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LMI-based Periodically Time-Varying Dynamical Controller Synthesis for Discrete-Time Uncertain Linear Systems ⋆

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Abstract: In this paper, we propose a new LMI-based method for robust state-feedback controller synthesis of discrete-time linear periodic/time-invariant systems subject to polytopic uncertainties. In stark contrast with existing approaches that are confined to static controller synthesis, we explore dynamic controller synthesis and reveal a particular periodically time-varying dynamical controller structure that allows LMI-based synthesis. In particular, we prove rigorously that the proposed design method encompasses the well-known extended-LMI-based design methods as particular cases. Through numerical experiments, we demonstrate that the suggested design method is indeed effective to achieve less conservative results.

Keywords: Robust control, periodic systems, polytopic uncertainties, linear matrix inequalities.

1. INTRODUCTION

Robust controller synthesis against parametric uncertainties of the plant has been a challenging topic in the community of the control theory. In the past years, we have observed drastic theoretical advances in this study area, and we could say that linear matrix inequality (LMI) plays an important role for such development. Since basic approaches based on quadratic stability concept was established (Bernussou et al. [1989]), intensive research effort has been made to obtain LMI-based results that are less conservative and computationally less demanding.

In the late 90's, a striking contribution along this line was made by de Oliveira et al. [1999], where the authors investigated robust *static* state-feedback stabilization problems of discrete-time linear systems subject to polytopic uncertainties. More specifically, the authors provided an "extended" LMI that characterizes Schur stability of a matrix, which enables us to design robust controllers in a less conservative fashion than the quadratic-stability-based approaches. This result was successfully extended to other control problems such as robust performance synthesis (de Oliveira et al. [2002]), robust filtering (Geromel et al. [2002]), robust stability and performance analysis (Peaucelle et al. [2000], Henrion et al. [2003], Ebihara and Hagiwara [2005]) and continuous-time robust controller synthesis (Apkarian et al. [2001], Shimomura et al. [2001], Ebihara and Hagiwara [2004]). Moreover, recent results such as Leite and Peres [2003] succeeded in deriving sharpened robustness *analysis* conditions. We could say that LMI-based robustness analysis methods are now fully matured, and those distinct approaches in Chesi et al. [2005] and Scherer [2005] related to sum-of-squares decomposition of positive polynomials are also quite effective for robustness analysis.

Unfortunately, however, these powerful LMI-based analysis conditions do not preserve convexity when we deal with robust controller synthesis problems. Due to this technical reason, to the best of the authors' knowledge, there is no LMI-based synthesis methods that go beyond the original results in de Oliveira et al. [1999, 2002]. Under these situations, recently, Arzelier et al. [2005] and Farges et al. [2007] showed an intriguing extensions of de Oliveira et al. [1999, 2002] to robust controller synthesis of uncertain periodic systems. Similarly to the LTI case, less conservative extended-LMI-based synthesis methods of periodically time-varying *static* controllers were suggested.

Even though the approaches in de Oliveira et al. [1999, 2002], Arzelier et al. [2005] and Farges et al. [2007] are promising, they are still conservative and leave some rooms for improvement. Nevertheless, if we persist in *static* controller synthesis, it should be hard to obtain a systematic single-shot LMI-based design method that outperforms these existing results. In view of these facts, in this paper, we explore an LMI-based design method of robust *dynamical* controllers for discrete-time uncertain linear periodic/time-invariant systems. To achieve this, we first consider a stability analysis problem of discrete-time periodic systems that has a particular structure. Based on this analysis, we next reveal a specific periodically time-varying dynamical controller (PTVDC) structure that allows us to carry out LMI-based synthesis. In the context of robust stabilizing state-feedback controller synthesis, it turns out that the suggested controller structure and the associated LMI-based design method encompass the extended-LMI-based methods as particular cases. From numerical experiments, we demonstrate that the proposed design method is surely effective to obtain less conservative results than the extended-LMI-based approaches.

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We use the following notations in this paper. The symbols $\mathbf{1}$ and $\mathbf{0}$ stand for the identity and zero matrices of appropriate dimensions, respectively. The set of symmetric matrices and positive-definite symmetric matrices of the size n are denoted by $\mathbf{S}_n$ and $\mathbf{S}_n^+$, respectively. For a real square matrix A, we define $\mathrm{He}\{A\} := A + A^T$. The convex hull of the collection of N elements $A_1, \cdots, A_N$ is denoted by $\mathrm{co}\{A_1, \cdots, A_N\}$.

In this paper, we make extensive use of the next lemma. The proof is given in the appendix section.

Lemma 1. For given $P \in \mathbf{S}_n$, $Q, S \in \mathbf{S}_m$, $R \in \mathbf{S}_l$, $V \in \mathbf{R}^{n\times m}$ and $W \in \mathbf{R}^{m\times l}$, the following conditions are equivalent.

(1) There exists $\mathcal{X} \in \mathbf{S}_m$ such that
$$\begin{bmatrix} P & V \\ V^* & Q+\mathcal{X} \end{bmatrix} \prec \mathbf{0}, \quad \begin{bmatrix} S-\mathcal{X} & W \\ W^* & R \end{bmatrix} \prec \mathbf{0}. \tag{1}$$
(2) The following condition holds:
$$\begin{bmatrix} P & V & \mathbf{0} \\ V^* & Q+S & W \\ \mathbf{0} & W^* & R \end{bmatrix} \prec \mathbf{0}. \tag{2}$$

2. PTVDC SYNTHESIS FOR DISCRETE-TIME PERIODIC SYSTEMS

2.1 Stability Analysis of Periodic Systems of Particular Structure

First of all we describe our underlying ideas for PTVDC synthesis. To this end, let us consider the stability analysis problem of discrete-time periodic systems that has a particular structure. For simplicity, we confine our discussion to the 2-periodic case for the time being. Thus, the difference equation for the system of interest will be

$$\begin{aligned} x_{k+1} &= A_{1,1}x_k, \\ x_{k+2} &= A_{2,2}x_{k+1} + A_{2,1}x_k, \\ x_{k+3} &= A_{1,1}x_{k+2}, \\ x_{k+4} &= A_{2,2}x_{k+3} + A_{2,1}x_{k+2}, \\ &\vdots \end{aligned} \tag{3}$$

where $A_{1,1}, A_{2,2}, A_{2,1} \in \mathbf{R}^{n\times n}$. Contrary to the standard state-space description of periodic systems, here we introduced the matrix $A_{2,1}$. This may seem strange, but does not violate the causality of the system. As we see in the sequel, the introduction of $A_{2,1}$ plays a key role for PTVDC synthesis to be presented.

To assess the stability of the system (3), let us denote the "hidden" state by ξ_k and rewrite (3) in the standard state-space form as follows:

$$\begin{aligned} \begin{bmatrix} x_{k+1} \\ \xi_{k+1} \end{bmatrix} &= \begin{bmatrix} A_{1,1} & \mathbf{0} \\ \mathbf{1} & \mathbf{0} \end{bmatrix} \begin{bmatrix} x_k \\ \xi_k \end{bmatrix}, \\ \begin{bmatrix} x_{k+2} \\ \xi_{k+2} \end{bmatrix} &= \begin{bmatrix} A_{2,2} & A_{2,1} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} x_{k+1} \\ \xi_{k+1} \end{bmatrix}, \\ &\vdots \end{aligned} \tag{4}$$

Then the monodromy matrix of the system (4) becomes

$$\Phi_2 = \begin{bmatrix} A_{2,2} & A_{2,1} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} A_{1,1} & \mathbf{0} \\ \mathbf{1} & \mathbf{0} \end{bmatrix} = \begin{bmatrix} A_{2,2}A_{1,1} + A_{2,1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}.$$

Hence, from Bittanti and Colaneri [2000], we can conclude that the system (3) is stable if and only if $\mathcal{A}_2 := A_{2,2}A_{1,1} + A_{2,1}$ is Schur stable. What is important here is that the Schur stability of $\mathcal{A}_2$ can be characterized via LMI that preserves the matrix structure of (3). Namely, we see that $\mathcal{A}_2$ is Schur stable if and only if there exists $X_1 \in \mathbf{S}_n^+$ and $\mathcal{F} \in \mathbf{R}^{2n\times 3n}$ such that

$$\begin{bmatrix} -X_1 & \mathbf{0} & \mathbf{0} \\ * & \mathbf{0} & \mathbf{0} \\ * & * & X_1 \end{bmatrix} + \mathrm{He}\left\{ \begin{bmatrix} A_{2,2} & A_{2,1} \\ -\mathbf{1} & A_{1,1} \\ \mathbf{0} & -\mathbf{1} \end{bmatrix} \mathcal{F} \right\} \prec \mathbf{0}. \tag{5}$$

This result follows immediately from Elimination Lemma (Skelton et al. [1997]) if we note

$$\begin{bmatrix} A_{2,2} & A_{2,1} \\ -\mathbf{1} & A_{1,1} \\ \mathbf{0} & -\mathbf{1} \end{bmatrix}^{\perp} = [\, \mathbf{1} \;\; A_{2,2} \;\; \mathcal{A}_2 \,].$$

We see that, similarly to Ebihara et al. [2005], the LMI (5) is affine with respect to $A_{1,1}$, $A_{2,2}$ and $A_{2,1}$, even though the monodromy matrix $\mathcal{A}_2$ involves a product among $A_{1,1}$ and $A_{2,2}$.

2.2 Stabilization of Periodic Systems via PTVDC

To move on to PTVDC synthesis, let us consider the "standard" 2-periodic system described by

$$\begin{aligned} x_{k+1} &= A_1x_k + B_1u_k, \\ x_{k+2} &= A_2x_{k+1} + B_2u_{k+1}, \\ x_{k+3} &= A_1x_{k+2} + B_1u_{k+2}, \\ x_{k+4} &= A_2x_{k+3} + B_2u_{k+3}, \\ &\vdots \end{aligned} \tag{6}$$

For this system, the controller discussed in Arzelier et al. [2005] is the static 2-periodic state-feedback controller of the form

$$\begin{aligned} u_k &= K_1x_k, \\ u_{k+1} &= K_2x_{k+1}, \\ &\vdots \end{aligned} \tag{7}$$

The closed-loop system is represented by

$$\begin{aligned} x_{k+1} &= (A_1 + B_1K_1)x_k, \\ x_{k+2} &= (A_2 + B_2K_2)x_{k+1}, \\ &\vdots \end{aligned} \tag{8}$$

From Arzelier et al. [2005], we see that the closed-loop system (8) is stable if and only if there exists $X_i \in \mathbf{S}_n^+$ and $G_i \in \mathbf{R}^{n\times n}$ $(i = 1, 2)$ such that

$$\begin{bmatrix} -X_{i+1} & (A_i + B_iK_i)G_i \\ * & X_i - G_i - G_i^T \end{bmatrix} \prec \mathbf{0}, \quad X_3 = X_1. \tag{9}$$

This result basically comes from extended Lyapunov Lemma (Bittanti et al. [1985]), in conjunction with the extended LMI results using extra variables (de Oliveira et al. [1999], Peaucelle et al. [2000]).

At this stage, to smoothen our subsequent discussions, we rewrite (9) into a conformable form to (5). It can be readily seen from Lemma 1 that (9) holds if and only if there exists $X_1 \in \mathbf{S}_n^+$, $G_i \in \mathbf{R}^{n\times n}$ $(i = 1, 2)$ such that

$$\begin{bmatrix} -X_1 & (A_2+B_2K_2)G_2 & \mathbf{0} \\ * & -G_2-G_2^T & (A_1+B_1K_1)G_1 \\ * & * & X_1-G_1-G_1^T \end{bmatrix} \prec \mathbf{0} \tag{10}$$

or equivalently,

$$\begin{bmatrix} -X_1 & \mathbf{0} & \mathbf{0} \\ * & \mathbf{0} & \mathbf{0} \\ * & * & X_1 \end{bmatrix} + \mathrm{He}\left\{ \begin{bmatrix} A_2+B_2K_2 & \mathbf{0} \\ -\mathbf{1} & A_1+B_1K_1 \\ \mathbf{0} & -\mathbf{1} \end{bmatrix} \begin{bmatrix} \mathbf{0} & G_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & G_1 \end{bmatrix} \right\} \prec 0. \tag{11}$$

Thus, by Lemma 1, we can eliminate the variable X_2 and also reduce the size of the LMI [1] .

The matrix inequalities in (9) can be reduced into LMIs via change of variables $Y_i = K_iG_i$ $(i = 1, 2)$ and used for state-feedback controller synthesis (Arzelier et al. [2005], Farges et al. [2007]). In particular, when we deal with polytopic-type uncertain systems, these LMIs enable us to design state-feedback via parameter-dependent Lyapunov functions so that less conservative results can be achieved (see the discussion in Subsection 2.3). The inequality (11) indicates that we can obtain exactly the same result with reduced computational burden.

Even though the approaches in Arzelier et al. [2005] and Farges et al. [2007] are promising for robust static state-feedback controller synthesis, they are still conservative and leave some rooms for improvement. Nevertheless, if we persist in static controller synthesis, it should be hard to obtain systematic LMI-based design methods that go beyond these existing results. This motivates us to explore dynamical state-feedback controller synthesis. In particular, motivated by the analysis results in Subsection 2.1, we are interested in the PTVDC of the form

$$\begin{aligned} u_k &= K_{1,1}x_k, \\ u_{k+1} &= K_{2,2}x_{k+1} + K_{2,1}x_k, \\ u_{k+2} &= K_{1,1}x_{k+2}, \\ u_{k+3} &= K_{2,2}x_{k+3} + K_{2,1}x_{k+2}, \\ &\ \ \vdots \end{aligned} \tag{12}$$

This controller leads to the closed-loop system described by

$$\begin{aligned} x_{k+1} &= (A_1 + B_1K_{1,1})x_k, \\ x_{k+2} &= (A_2 + B_2K_{2,2})x_{k+1} + B_2K_{2,1}x_k, \\ &\ \ \vdots \end{aligned} \tag{13}$$

From the discussion in Subsection 2.1, it is obvious that this closed-loop system is stable if and only if $\mathcal{A}_{\mathrm{cl},2} := (A_2 + B_2K_{2,2})(A_1 + B_1K_{1,1}) + B_2K_{2,1}$ is Schur stable. In addition, from (5), we see that $\mathcal{A}_{\mathrm{cl},2}$ is Schur stable if and only if there exist $X_1 \in \mathbf{S}_n^+$ and $\mathcal{F} \in \mathbf{R}^{2n\times 3n}$ such that

$$\begin{bmatrix} -X_1 & \mathbf{0} & \mathbf{0} \\ * & \mathbf{0} & \mathbf{0} \\ * & * & X_1 \end{bmatrix} + \mathrm{He}\left\{ \begin{bmatrix} A_2+B_2K_{2,2} & B_2K_{2,1} \\ -\mathbf{1} & A_1+B_1K_{1,1} \\ \mathbf{0} & -\mathbf{1} \end{bmatrix} \mathcal{F} \right\} \prec \mathbf{0}. \tag{14}$$

Consequently, we can assess the stability of the closed-loop system by this LMI.

Unfortunately, the inequality (14) is not suitable for controller synthesis due to the multiple bilinear terms between the variables $K_{1,1}, K_{2,2}, K_{2,1}$ and $\mathcal{F}$. To get around this difficulty, we restrict the variable $\mathcal{F}$ as follows:

$$\begin{bmatrix} -X_1 & \mathbf{0} & \mathbf{0} \\ * & \mathbf{0} & \mathbf{0} \\ * & * & X_1 \end{bmatrix} + \mathrm{He}\left\{ \begin{bmatrix} A_2+B_2K_{2,2} & B_2K_{2,1} \\ -\mathbf{1} & A_1+B_1K_{1,1} \\ \mathbf{0} & -\mathbf{1} \end{bmatrix} \mathcal{G} \right\} \prec \mathbf{0}, \tag{15}$$

$\mathcal{G} = \begin{bmatrix} \mathbf{0}_{2n,n} & \text{block-diag}(G_2, G_1) \end{bmatrix}, \quad G_i \in \mathbf{R}^{n\times n}\ (i = 1, 2).$

It follows that the closed-loop system is stable *if* there exist $X_1 \in \mathbf{S}_n^+$ and G_i $(i = 1, 2)$ such that (15) holds. We see that (15) can be reduced into an LMI via change of variables $Y_{i,i} = K_{i,i}G_i$ $(i = 1, 2)$ and $Y_{2,1} = K_{2,1}G_1$.

2.3 Robustly Stabilizing Controller Synthesis

Now we are ready to state the advantage of the controller (12) over the conventional form (7). Let us consider the case where the system (6) is subject to the polytopic uncertainties as follows:

$$\begin{bmatrix} A_1 & B_1 \\ A_2 & B_2 \end{bmatrix} \in \mathrm{co}\left\{ \begin{bmatrix} A_1^{[1]} & B_1^{[1]} \\ A_2^{[1]} & B_2^{[1]} \end{bmatrix}, \cdots \begin{bmatrix} A_1^{[L]} & B_1^{[L]} \\ A_2^{[L]} & B_2^{[L]} \end{bmatrix} \right\}.$$

To design robustly stabilizing static controllers, several LMI-based methods are suggested in Arzelier et al. [2005]. Among them, the result in Theorem 5 of Arzelier et al. [2005] is essentially equivalent to solving the LMIs:

$$\begin{gathered} \begin{bmatrix} -X_1^{[p]} & \mathbf{0} & \mathbf{0} \\ * & \mathbf{0} & \mathbf{0} \\ * & * & X_1^{[p]} \end{bmatrix} \\ +\mathrm{He}\left\{ \begin{bmatrix} A_2^{[p]}G_2 + B_2^{[p]}Y_{2,2} & \mathbf{0} \\ -G_2 & A_1^{[p]}G_1 + B_1^{[p]}Y_{1,1} \\ \mathbf{0} & -G_1 \end{bmatrix} \begin{bmatrix} \mathbf{0}\,\mathbf{1}\,\mathbf{0} \\ \mathbf{0}\,\mathbf{0}\,\mathbf{1} \end{bmatrix} \right\} \prec \mathbf{0}. \end{gathered} \tag{16}$$

Here, $p \in \{1, \cdots L\}$. If (16) is feasible, the robustly stabilizing feedback gains of the form (7) are given by $K_i = Y_{i,i}G_i^{-1}$ $(i = 1, 2)$.

On the other hand, it is clear from (15) that we can design robustly stabilizing PTVDC of the form (12) by solving

$$\begin{gathered} \begin{bmatrix} -X_1^{[p]} & \mathbf{0} & \mathbf{0} \\ * & \mathbf{0} & \mathbf{0} \\ * & * & X_1^{[p]} \end{bmatrix} \\ +\mathrm{He}\left\{ \begin{bmatrix} A_2^{[p]}G_2 + B_2^{[p]}Y_{2,2} & B_2^{[p]}Y_{2,1} \\ -G_2 & A_1^{[p]}G_1 + B_1^{[p]}Y_{1,1} \\ \mathbf{0} & -G_1 \end{bmatrix} \begin{bmatrix} \mathbf{0}\,\mathbf{1}\,\mathbf{0} \\ \mathbf{0}\,\mathbf{0}\,\mathbf{1} \end{bmatrix} \right\} \prec \mathbf{0}, \end{gathered} \tag{17}$$

$p \in \{1, \cdots L\}.$

By comparing (16) and (17), it is obvious that if (16) holds, then (17) holds with the same $X_1^{[p]}$, G_i, $Y_{i,i}$ $(i = 1, 2)$ and $Y_{2,1} = 0$. Hence, in the context of robust stabilizing controller synthesis for polytopic-type uncertain systems, we can obtain no more conservative results by (17). In fact, the controller synthesis based on (17) is surely effective as wee see in the next numerical examples.

Numerical Examples: To illustrate the effectiveness of the suggested method, we solved the problem discussed in Arzelier et al. [2005] [2] . More precisely, we solved the 2-periodic case problem in Example 1 of Arzelier et al. [2005], where our goal is to maximize a properly defined stability margin (i.e., the allowable maximal absolute value of an uncertain parameter). It was shown that the LMIs in (16) ensures the stability margin $\bar{\alpha}^{\max} = 0.80$. On the

[1] Note that the total size of the LMIs in (9) is $4n$.

[2] In this paper, every LMI-related computation is carried out with SeDuMi and Matlab, on PC with CPU Pentium IV 3.6 GHz.

other hand, maximizing the stability margin subject to (17), we obtained $\bar{\alpha}^{\max} = 0.90$. The resulting gains are

$$K_{1,1} = [\,2.8218\ \ -2.0095\,],$$
$$K_{2,2} = [\,1.2485\ \ -1.7025\,], \quad K_{2,1} = [\,0.3700\ \ 0.0008\,].$$

Since the suggested controller is dynamic, the resulting control performance could depend on the timing of the implementation. To examine this point, we solved the same problem by regarding (A_1, B_1) as (A_2, B_2) and vice versa. Then, we obtained $\bar{\alpha}^{\max} = 0.84$ and the gains

$$K_{1,1} = [\,1.1770\ \ -1.6873\,],$$
$$K_{2,2} = [\,3.1063\ \ -2.2083\,], \quad K_{2,1} = [\,0.3446\ \ -0.0861\,].$$

Thus, irrespective of the timing of the implementation, we can confirm the effectiveness of the PTVDC structure (12) when dealing with robust stabilization problems of polytopic-type uncertain systems.

2.4 Extension to the N-periodic Case

To extend the preceding results to general N-periodic case, let us consider the N-periodic system described by

$$\begin{array}{lll} x_{k+1} & = A_{1,1}x_k, & \\ x_{k+2} & = A_{2,2}x_{k+1} & + A_{2,1}x_k, \\ & \ \vdots & \\ x_{k+N} & = A_{N,N}x_{k+N-1} & + A_{N,N-1}x_{k+N-2} + \cdots A_{N,1}x_k, \\ x_{k+N+1} & = A_{1,1}x_{k+N}, & \\ & \ \vdots & \end{array} \tag{18}$$

We denote the associated transition matrix from x_k to x_{k+N} by $\mathcal{A}_N$. In addition, we denote the transition matrix from x_{k+p} $(p = 1, \cdots, N-1)$ to x_{k+N} in the case of $A_{i,j} = 0$ $(i = p+1, \cdots, N, j = 1, \cdots, p)$ by $\mathcal{A}_{N,p}$. Under these notations, the next lemma follows.

Lemma 2. For the the N-periodic system (18), let us define $\widehat{\mathcal{A}}_N \in \mathbf{R}^{(N+1)n \times Nn}$ by

$$\widehat{\mathcal{A}}_N := \begin{bmatrix} A_{N,N} & A_{N,N-1} & \cdots & \cdots & \cdots & A_{N,1} \\ -\mathbf{1} & A_{N-1,N-1} & A_{N-1,N-2} & \cdots & \cdots & A_{N-1,1} \\ \mathbf{0} & -\mathbf{1} & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & A_{2,2} & A_{2,1} \\ \vdots & & & \ddots & -\mathbf{1} & A_{1,1} \\ \mathbf{0} & \cdots & \cdots & \cdots & \mathbf{0} & -\mathbf{1} \end{bmatrix}. \tag{19}$$

Then, we have $\widehat{\mathcal{A}}_N^{\perp} = [\,\mathbf{1}\ \mathcal{A}_{N,N-1}\ \cdots\ \mathcal{A}_{N,1}\ \mathcal{A}_N\,]$.

Proof 3. From the definition of $\mathcal{A}_{N,p}$ $(p = 1, \cdots, N-1)$, we see that

$$\mathcal{A}_{N,N-1} = A_{N,N},$$
$$\mathcal{A}_{N,p} = \mathcal{A}_{N,p+1} + \sum_{i=p+1}^{N-1} \mathcal{A}_{N,i}A_{i,p+1}\ (p = 1, \cdots, N-2),$$
$$\mathcal{A}_N = \mathcal{A}_{N,1} + \sum_{i=1}^{N-1} \mathcal{A}_{N,i}A_{i,1}.$$

It follows that

$$\begin{aligned} \widehat{\mathcal{A}}_N^{\perp} &= [\,\mathbf{1}\ \ \ A_{N,N}\ \ \ A_{N,N-1} + \mathcal{A}_{N,N}A_{N-1,N-1}\ \cdots\ \cdots\ \cdots\,] \\ &= [\,\mathbf{1}\ \mathcal{A}_{N,N-1}\ \ \ \mathcal{A}_{N,N-2}\ \cdots\ \mathcal{A}_{N,1}\ \mathcal{A}_N\,]. \end{aligned}$$

This completes the proof. □

As in the 2-periodic case, we can confirm that the system (18) is stable if and only if $\mathcal{A}_N$ is Schur stable. With this fact and Lemma 2, we can obtain the next result.

Theorem 4. The N-periodic system (18) is stable if and only if there exist $X_1 \in \mathbf{S}_n^+$, $\mathcal{F} \in \mathbf{R}^{Nn \times (N+1)n}$ such that

$$\text{block-diag}(-X_1, \mathbf{0}_{(N-1)n,(N-1)n}, X_1) + \text{He}\{\widehat{\mathcal{A}}_N\mathcal{F}\} \prec \mathbf{0}.$$

Based on Theorem 4, we next consider to design N-PTVDCs. To this end, let us consider the "standard" N-periodic system described by

$$\begin{array}{lll} x_{k+1} & = A_1x_k & + B_1u_k, \\ x_{k+2} & = A_2x_{k+1} & + B_2u_{k+1}, \\ & \ \vdots & \\ x_{k+N} & = A_Nx_{k+N-1} & + B_Nu_{k+N-1}, \\ x_{k+N+1} & = A_1x_{k+N} & + B_1u_{k+N}, \\ & \ \vdots & \end{array} \tag{20}$$

For this system, we design the N-PTVDC of the form

$$\begin{array}{ll} u_k & = K_{1,1}x_k, \\ u_{k+1} & = K_{2,2}x_{k+1} + K_{2,1}x_k, \\ & \ \vdots \\ u_{k+N-1} & = K_{N,N}x_{k+N} + K_{N,N-1}x_{k+N-1} \\ & \qquad + K_{N,N-2}x_{k+N-2} + \cdots + K_{N,1}x_k \\ u_{k+N} & = K_{1,1}x_{k+N}, \\ & \ \vdots \end{array} \tag{21}$$

Then, the closed-loop system can be described by

$$\begin{array}{ll} x_{k+1} & = (A_1 + B_1K_{1,1})x_k, \\ x_{k+2} & = (A_2 + B_2K_{2,2})x_{k+1} + B_2K_{2,1}x_k, \\ & \ \vdots \\ x_{k+N} & = (A_N + B_NK_{N,N})x_{k+N-1} \\ & \qquad + B_NK_{N,N-1}x_{k+N-1} + \cdots + B_NK_{N,1}x_k, \\ x_{k+N+1} & = (A_1 + B_1K_{1,1})x_{k+N}, \\ & \ \vdots \end{array} \tag{22}$$

Thus, from Theorem 4, we can obtain the next results.

Theorem 5. For the N-periodic system (20) and N-PTVDC (21), let us define $\widehat{\mathcal{A}}_{\text{cl},N} \in \mathbf{R}^{(N+1)n \times Nn}$ as given at the top of the next page. Then, the closed-loop system constructed from (20) and (21) is stable if and only if there exist $X_1 \in \mathbf{S}_n^+$, $\mathcal{F} \in \mathbf{R}^{Nn \times (N+1)n}$ such that

$$\text{block-diag}(-X_1, \mathbf{0}_{(N-1)n,(N-1)n}, X_1) + \text{He}\{\widehat{\mathcal{A}}_{\text{cl},N}\mathcal{F}\} \prec \mathbf{0}.$$

Corollary 6. The closed-loop system constructed from (20) and (21) is stable *if* there exist $X_1 \in \mathbf{S}_n^+$, $G_i \in \mathbf{R}^{n \times n}$ $(i = 1, \cdots, N)$ such that

$$\text{block-diag}(-X_1, \mathbf{0}_{(N-1)n,(N-1)n}, X_1) + \text{He}\{\widehat{\mathcal{A}}_{\text{cl},N}\mathcal{G}\} \prec \mathbf{0} \tag{24}$$

where $\mathcal{G} = [\,\mathbf{0}_{Nn,n}\ \text{block-diag}(G_N, \cdots, G_1)\,]$. The matrix inequality (24) can be reduced into an LMI via change of variables $Y_{i,j} = K_{i,j}G_j$ $(i = 1, \cdots, N, j = 1, \cdots, i)$.

As we have seen, the particular structure (21) allows us to carry out LMI-based controller synthesis. We note that (24) with $K_{i,j} = \mathbf{0}$ $(i \neq j)$ is essentially equivalent to the condition given in Theorem 5 of Arzelier et al. [2005], even though here we have succeeded in eliminating $N-1$ Lyapunov matrices and reducing the size of the LMIs. In

$$\widehat{\mathcal{A}}_{\mathrm{cl},N} := \begin{bmatrix} A_N + B_N K_{N,N} & B_N K_{N,N-1} & \cdots & \cdots & \cdots & B_N K_{N,1} \\ -\mathbf{1} & A_{N-1} + B_{N-1} K_{N-1,N-1} & B_{N-1} K_{N-1,N-2} & \cdots & \cdots & B_{N-1} K_{N-1,1} \\ \mathbf{0} & -\mathbf{1} & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & A_2 + B_2 K_{2,2} & B_2 K_{2,1} \\ \vdots & & & \ddots & -\mathbf{1} & A_1 + B_1 K_{1,1} \\ \mathbf{0} & \cdots & \cdots & \cdots & \mathbf{0} & -\mathbf{1} \end{bmatrix}. \tag{23}$$

the context of robust stabilizing controller synthesis for polytopic-type uncertain systems, it is obvious that we can obtain no more conservative results than Arzelier et al. [2005] by means of (24).

Numerical Examples: To illustrate the effectiveness of the suggested robust controller synthesis method, we solved the 3-periodic case problem in Example 1 of Arzelier et al. [2005]. Maximizing the stability margin subject to (24), we obtained $\bar{\alpha}^{\max} = 0.63$ and feedback gains

$K_{1,1} = [\, 3.1022 \ \ -2.2131 \,]$,
$K_{2,2} = [\, 1.1555 \ \ -2.1051 \,]$, $\quad K_{2,1} = [\, 0.2791 \ \ -0.0092 \,]$,
$K_{3,3} = [\, -2.4095 \ \ -2.4242 \,]$, $\quad K_{3,2} = [\, -0.3164 \ \ 0.0178 \,]$,
$K_{3,1} = [\, -0.0406 \ \ 0.1783 \,]$.

This result surely goes beyond the stability margin $\bar{\alpha}^{\max} = 0.49$ obtained in Arzelier et al. [2005]. When we change the timing of implementation, we obtained $\bar{\alpha}^{\max} = 0.67$ and $\bar{\alpha}^{\max} = 0.57$, respectively.

3. APPLICATION TO LTI SYSTEM SYNTHESIS

In this section, we clarify that the suggested PTVDC structure and the associated LMI-based design method are promising when dealing with LTI systems as well. It is of course meaningless to consider the complicated controller structure (21) for nominal system stabilization. However, when we consider robust stabilization problems of polytopic uncertain systems (de Oliveira et al. [1999]) to which definite solution is not currently available, the PTVDC brings some improvements over the existing methods (at the expense of complicated controller structure).

Let us consider the polytopic-type uncertain LTI system described by

$$x_{k+1} = Ax_k + Bu_k. \tag{25}$$

where $[A \ B] \in \mathrm{co}\left\{[A^{[1]} \ B^{[1]}], \cdots, [A^{[L]} \ B^{[L]}]\right\}$. By regarding this LTI system as N-periodic (i.e, $A_i = A$, $B_i = B$ $(i = 1, \cdots, N)$ in (20)), we see from (24) that the closed-loop system constructed from (25) and (21) is robustly stable if there exist $X_1^{[p]} \in \mathbf{S}_n^+$, $G_i \in \mathbf{R}^{n\times n}$ $(i = 1, \cdots, N)$ such that

$$\begin{aligned} &\text{block-diag}(-X_1^{[p]}, \mathbf{0}_{(N-1)n,(N-1)n}, X_1^{[p]}) \\ &\qquad\qquad + \mathrm{He}\{\widehat{\mathcal{A}}_{\mathrm{cl},N}^{[p]} \mathcal{G}\} \prec \mathbf{0}, \end{aligned} \tag{26}$$

$p \in \{1, \cdots L\}$.

Here, $\widehat{\mathcal{A}}_{\mathrm{cl},N}^{[p]}$ is defined by (23) with A_i, B_i $(i = 1, \cdots, N)$ replaced by $A^{[p]}, B^{[p]}$, respectively. As before, the inequality (26) can be used for controller synthesis via change of variables $Y_{i,j} = K_{i,j} G_j$ $(i = 1, \cdots, N, j = 1, \cdots, i)$.

The advantage of (26) can be stated in light of the extended LMI given as follows (de Oliveira et al. [1999]):

$$\begin{bmatrix} -X^{[p]} & A^{[p]}G + B^{[p]}Y \\ * & X^{[p]} - G - G^T \end{bmatrix} \prec \mathbf{0}, \quad p \in \{1, \cdots L\}. \tag{27}$$

If (27) holds, we can confirm that the uncertain system (25) can be robustly stabilized via time-invariant static state-feedback $u_k = Kx_k$ where $K = YG^{-1}$.

When comparing these two LMI conditions, we see again from Lemma 1 that if (27) holds, then (26) holds with $G_i = G$ $(i = 1, \cdots, N)$, $X_1^{[p]} = X^{[p]}$ $(p \in \{1, \cdots L\})$, $Y_{i,i} = Y$ $(i = 1, \cdots, N)$ and $Y_{i,j} = 0$ $(i \neq j)$. Thus, again, it is ensured that we can obtain no more conservative results by means of (26). In fact, from numerical examples shown below, we can confirm that it is surely possible to reduce the conservatism of de Oliveira et al. [1999] by designing PTVDCs.

Remark 7. Note that we *cannot* expect improvement of control performance over de Oliveira et al. [1999] if we design N-periodically time-varying *static* controllers. To see this, let us consider the case where we regard the original LTI polytopic system as 2-periodic and consider the following LMIs for 2-periodic static controller synthesis:

$$\begin{bmatrix} -X_{i+1}^{[p]} & A^{[p]}G_i + B^{[p]}Y_i \\ * & X_i^{[p]} - G_i - G_i^T \end{bmatrix} \prec \mathbf{0}, \ i = 1, 2, \ X_3^{[p]} = X_1^{[p]}. \tag{28}$$

Then, we see that if (28) holds, then (27) holds with $X^{[p]} = X_1^{[p]} + X_2^{[p]}$ $(p \in \{1, \cdots L\}$, $G = G_1 + G_2$ and $Y = Y_1 + Y_2$. This implies that if we can find a robustly stabilizing 2-periodic static controller by (28), it is always possible to find a time-invariant static robustly stabilizing controller by (27). Thus, for the improvement of control performance, it is essential to make the controllers to be dynamic.

In the above discussions, we have clarified the advantage of PTVDCs by regarding LTI polytopic systems as N-periodic. When dealing with N_o-periodical polytopic systems, we can state a similar advantage in the following way. Namely, suppose we regard the original N_o-periodic system as rN_o-periodic and assume that the robust version of the LMI (24) corresponding to $N = rN_o$ is feasible. Then, the robust version of (24) obtained by regarding the system as sN_o-periodic is always feasible if s is a multiple of r.

Numerical Examples: To illustrate the effectiveness of the suggested design method, we first solved the robust state-feedback stabilization problem for polytopic-type uncertain LTI systems discussed in Section 4 of de Oliveira et al. [1999]. By regarding the system as N-periodic, we maximized the stability margin γ_N. Then, we obtained

$\gamma_1 = 0.88$ (this is obtained in de Oliveira et al. [1999]), $\gamma_2 = 0.90$, $\gamma_3 = 0.98$, $\gamma_4 = 1.02$, $\gamma_5 = 1.04$ and $\gamma_6 = 1.04$. The CPU time were 0.34, 0.40, 0.53, 0.74, 1.00 and 1.34 [sec], respectively.

We next solved the 2-periodic case problem in Example 1 of Arzelier et al. [2005], by regarding the system as 2-periodic, 4-periodic and 6-periodic. Maximizing the stability margin subject to (24), we obtained $\bar{\alpha}_2^{\max} = 0.9$, $\bar{\alpha}_4^{\max} = 0.93$ and $\bar{\alpha}_6^{\max} = 0.94$. The CPU time were 0.29, 0.38 and 0.50 [sec], respectively.

Finally we solved the 3-periodic case problem in Example 1 of Arzelier et al. [2005], by regarding the system as 3-periodic and 6-periodic. Maximizing the stability margin subject to (24), we obtained $\bar{\alpha}_3^{\max} = 0.63$ and $\bar{\alpha}_6^{\max} = 0.69$. The CPU time were 0.37 and 0.52 [sec], respectively.

4. CONCLUSION

In this paper, we proposed an LMI-based design method of periodically time-varying dynamical state-feedback controllers for robust stabilization of discrete-time uncertain linear periodic/time-invariant systems. Through numerical experiments, we confirmed that the suggested design method is indeed effective to obtain less conservative results. The suggested controller structure and the associated LMI-based synthesis method work effectively in other control problems, for which definite solutions are not available in the current state of the art. We will report this result elsewhere in the near future.

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Appendix A. PROOF OF LEMMA 1

Proof 8. 1.→2. It is obvious that (1) holds if and only if

$$\begin{bmatrix} P & V & \mathbf{0} & \mathbf{0} \\ V^* & Q+\mathcal{X} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & S-\mathcal{X} & W \\ \mathbf{0} & \mathbf{0} & W^* & R \end{bmatrix} \prec \mathbf{0}. \tag{A.1}$$

Thus we have

$$\begin{bmatrix} \mathbf{1}_n & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1}_m & \mathbf{1}_m & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1}_l \end{bmatrix} \begin{bmatrix} P & V & \mathbf{0} & \mathbf{0} \\ V^* & Q+\mathcal{X} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & S-\mathcal{X} & W \\ \mathbf{0} & \mathbf{0} & W^* & R \end{bmatrix} \begin{bmatrix} \mathbf{1}_n & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1}_m & \mathbf{1}_m & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1}_l \end{bmatrix}^T$$
$$= \begin{bmatrix} P & V & \mathbf{0} \\ V^* & Q+S & W \\ \mathbf{0} & W^* & R \end{bmatrix} \prec \mathbf{0}.$$

2.→1. If (2) holds, then we have

$$\begin{bmatrix} P & V \\ V^* & Q+S-WR^{-1}W^* \end{bmatrix} \prec \mathbf{0}. \tag{A.2}$$

It follows that there exists $\varepsilon > 0$ such that

$$\begin{bmatrix} P & V \\ V^* & Q+S-WR^{-1}W^* + \varepsilon \mathbf{1}_m \end{bmatrix} \prec \mathbf{0}. \tag{A.3}$$

If we let $\mathcal{X} = S - WR^{-1}W^* + \varepsilon \mathbf{1}_m \in \mathbf{S}_m$, we see that the two inequalities in (1) follow. This completes the proof. □