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Chaotification for a Class of Nonlinear Systems with Backlash Functions ⋆

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Abstract: The problem of creating chaos for a class of nonlinear systems with backlash function is studied in this paper. By utilizing the characteristic of backlash function, some criteria for guaranteing the nonlinear system to be horseshoe type of chaos are derived based on Ši'lnikov heterclinic theorem and Ši'lnikov homoclinic theorem. Examples and simulations are given to illustrate the effectiveness of the theoretical results.

1. INTRODUCTION

Chaos control of dynamical systems has received a great deal of attention from the nonlinear dynamical community. It has demonstrated that chaos is actually useful and can also be well controlled. Contrary to suppress chaos, chaotification (also called "anticontrol of chaos") by mean of making an originally non-chaotic system chaotic, or enhancing a retaining existing chaos, has attracted special attention lately due to the great potential of chaos in non-traditional application. Chaos, when under appropriately monitoring, can provide a system designer with a variety of special properties, richness of flexibility, and a cornucopia of opportunities. Dynamical systems with chaotic behaviors have been reported very useful in some real world applications, such as human brain research, human heartbeat regulation, encryption, digital communication, information processing, particularly biological and medical systems, to just mention a few. This provides a strong motivation forth e current research on chaotification of the discrete-tune dynamical systems and the continuous-time dynamical systems.

Over the last decade, knowing that chaos can actually be useful and can be well controlled, the intensive study of chaotic dynamics has evolved from the traditional trend of understanding and analyzing chaos to the new attempt of controlling and utilizing it. A systematic design approach was proposed based on time-delay feedback for chaotification of a continuous-time, feedback linearizable system, see Wang et al. (2003); They also proposes a unified approach for generating chaos in n-dimensional continuous-time affine systems, with $n > 3$, based on the normal form of chaotic systems and nonlinear control theory, see Wang et al. (2003). G. Chen summarized the methods of generating chaos simply either in discrete systems or continuous systems, see Chen (2003). Meanwhile, some people applied a switching piecewise-linear controller to create chaotic attractors see Yang et al. (2002), Lu et al. (2002) and Zheng et al. (2004), especially some attractor are n-scroll attractors, see Tang et al. (2001) and Suykens et al. (1993).

On the other hand, in a dissipative system, if two distinct saddle points converge to different point when $t \to +\infty$ and $t \to -\infty$, respectively, this fold is called heterclinic orbit; if for a saddle point, when $t \to \pm\infty$, the orbit would converge to a same point, then it would compose an orbit from the saddle focus to itself, this fold would be called homoclinic orbit. An important homoclinic and heteroclinic situation have been analyzed by Ši'lnikov and the resulting behavior is called chaos, see Ši'lnikov (1965). In this case, the saddle point is said to be a saddle focus. Furthermore, P. Silva has put forward the Ši'lnikov homoclinic and heteroclinic method to make systems have the horseshoe type of chaos, see Silva (1993). In this paper, we will apply Ši'lnikov criterion to guarantee that a class of nonlinear system with backlash function has the horseshoes type of chaos via parameter analysis.

This paper is organized as follows. Section 2 introduces the concept of backlash function and presents some characteristics. In section 3, some analysis of the parameters for the nonlinear system are present and some theoretical results are obtained based on Ši'lnikov theorem. In Section 4, examples and simulations are given to prove the effectiveness of the theoretical results. Finally, some concluding remarks are given in Section 5.

2. BACKLASH FUNCTION DESCRIPTION

This section reviews the backlash function concept and presents some fundamental characteristic. It is well known that backlash function is one of the typical nonlinear function, as well as the stair function, the saturated function and the hysteretic function. It is depicted as following

$$f(x) = \begin{cases} k(x-q), & \dot{x} > 0, x > -(p-2q); \\ k(p-q), & \dot{x} < 0, x > (p-2q); \\ k(x+q), & \dot{x} < 0, x < (p-2q); \\ k(-p+q), & \dot{x} > 0, x < -(p-2q), \end{cases} \quad (1)$$

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where k is a positive constant, which is the slope of the backlash function, and p, q are real parameters. Backlash function is a piecewise-linear continuous function and it would switch on the state space $x = \pm(p-2q)$.

Generally speaking, the existence of backlash could cause disadvantage over the stability of systems, e.g., it can make the stability error larger, make the dynamical stable characteristic worse and then causes the vibration to intensify. Based on this, in this paper we will analyze a class of nonlinear system with backlash function to produce chaotic behavior under some conditions.

3. PARAMETER ANALYSIS AND THEORY RESULTS

Consider the following nonlinear system

$$\dot{X} = AX + F(x), \tag{2}$$

with

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix},$$

where the elements of matrix A are all real parameters, $X = [x, y, z]^T$ are state variables, $F(x) = [0, 0, f(x)]^T$ with $f(x)$ given in (1), and k, p, q are real parameters.

For a chaotic system, it is dissipative and its trajectory would fold and stretch itself repeatedly. Hence, its volume would run in a defined area and would not extend into the whole space. In this paper, we would use dissipative theory to analyze the system (2).

Considering the system (2), the time variation rate of the volume $V(t)$ or the trace of the system Jacobian (divergence of the vector fields) is

$$\frac{1}{V}\frac{dV}{Dt} = \partial\dot{x}/\partial x + \partial\dot{y}/\partial y + \partial\dot{z}/\partial z = a_{11} + a_{22} + a_{33}.$$

As a result, all system orbits will converge to a defined phase space at an exponential contraction rate $e^{(a_{11}+a_{22}+a_{33})t}$, if and only if $a_{11} + a_{22} + a_{33} < 0$.

Remark 1. A volume element V_0 is contracted by the flow into a volume element $V(0)e^{(a_{11}+a_{22}+a_{33})t}$ in time t. That is to say, each volume containing the system orbit shrinks to zero as $t \longrightarrow \infty$ at an exponential rate $e^{(a_{11}+a_{22}+a_{33})t}$ with $a_{11} + a_{22} + a_{33} < 0$, which is independent of x, y, z. Accordingly, all system orbits will ultimately be converged into a specific subset of zero volume and the asymptotic motion settles onto an attractor.

Then, we analysis the equilibria of system (2) and obtain the following results:
(i)if $\dot{x} > 0$ and $x > -(p-2q)$, the equilibrium is $x_e^1 = (-\frac{a_{11}a_{22}a_{33}}{detJ}q + q, -\frac{ka_{11}a_{23}}{detJ}q, \frac{ka_{11}a_{22}}{detJ}q)$;
(ii)if $\dot{x} < 0$ and $x > (p-2q)$, the equilibrium is $x_e^2 = (-\frac{ka_{11}a_{23}}{a_{11}a_{22}a_{33}}(p-q) + \frac{ka_{13}}{a_{11}a_{33}}(p-q), -\frac{ka_{23}}{a_{22}a_{33}}(p-q), -\frac{k}{a_{33}}(p-q))$;
(iii)if $\dot{x} < 0$ and $x < (p-2q)$, the equilibrium is $x_e^3 = (\frac{a_{11}a_{22}a_{33}}{detJ}q - q, \frac{ka_{11}a_{23}}{detJ}q, -\frac{ka_{11}a_{22}}{detJ}q)$;
(iv)if $\dot{x} > 0$ and $x < -(p-2q)$, the equilibrium is $x_e^4 = (\frac{ka_{11}a_{23}}{a_{11}a_{22}a_{33}}(p-q) - \frac{ka_{13}}{a_{11}a_{33}}(p-q), \frac{ka_{23}}{a_{22}a_{33}}(p-q), \frac{k}{a_{33}}(p-q))$,
where J is given below.

For the equilibria x_e^2 and x_e^4, the system Jacobian J' is

$$J' = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix}.$$

The character equation is

$$(\lambda' - a_{11})(\lambda' - a_{22})(\lambda' - a_{33}) = 0,$$

then the character value is $\lambda'_1 = a_{11}, \lambda'_2 = a_{22}, \lambda'_2 = a_{33}$. As matrix A is a real matrix, that is to say, the elements of matrix A are all real parameters. So, when the equilibria are x_e^2 and x_e^4, it is impossible to generate chaos for system (2).

Similarly, for the equilibria x_e^1 and x_e^3, the system Jacobian J is

$$J = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ k & 0 & a_{33} \end{bmatrix}.$$

The character equation is

$$\begin{aligned} \lambda^3 &- (a_{11} + a_{22} + a_{33})\lambda^2 + (a_{11}a_{22} + a_{11}a_{33} \\ &+ a_{22}a_{33} - ka_{13})\lambda - \det J = 0. \end{aligned} \tag{3}$$

Let

$$\begin{aligned} P_1 &= -(a_{11} + a_{22} + a_{33}), \\ P_2 &= a_{11}a_{22} + a_{11}a_{33} + a_{22}a_{33} - ka_{13}, \\ P_3 &= -\det J. \end{aligned}$$

Then, (3) could be written as

$$\lambda^3 + P_1\lambda^2 + P_2\lambda + P_3 = 0. \tag{4}$$

To calculate the characteristic eigenvalues, letting $\lambda = \mu + \frac{P_1}{3}$ yields

$$\mu^3 + A\mu + B = 0,$$

where

$$A = P_2 - \frac{P_1^2}{3}, B = P_3 - \frac{P_1P_2}{3} + \frac{2P_1^2}{27}.$$

For simplification, let

$$\Delta = \frac{B}{2}^2 + \frac{A}{3}^3.$$

Then, calculation yields

$$\Delta = \frac{P_3^2}{4} - \frac{P_1^2P_2^2}{108} + \frac{P_3P_1^3}{27} + \frac{P_1P_2P_3}{6} - \frac{P_2^3}{27}.$$

Equation (4) has a unique real root, along with a pair of complex conjugate roots; or equivalently, the original equation (3) has one real root and one pair of complex conjugate roots: $\lambda_1 = -\frac{P_1}{3} + \alpha - \beta$, and $\lambda_{2,3} = -\frac{P_1}{3} - \frac{\alpha-\beta}{2} \pm \frac{\sqrt{3}(\alpha+\beta)i}{2}$, where $\alpha = \sqrt[3]{\frac{-B}{2} + \sqrt{\Delta}}$, $\beta = \sqrt[3]{\frac{B}{2} + \sqrt{\Delta}}$

and $i = \sqrt{-1}$.

Lemma 1 [Ši'lnikov(1965)]
Suppose that two distinct equilibrium points of system (2) denoted by x_e and x'_e respectively are saddle foci, whose characteristic value γ_k and $\sigma_k \pm i\omega_k$ satisfy the following inequality:

$$|\gamma_k| > |\sigma_k| > 0, k = 1, 2$$

under constraint

$$\sigma_1\sigma_2 > 0, or \gamma_1\gamma_2 > 0.$$

If there exists a heteroclinic orbit jointing x_e and x'_e. Then:
(I) the Ši'lnikov map, defined in a neighborhood of the heteroclinic orbit, has a countable number of Smale horseshoes in its discrete dynamics;
(II) For any sufficiently small $C-$ perturbation g of f, the perturbed system

$$\frac{dx}{dt} = g(x), x \in R^3,$$

has at least a finite number of Smale horseshoes in the discrete Ši'lnikov map defined near the heteroclinic orbit;
(III) Both the original system (2) and the perturbed system have horseshoe type of chaos.

Remark 2. x_e and x'_e must be one pair of symmetric equilibria. Meanwhile, at the two equilibria x_e and x'_e, we have the same characteristic polynomials of the Jacobian matrices. The three corresponding eigenvalues have the following properties: one is negative (e.g. $\lambda_1 < 0$ in (3)), and the other two are a pair of complex conjugate values with positive real parts (e.g. $\mathrm{Re}[\lambda_{2,3}] > 0$ in (3)). Outside trajectories are being attracted in to the vicinity of its steady state and they are alternatively swirling between the two equilibria in the chaotic attractor, if a system has a two-scroll chaotic attractor.

Lemma 2 [Ši'lnikov(1965)]
Suppose that x_e is an equilibrium point of system (2), which ia a saddle focus. And its characteristic value are λ_k and $\sigma_k \pm i\omega_k$, assume it satisfy the following Ši'lnikov inequality

$$|\lambda_k| > |\sigma_k| > 0.$$

If there exists a homoclinic orbit connected at x_e, Then:
(i) the Ši'lnikov map, defined in a neighborhood of the heteroclinic orbit, has a countable number of Smale horseshoes in its discrete dynamics;
(ii) For any sufficiently small $C-$ perturbation g of f, the perturbed system

$$\frac{dx}{dt} = g(x), x \in R^3,$$

has at least a finite number of Smale horseshoes in the discrete Ši'lnikov map defined near the heteroclinic orbit;
(iii) Both the original system (2) and the perturbed system have horseshoe type of chaos.

Remark 3. We all know that, it is two main features of a Smale map that has one contraction and one prolongation. In Ši'lnikov theorems, the equilibria are of saddle-foci type, which is important since it has some basic properties of Smale horseshoes: the negative characteristic value corresponds to the contraction direction while the positive real part of the conjugate pair of complex characteristic values corresponds to the prolongation direction in the Smale map, and the contraction rate should be bigger than the prolongation rate. The combination of contraction and prolongation results in the Smale horseshoe structure of chaos of the Ši'lnikov types.

Theorem 1. Suppose that there are two distinct equilibrium points x_e^1 and x_e^3 of system (2) are saddle foci. if $\Delta > 0$ and $B = P_3 - \frac{P_1P_2}{3} + \frac{2P_1^2}{27} > 0$ are satisfied, then there exists a heterclinic orbit connecting the two equilibria and the system (2) possesses the horseshoe type of chaos.

Proof. For the equilibria x_e^1 and x_e^3 of system (2), which satisfy the assumption of theorem 1, we can find that

$$\lambda_1 \mathrm{Re}[\lambda_{2,3}] < 0, \mathrm{Im}[\lambda_{2,3}] = \frac{\sqrt{3}}{2}(\alpha + \beta) \neq 0; \qquad (5)$$

and $\Delta > 0$.

From the condition $B > 0$, we can obtain that

$$\alpha - \beta < 0. \qquad (6)$$

Because

$$P_1 = -(a_{11} + a_{22} + a_{33}) > 0, \qquad (7)$$

then from equation (6) and (7), we can obtain that

$$\lambda_1 = -\frac{P_1}{3} + \alpha - \beta < 0, \qquad (8)$$

and

$$|\lambda_1| = -\lambda_1 = \frac{P_1}{3} - (\alpha - \beta) > 0. \qquad (9)$$

substituting (8) into (5), we can find that

$$\mathrm{Re}[\lambda_{2,3}] = |\mathrm{Re}[\lambda_{2,3}]| = -\frac{P_1}{3} - \frac{\alpha - \beta}{2} > 0, \qquad (10)$$

Therefore, from (9) and (10) we can obtain that

$$\begin{aligned} |\lambda_1| - |\mathrm{Re}[\lambda_{2,3}]| &= \frac{4}{3}P_1 - \frac{1}{2}(\alpha - \beta) \\ &= \frac{1}{2}[\frac{1}{3}P_1 - (\alpha - \beta)] + \frac{7}{6}P_1 \\ &= -\frac{1}{2}\lambda_1 + \frac{7}{6}P_1 > 0, \end{aligned}$$

namely,

$$|\lambda_1| > |\mathrm{Re}[\lambda_{2,3}]|.$$

Therefore, applying Lemma 1, we can conclude that if $\Delta > 0$, $B = P_3 - \frac{P_1P_2}{3} + \frac{2P_1^2}{27} > 0$, and $\frac{P_1}{3} < -\frac{1}{4}(\alpha - \beta)$ are satisfied, a heterclinic orbit connecting x_e^1 and x_e^3 will exist.

Theorem 2. Suppose that there is an equilibrium point x_e^1 or x_e^3 for system (2), respectively, which is a saddle focus; if $\Delta > 0$, $B = P_3 - \frac{P_1 P_2}{3} + \frac{2P_1^2}{27} < 0$ and $0 < P_1 < \frac{3}{4}(\alpha - \beta)$ are satisfied, then there is a homoclinic orbit based at the equilibrium x_e^1 or x_e^3 and system (2) has the horseshoe type of chaos.

Proof. In this case, we can judge that

$$\lambda_1 = |\lambda_1| = -\frac{P_1}{3} + \alpha - \beta > 0, \qquad (11)$$

Simultaneously, for the equilibria x_e^1 or x_e^3 of system (2), which satisfy the assumption of theorem 2, we can find that

$$\lambda_1 \mathrm{Re}[\lambda_{2,3}] < 0, \mathrm{Im}[\lambda_{2,3}] = \frac{\sqrt{3}}{2}(\alpha + \beta) \neq 0; \qquad (12)$$

and $\Delta > 0$.

Substituting (11) into (12), we can find that

$$\mathrm{Re}[\lambda_{2,3}] = -\frac{P_1}{3} - \frac{\alpha - \beta}{2} < 0, \qquad (13)$$

then,

$$|\mathrm{Re}[\lambda_{2,3}]| = -\mathrm{Re}[\lambda_{2,3}] = \frac{P_1}{3}\frac{\alpha - \beta}{2} > 0. \qquad (14)$$

Therefore, from (11) and (14) we can have

$$|\lambda_1| - |\mathrm{Re}[\lambda_{2,3}]| = -\frac{2}{3}P_1 + \frac{1}{2}(\alpha - \beta) \qquad (15)$$

As $0 < P_1 < \frac{3}{4}(\alpha - \beta)$, substituting it into (15), we can obtain that

$$\begin{aligned} |\lambda_1| - |\mathrm{Re}[\lambda_{2,3}]| &= -\frac{2}{3}P_1 + \frac{1}{2}(\alpha - \beta) \\ &= -\frac{2}{3}(P_1 - \frac{3}{4}(\alpha - \beta)) > 0. \end{aligned}$$

Therefore, applying Lemma 2, we can conclude that if $\Delta > 0$, $B = P_3 - \frac{P_1 P_2}{3} + \frac{2P_1^2}{27} < 0$, and $0 < P_1 < \frac{3}{4}(\alpha - \beta)$ are satisfied, a homoclinic orbit based at the equilibrium x_e^1 or x_e^3 will exist.

4. EXAMPLES AND SIMULATIONS

Example 1. Consider system (2) with the following parameters $p = 3, q = 1, k = 1.5$ and

$$A = \begin{bmatrix} 1 & -10 & 2 \\ 0 & -3.5 & 3 \\ 0 & 0 & -5 \end{bmatrix}.$$

The chaotic attractor which possesses a heterclinic orbit is shown in Fig. 2. Furthermore, using the method of Wolf reconstructionwe could calculate the maximum Lyapunov exponent of this attractor $LE = 0.0143 > 0$. It proves further the theorem's effectiveness.

Example 2. Let the parameters for system (2) are $p = 3, q = 1, k = 1.5$ and

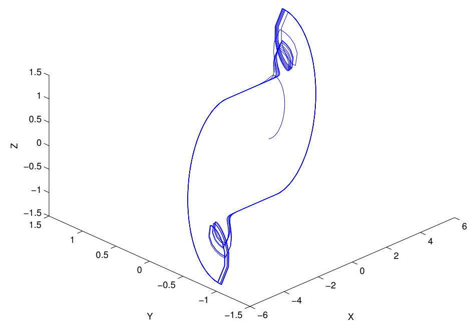

Fig. 1. The two-scroll chaotic attractor

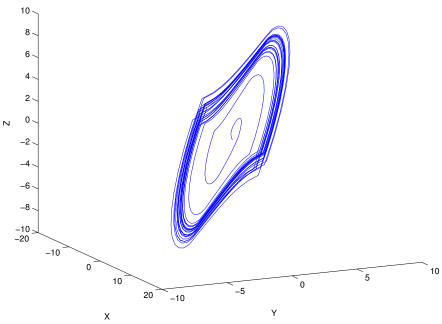

Fig. 2. The one-scroll chaotic attractor

$$A = \begin{bmatrix} -4.5 & -10 & 0.4 \\ 0 & 0.45 & 1.25 \\ 0 & 0 & -0.05 \end{bmatrix}.$$

Applying theorem 2, the corresponding system is chaotic, namely, there is a homoclinic orbit based at the equilibrium. We also find that the characteristic eigenvalues are $\lambda_1 = -5.3387, \lambda_{2,3} = 0.1693 \pm 1.8579j$, thus satisfying the Ši'lnikov inequality. Meanwhile, using the method of Wolf reconstruction, we could calculate that the maximum Lyapunov exponent of this attractor is $LE = 0.0394 > 0$, and it proves further that the system is chaotic under control of the backlash function. The attractor is shown in Fig. 3.

5. CONCLUSIONS

In this paper, we have investigated the problem of generating chaos for a class of nonlinear system with backlash function. Based on Ši'lnikov heterclinic or homoclinic theorem, we have proved that the existence of the horseshoe type of chaos for this kind of nonlinear system provided that the parameters satisfy the conditions.

It has been demonstrated that abundant complex dynamical behaviors can be generated if designed appropriately. Although this paper provides a class of nonlinear systems that fall into this category, the new finding is quite interesting both theoretically and practically, especially in the regard of possible future engineering applications of chaos generation.

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