aligned} a(z, \alpha, \beta) = &-\tfrac{1}{2}\mathrm{s}(z_1){z_4}^2 - 4\,\mathrm{s}(z_2){z_5}^2 + 4\,\mathrm{c}(z_1/2)\mathrm{s}(z_3){z_6}^2 \\ &- 4\,k_2 z_2 \mathrm{c}(z_2) - 4\,\mathrm{c}(z_2)k_5 z_5 + 4\,k_3 z_3 \mathrm{c}(z_1/2)\mathrm{c}(z_3) \\ &+ 4\,\mathrm{c}(z_1/2)\mathrm{c}(z_3)k_6 z_6 - 4\,\alpha^2\mathrm{s}(z_1)\mathrm{c}^2(z_2) \\ &- 4\,\alpha\,\beta\,\mathrm{c}(z_2)\mathrm{s}(z_1/2)\mathrm{c}(z_3) + 8\,\alpha\,\beta\,\mathrm{c}(z_2)\mathrm{s}(z_1)\mathrm{c}(z_1/2)\mathrm{c}(z_3) \\ &+ 4\,\beta^2\mathrm{s}(z_1/2)\mathrm{c}^2(z_3)\mathrm{c}(z_1/2) - 4\,\beta^2\mathrm{c}^2(z_1/2)\mathrm{c}^2(z_3\mathrm{s}(z_1). \end{aligned}$$

Observe that if k_2, k_5, k_3 and k_6 are strictly positive, then the components of the state z_2, z_5, z_3 and z_6 converge to zero exponentially. Hence we may focus on components z_1 and z_4 of the first order average truncate, under the assumption that the remaining components equal zero, namely

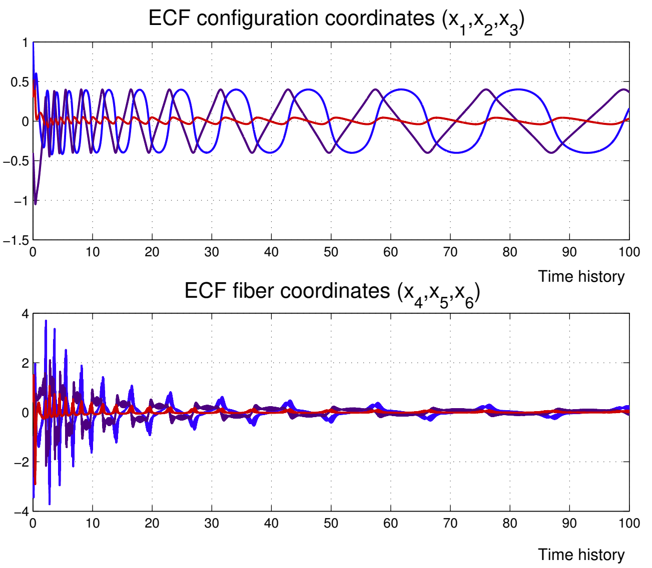
$$\begin{aligned} \dot{z}_1 &= z_4 \\ \dot{z}_4 &= -\tfrac{1}{2}\mathrm{s}(z_1)z_4^2 - 4\alpha^2\,\mathrm{s}(z_1) + 4\alpha\beta\,\mathrm{s}(3/2z_1) - \beta^2\,\mathrm{s}(2z_1). \end{aligned} \tag{25}$$

It is not intuitively evident how to design α and β as functions of z such that z_4 converges to zero. However, we may set them such that $\dot{z}_4$ is quadratic w.r.t. z_4, and such that the term having the product $\alpha\beta$ in (25) introduces "dissipation." For example, consider

$$\begin{aligned} \alpha &= -k_1\mathrm{s}(3/2z_1)\,\mathrm{sgn}(z_4)z_4, \\ \beta &= k_4 z_4, \end{aligned} \tag{26}$$

with k_1, k_4 positive. The closed-loop zero dynamics (23) derived from the time-varying feedback given by (24) and (26) turns out to be locally Lipschitz w.r.t. z, ensuring the local existence and uniqueness of trajectories. Numerical simulations with appropriately fixed parameters λ, $k_1, \ldots, k_6$, suggest that the zero-section is locally attractive. However, at present, we have no proof of the attractiveness of the zero-section.

Furthermore, numerical simulations for the *compound* system with control feedback $(u_1, u_2, u_3)(z, \theta, \omega, u_4, u_5)$ designed using the generalized vertically transverse function associated with (22), and $(u_4, u_5)(z, t)$ designed by means of high-order averaging in equations (24) and (26) suggest that the zero-section is locally attractive. Indeed, the simulation shown in the figure below is representative of the qualitative behavior exhibited by the compound system with different initial conditions. As depicted, the configuration components seem to enter a prescribed bounded neighborhood of e whereas the velocities seem to vanish asymptotically.



6. CONCLUSIONS.

This paper, reporting ongoing research, arises as an attempt to further our program to apply vertically transverse functions for control of (critical) underactuated mechanical systems. In particular, our goal is to introduce dissipation in a system which, under appropriate conditions, presents an ultimately oscillatory behavior. If this goal is accomplished, the target trajectories exhibit a behavior that enforces asymptotic vanishing of their velocity coordinates, a desirable feature in the control of mechanical systems. The problem is formulated as that of rendering a compact subset (the zero-section) of the auxiliary state manifold locally attractive. Given the nature of the control objective and the applicable obstructions, the solution, if any, calls for rather sophisticated techniques such as time-varying feedback. Our approach is based on the introduction of additional inputs for the auxiliary system, thanks to the use of generalized (vertically) transverse functions, followed by an application of high-order averaging techniques. The general problem remains to be solved, nonetheless numerical simulations support our conjecture that an appropriately chosen generalized vertically transverse function, coupled with the design of the *extra* control inputs for the zero dynamics via high-order averaging, may derive in the asymptotic vanishing of the velocities and therefore in practical configuration stabilization for a class of SMS.

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