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Non-asymptotic Model Quality Assessment of Transfer Functions at Multiple Frequency Points ⋆

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Abstract: In this paper we develop methods for evaluating uncertainties in the frequency response of a dynamical system based on finitely many input-output data points. We extend the "Leave-out Sign-dominant Correlation Regions" (LSCR) algorithm to deliver confidence regions with a guaranteed probability for the frequency response at multiple frequency points, and we introduce a computationally efficient scheme which enables confidence regions to be constructed separately at each frequency. Simulation examples illustrating the usefulness of the developed algorithm are provided.

1. INTRODUCTION

In dynamical system identification, providing a description of the uncertainties associated with the nominal system model is as important as obtaining the nominal model itself, especially for the synthesis of robust controllers. A popular technique for evaluating the model quality is based on constructing *asymptotic* statistical confidence regions. This is a well-matured approach and the confidence regions can be computed relatively easily (see Ljung (1999)). However in some cases, using asymptotic theory may lead to unreliable results (Garatti, Campi & Bittanti 2004) when applied to a finite number of data points.

In this paper, we consider a *non-asymptotic* method based on finitely many data points as, e.g., considered in Bayard (1993), Goodwin, Gevers & Ninnes (1992), and Campi & Weyer (2005). In Campi & Weyer (2005) and Campi & Weyer (2006), the "Leave-out Sign-dominant Correlation Regions" (LSCR) algorithm was introduced for constructing confidence regions for system parameters with guaranteed probabilities.

In Ko, Weyer & Campi (2007), we extended the LSCR technique to provide confidence regions for the frequency response at multiple frequencies based on a finite number of (multi-sine) input-output data points. For this, *a priori* information about the tail of the impulse response sequence of the system, which cannot be deduced from finitely many data points, was incorporated into the algorithm. Generally, since the confidence region at each frequency is dependent on those of the other frequencies, it is computationally prohibitive to construct the confidence regions for the case of multi-sine inputs. In order to substantially reduce this difficulty, we devise a fast algorithm for the construction of the confidence regions at each frequency.

In the next section, the general procedure for construction of simultaneous confidence regions in case of multi-sine input is presented. In Section 3, a computationally inexpensive algorithm is introduced. Two simulation examples demonstrating the usefulness of the proposed approach are given in Section 4. Section 5 concludes the study.

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2. MAIN ALGORITHM

Here we extend the algorithm for discrete-time systems introduced in Ko et al. (2007) to provide confidence regions for a continuous-time transfer function at multiple frequencies.

2.1 Problem definition

Data generating system and input: Consider the following linear continuous-time system with additive noise

$$y(t) = \int_0^\infty g^0(\tau)u(t-\tau)d\tau + v(t) \qquad (1)$$

where $g^0(\tau)$ is the impulse response of the true system. The transfer function is the Laplace transform of $g^0(\tau)$ and given by

$$G^0(s) = \int_0^\infty g^0(t)e^{-st}dt.$$

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The following multi-sine input is applied to the system

$$u(t)=\begin{cases}\sum_{m=1}^{L} A_m \cos\varphi_m(t)=\sum_{m=1}^{L} A_m \cos(\Omega_m t+\psi_m), & t\geqslant 0\\ 0, & t<0.\end{cases} \tag{2}$$

We can express the output $y(t)$ as

$$\begin{aligned} y(t) &= \sum_{m=1}^{L} A_m \int_0^t g^0(\tau)\cos\varphi_m(t-\tau)\, d\tau + v(t)\\ &= \sum_{m=1}^{L} A_m \left[a_m^0 \cos\varphi_m(t) - b_m^0 \sin\varphi_m(t) + \bar{y}_m(t)\right] + v(t), \end{aligned} \tag{3}$$

where

$$\begin{aligned} a_m^0 &\triangleq \mathrm{Re}\left\{G^0(j\Omega_m)\right\}, \quad b_m^0 \triangleq \mathrm{Im}\left\{G^0(j\Omega_m)\right\},\\ \bar{y}_m(t) &\triangleq -\mathrm{Re}\left\{\int_t^\infty g^0(\tau)e^{-j\Omega_m\tau}d\tau \cdot e^{j\varphi_m(t)}\right\}. \end{aligned} \tag{4}$$

Here $\bar{y}_m(t)$ is the transients due to that $u(t)=0$ for all $t<0$.

The magnitude and phase of the frequency response at a frequency Ω_m are given by

$$\begin{aligned} |G^0(j\Omega_m)| &= \sqrt{a_m^{02} + b_m^{02}},\\ \angle G^0(j\Omega_m) &= \tan^{-1}\left(b_m^0/a_m^0\right). \end{aligned} \tag{5}$$

Let the input and output be sampled at time instants $t=kT$ for $k=0,1,2,\ldots,N_1$ with sampling period T. We collect input-output data $\left\{u(kT),\ y(kT)\right\}_{k=0,1,2,\ldots,N_1}$.

Assumptions:

(**A1**) $|g^0(\tau)| \leqslant M_g e^{-\rho\tau}$, for some $0< M_g <\infty$ and $\rho>0$, where M_g and ρ are known *a priori*.

(**A2**) The sampled noise $v(kT)$ is an independent random variable with symmetric distribution around zero, and all $v(kT)$ admit densities.

Due to the finite number of input-output data points we cannot extract information about the tail of the impulse response $g^0(\tau)$ from the measured data and hence the only way the effect of the tail can be taken into account is via *a priori* information. Thus we assume that system information (**A1**) is available. An iterative method for estimating the bounds is proposed in de Vries & Van den Hof (1995).

$\bar{y}_m(t)$ in (4) which is due to the tail is unknown but we can bound it using Assumption (**A1**)

$$\begin{aligned} |\bar{y}_m(t)| &\leqslant \left|\int_t^\infty g^0(\tau)d\tau\right|\\ &\leqslant M_g \int_t^\infty e^{-\rho\tau}d\tau = \frac{M_g e^{-\rho t}}{\rho} \triangleq \gamma(t). \end{aligned} \tag{6}$$

In the case of non-zero initial conditions due to the unknown past input $u(t)$ for $t<0$, we have the additional term $\int_{-\infty}^0 g^0(t-\tau)u(\tau)d\tau$ in (3). To bound this unknown term, we need *a priori* information about $u(t),\ t<0$, e.g., $|u(t)|\leqslant M_u,\ t<0$. However for simplicity in this paper we consider only the case $u(t)=0,\ t<0$.

Objective: The goal is to provide guaranteed confidence regions for $\boldsymbol{\theta}^0 \triangleq \left[a_1^0, b_1^0, \ldots, a_L^0, b_L^0\right]^T$ using $N_1-\ell$ input-output data pairs $\{u(kT), y(kT)\}_{k=\ell+1,\ldots,N_1}$ measured after waiting $\ell\cdot T$ seconds to reduce the effect of the transient response of the system. Confidence regions for the magnitude and phase can subsequently be obtained using (5).

2.2 Construction of confidence regions

This section describes the procedures for constructing confidence regions for the parameter $\boldsymbol{\theta}^0$ using the correlation between the output prediction error and the input.

Procedure for the construction of confidence regions:

(**P1**) Compute the predictor and the corresponding prediction error

$$\begin{aligned} \hat{y}_k(\boldsymbol{\theta}) &= \sum_{m=1}^{L} A_m\left[a_m \cos\varphi_m(kT) - b_m \sin\varphi_m(kT)\right],\\ \epsilon_k(\boldsymbol{\theta}) &= y(kT) - \hat{y}_k(\boldsymbol{\theta}),\ \boldsymbol{\theta} \triangleq \left[a_1, b_1, \ldots, a_L, b_L\right]^T, \end{aligned} \tag{7}$$

for $k=\ell+1,\ldots,N_1$.

(**P2**) Compute the correlation functions for $r=1,\ldots,L$ and $k=\ell+1,\ldots,N_1$

$$f_{r,k}^a(\boldsymbol{\theta}) \triangleq \epsilon_k(\boldsymbol{\theta})\cos\varphi_r(kT),\ \ f_{r,k}^b(\boldsymbol{\theta}) \triangleq \epsilon_k(\boldsymbol{\theta})\sin\varphi_r(kT). \tag{8}$$

(**P3**) Select a positive integer M and construct M binary (0,1) stochastic strings of length $N \triangleq N_1-\ell$ as follows: Let $h_0 = h_{0,\ell+1}, \ldots, h_{0,N_1}$ be the string of all zeros. Every element of the remaining $M-1$ strings takes the value 0 or 1 with probability 0.5 each, and the elements are independent of each other. However, if a string turns out to be equal to an already constructed string, remove this string and construct another string according to the same rule to be used in its place. Name the constructed non-zero strings $h_{1,\ell+1}, \ldots, h_{1,N_1}; h_{2,\ell+1}, \ldots, h_{2,N_1}; \ldots; h_{M-1,\ell+1}, \ldots, h_{M-1,N_1}$. Each of the constructed stochastic strings determines a set of time indices to be used for calculating the corresponding empirical correlation functions in Step (**P4**).

(**P4**) Compute the scaled empirical correlation functions for $i=0,\ldots,M-1$

$$C_{r,i}^a(\boldsymbol{\theta}) \triangleq \sum_{k=\ell+1}^{N_1} h_{i,k} f_{r,k}^a(\boldsymbol{\theta}),\ C_{r,i}^b(\boldsymbol{\theta}) \triangleq \sum_{k=\ell+1}^{N_1} h_{i,k} f_{r,k}^b(\boldsymbol{\theta}), \tag{9}$$

which can be expressed as (10) on the top of the next page.

(**P5**) For a fixed $r \in \{1,\ldots,L\}$ select an integer q in the interval $[1,(M+1)/2)$ and find the region $\boldsymbol{\Theta}_r^a$ ($\boldsymbol{\Theta}_r^b$) such that for all $\boldsymbol{\theta} \in \boldsymbol{\Theta}_r^a$ ($\boldsymbol{\theta} \in \boldsymbol{\Theta}_r^b$) at least q of the empirical correlation estimates $C_{r,i}^a(\boldsymbol{\theta})$ ($C_{r,i}^b(\boldsymbol{\theta})$) satisfy $C_{r,i}^a(\boldsymbol{\theta}) - \Gamma_{r,i}^a < 0$ and $C_{r,i}^a(\boldsymbol{\theta}) + \Gamma_{r,i}^a > 0$ ($C_{r,i}^b(\boldsymbol{\theta}) - \Gamma_{r,i}^b < 0$ and $C_{r,i}^b(\boldsymbol{\theta}) + \Gamma_{r,i}^b > 0$) where

$$\begin{aligned} \Gamma_{r,i}^a &\triangleq A \sum_{k=\ell+1}^{N_1} h_{i,k}\gamma(kT)\left|\cos\varphi_r(kT)\right|,\\ \Gamma_{r,i}^b &\triangleq A \sum_{k=\ell+1}^{N_1} h_{i,k}\gamma(kT)\left|\sin\varphi_r(kT)\right|,\ \ A \triangleq \sum_{m=1}^{L} A_m. \end{aligned}$$

Here $\gamma(kT)$ is evaluated at $t=kT$ using (6).

$$
\begin{aligned}
C^a_{r,i}(\boldsymbol{\theta}) &= \sum_{m=1}^{L} A_m \Bigg\{ (a^0_m - a_m) \Bigg[\sum_{k=\ell+1}^{N_1} h_{i,k} \cos\varphi_m(kT)\cos\varphi_r(kT) \Bigg] - (b^0_m - b_m) \Bigg[\sum_{k=\ell+1}^{N_1} h_{i,k} \sin\varphi_m(kT)\cos\varphi_r(kT) \Bigg] \Bigg\} \\
&\quad + \sum_{k=\ell+1}^{N_1} h_{i,k} \sum_{m=1}^{L} A_m \bar{y}_m(kT)\cos\varphi_r(k) + \sum_{k=\ell+1}^{N_1} h_{i,k}\, v(kT)\cos\varphi_r(kT), \quad i=0,\dots,M-1 \\
C^b_{r,i}(\boldsymbol{\theta}) &= \sum_{m=1}^{L} A_m \Bigg\{ (a^0_m - a_m) \Bigg[\sum_{k=\ell+1}^{N_1} h_{i,k} \cos\varphi_m(kT)\sin\varphi_r(kT) \Bigg] - (b^0_m - b_m) \Bigg[\sum_{k=\ell+1}^{N_1} h_{i,k} \sin\varphi_m(kT)\sin\varphi_r(kT) \Bigg] \Bigg\} \\
&\quad + \sum_{k=\ell+1}^{N_1} h_{i,k} \sum_{m=1}^{L} A_m \bar{y}_m(kT)\sin\varphi_r(kT) + \sum_{t=\ell+1}^{N_1} h_{i,k}\, v(kT)\sin\varphi_r(kT), \quad i=0,\dots,M-1
\end{aligned}
\tag{10}
$$

The intuitive idea of Step (**P5**) is that for the true parameter, i.e., $\boldsymbol{\theta} = \boldsymbol{\theta}^0$, the terms in the parenthesis $\{\cdot\}$ in (10) disappear, and the next term after each parenthesis can be bounded using (6)

$$
\left| \sum_{k=\ell+1}^{N_1} h_{i,k} \sum_{m=1}^{L} A_m \bar{y}_m(kT)\cos\varphi_r(kT) \right| \leqslant \Gamma^a_{r,i}
$$

$$
\left| \sum_{k=\ell+1}^{N_1} h_{i,k} \sum_{m=1}^{L} A_m \bar{y}_m(kT)\sin\varphi_r(kT) \right| \leqslant \Gamma^b_{r,i}.
$$

Then the empirical correlation functions for the true parameter satisfy for $i = 0, \dots, M-1$

$$
\begin{aligned}
C^a_{r,i}(\boldsymbol{\theta}^0) - \Gamma^a_{r,i} &\leqslant \sum_{k=\ell+1}^{N_1} h_{i,k} v(kT)\cos\varphi_r(kT) \leqslant C^a_{r,i}(\boldsymbol{\theta}^0) + \Gamma^a_{r,i} \\
C^b_{r,i}(\boldsymbol{\theta}^0) - \Gamma^b_{r,i} &\leqslant \sum_{k=\ell+1}^{N_1} h_{i,t} v(kT)\sin\varphi_r(kT) \leqslant C^b_{r,i}(\boldsymbol{\theta}^0) + \Gamma^b_{r,i}.
\end{aligned}
\tag{11}
$$

Since $v(kT)$ is symmetrically distributed around zero, it is unlikely that nearly all of $C^a_{r,i}(\boldsymbol{\theta}^0) + \Gamma^a_{r,i}$ (or $C^b_{r,i}(\boldsymbol{\theta}^0) + \Gamma^b_{r,i}$) take on negative values or nearly all of $C^a_{r,i}(\boldsymbol{\theta}^0) - \Gamma^a_{r,i}$ (or $C^b_{r,i}(\boldsymbol{\theta}^0) - \Gamma^b_{r,i}$) take on positive values. In Step (**P5**) above we exclude the regions in parameter space where all $C^a_{r,i}(\boldsymbol{\theta}) + \Gamma^a_{r,i}$'s (or $C^b_{r,i}(\boldsymbol{\theta}) + \Gamma^b_{r,i}$'s) are negative or all $C^a_{r,i}(\boldsymbol{\theta}) - \Gamma^a_{r,i}$'s (or $C^b_{r,i}(\boldsymbol{\theta}) - \Gamma^b_{r,i}$'s) are positive except for a small number q. We therefore expect that $\boldsymbol{\theta}^0 \in \boldsymbol{\Theta}^a_r$ ($\boldsymbol{\theta}^0 \in \boldsymbol{\Theta}^b_r$) with high probability which is indeed the case as shown in the following theorem.

Theorem 1. Under assumptions (**A1**) *and* (**A2**), *the sets* $\boldsymbol{\Theta}^a_r$ *and* $\boldsymbol{\Theta}^b_r$ *constructed above are such that*

$$
\Pr\{\boldsymbol{\theta}^0 \in \boldsymbol{\Theta}^a_r\} \geqslant 1 - \frac{2q}{M} \text{ and } \Pr\{\boldsymbol{\theta}^0 \in \boldsymbol{\Theta}^b_r\} \geqslant 1 - \frac{2q}{M}.
$$

Proof. See Ko, Weyer & Campi (2008) □

Since each one of the sets $\boldsymbol{\Theta}^a_r$ and $\boldsymbol{\Theta}^b_r$ can be unbounded in some directions of the parameter space, we construct a simultaneous confidence region for all frequency points by intersecting all of the confidence regions

$$
\hat{\boldsymbol{\Theta}}_{2L} = \bigcap_{r=1}^{L} \left(\boldsymbol{\Theta}^a_r \bigcap \boldsymbol{\Theta}^b_r \right).
$$

The following theorem is immediate from Theorem 1 using the Bonferroni inequality.

Theorem 2. Under assumptions (**A1**) *and* (**A2**),

$$
\Pr\{\boldsymbol{\theta}^0 \in \hat{\boldsymbol{\Theta}}_{2L}\} \geqslant 1 - 2L \cdot \frac{2q}{M}.
$$

Remark 1. (**Classical correlation method**). The current method for constructing confidence regions is closely connected to the classical frequency analysis by the correlation method (Ljung 1999, p.171), where estimates of of the frequency response are obtained by considering the correlations between the *output* and cosines and sines of the same frequency as the input signal. Here, in order to evaluate the uncertainties of the frequency response, we use the correlations between the *output prediction error* and cosines and sines of the input frequency. □

3. COMPUTATIONAL ASPECT: DECOUPLING STRING GENERATION

Using the procedure in the previous section, theoretically we can construct non-asymptotic confidence regions for the frequency response at multiple frequencies. However, each of the empirical correlation functions (10) depends on the whole set of parameters and thus the resulting confidence region for each parameter is dependent on all the other parameters. Therefore, constructing the simultaneous confidence region $\hat{\boldsymbol{\Theta}}_{2L}$ can be computationally prohibitive.

In this section we develop a method for the generation of decoupling binary strings which enable us to construct the confidence regions for a^0_r and b^0_r at frequency Ω_r independent of the other parameters $\{a_m, b_m\}_{m=1,\dots,L\ (m\neq r)}$ and thus we have $C^a_{r,i}(\boldsymbol{\theta}) = C^a_{r,i}(a_r)$ and $C^b_{r,i}(\boldsymbol{\theta}) = C^b_{r,i}(b_r)$.

Before generating such decoupling binary strings, we conduct the following *experiment design*:

(P0) Experiment design for uncorrelated confidence regions:

- **(a)** The allowable set of frequencies in the multi-sine input (2) are integer multiples of a baseline frequency Ω_0
$$\Omega_m = i_m \cdot \Omega_0 \text{ for } i_m \in \mathbb{N},\ m = 1, 2, \dots, L. \tag{12}$$
- **(b)** The sampling period T is chosen such that we get about $4 \cdot S$ samples per period of the highest frequency present in the input signal. To be specific, choose a positive integer S (which should be at least 2 or 3) and calculate the sampling interval
$$T = \frac{T_0}{S \cdot 2^P}, \tag{13}$$

where $T_0 = 2\pi/\Omega_0$ is the period of the baseline frequency and P is given by

$$P = \lfloor \log_2(2 \cdot i_{\max}) \rfloor + 1. \tag{14}$$

Here

$$i_{\max} = \max_{m=1,\ldots,L} \{i_m\} \tag{15}$$

and $\lfloor(\cdot)\rfloor$ denotes that the number $(\cdot)$ is rounded to the nearest integer towards zero. With this notation, $i_{\max}$ is expressed as

$$2 \cdot i_{\max} = 2^{P-1} + Q$$

with $0 \leqslant Q = 2i_{\max} - 2^{P-1} < 2^{P-1}$. We denote the number of the samples within one period of the baseline sinusoid as

$$N_0 = S \cdot 2^P. \tag{16}$$

(**c**) Choose a positive integer n for the total length of the samples N

$$N \triangleq N_1 - \ell = n \cdot N_0. \tag{17}$$

In order to compute the confidence regions for each parameter separately, it can be seen from (10) that we need for $m, r = 1, \ldots, L$ with $r \neq m$

$$\sum_{k=\ell+1}^{N_1} h_{i,k} \cos\varphi_m(kT) \cos\varphi_r(kT) = 0,$$

$$\sum_{k=\ell+1}^{N_1} h_{i,k} \sin\varphi_m(kT) \sin\varphi_r(kT) = 0,$$

and for $m, r = 1, \ldots, L$

$$\sum_{k=\ell+1}^{N_1} h_{i,k} \sin\varphi_m(kT) \cos\varphi_r(kT) = 0,$$

where $\varphi_m(kT) = i_m\Omega_0 kT + \psi_m$. Expressing each product of two trigonometric functions in terms of a sum of two trigonometric functions, we find that the highest frequency generated from these products of trigonometric functions is $\Omega_{\max} = 2 \cdot i_{\max} \cdot \Omega_0$. For the decoupling-string generation, it suffices to find a set of time indices $\{k_j\} \subset \{\ell+1, \ell+2, \ldots, N_1\}$ such that

$$\sum_{\{k_j\}} \sin(i_m \Omega_0 T k_j) = 0 \quad \text{for all } i_m \in \{1, \ldots, 2 \cdot i_{\max}\}. \tag{18}$$

For this, instead of Step (**P3**) in Section 2.2 we use the following new step (**P3′**) for generating a set of decoupling binary strings.

(P3′) Algorithm for decoupling string generation: The idea for generating the decoupling strings is as follows: We divide each period of the baseline sinusoid into 2^P equal segments consisting of S time indices each. Since we have n periods of the baseline sinusoid, we get $n \cdot 2^P$ segments. For the first segment in each period, we randomly select a set of time indices (out of the S time indices), and we denote these sets as $\mathbf{K}_{1,p}$ for $p = 1, \ldots, n$. We determine the binary string corresponding to $\mathbf{K}_{1,p}$ and then use this string for all the 2^{P-1} remaining segments in the p-th period. This way we obtain one binary string for the whole sample length. This procedure is repeated $M-2$ times and a binary string of all zeros is added. The procedure is summarized below.

(**1**) Determine n index sets $\mathbf{K}_{1,p}$ for $p = 1, \ldots, n$ such that each index set $\mathbf{K}_{1,p}$ consists of the elements from $\{(p-1)N_0+1, \ldots, (p-1)N_0+S\}$ by randomly choosing with distribution

$$\begin{cases} k \in \mathbf{K}_{1,p}, \text{ with probability } 0.5 \\ k \notin \mathbf{K}_{1,p}, \text{ with probability } 0.5 \end{cases} \tag{19}$$

for all $k \in \{(p-1)N_0+1, \ldots, (p-1)N_0+S\}$. Let $\mathbf{K}_{1,p} = \{k_{1,p}, \ldots, k_{q_p,p}\}$ with $q_p \leqslant S$ and

$$\mathbf{K}_{j,p} = \{k_{1,p} + (j-1)S,\ k_{2,p} + (j-1)S, \ldots, k_{q_p,p} + (j-1)S\} \tag{20}$$

for $j = 2, \ldots, 2^P$ and $p = 1, \ldots, n$. Then, construct

$$\mathbf{J}_p = \{\mathbf{K}_{1,p},\ \mathbf{K}_{2,p},\ \mathbf{K}_{3,p}, \ldots,\ \mathbf{K}_{2^P,p}\} \tag{21}$$

for $p = 1, \ldots, n$. By concatenating the sets $\mathbf{J}_p$, we generate

$$\mathcal{J}_1 = \Big\{\mathbf{J}_1,\ \mathbf{J}_2, \ldots,\ \mathbf{J}_n\Big\}. \tag{22}$$

This is a set of time indices which satisfies the decoupling requirement (18) (for the proof see Ko et al. (2008)).

(**2**) By repeating Step (**1**) $M-2$ times and adding a null set $\mathcal{J}_0 = \emptyset$, we construct the set

$$\mathcal{J} = \left\{ \begin{array}{c} \mathcal{J}_0 \\ \mathcal{J}_1 \\ \vdots \\ \mathcal{J}_{M-1} \end{array} \right\}. \tag{23}$$

However, if an index set turns out to be equal to an already constructed set, remove this set and construct another set according to Step (**1**) to be used in its place. From $\mathcal{J}$, construct the corresponding binary (0,1) strings $h_i = h_{i,\ell+1}, h_{i,\ell+2}, \ldots, h_{i,N_1}$ of length N such that

$$\begin{cases} h_{i,\ell+k} = 1, \text{ if } k \in \mathcal{J}_i \\ h_{i,\ell+k} = 0, \text{ if } k \notin \mathcal{J}_i \end{cases} \tag{24}$$

for $k = 1, \ldots, N$ and $i = 0, 1, \ldots, M-1$.

Remark 2. (**Theorem 1 and 2**). It can be easily shown that even if we use the procedure (**P3′**) for generating the binary strings, the results in Theorem 1 and 2 still hold. □

Remark 3. (**Shape of the confidence regions**). With the use of decoupling binary strings, each correlation function depends only on one parameter, i.e., $C^a_{r,i}(\boldsymbol{\theta}) = C^a_{r,i}(a_r)$, $C^b_{r,i}(\boldsymbol{\theta}) = C^b_{r,i}(b_r)$. This means that each correlation function determines the maximum and minimum values of the corresponding parameter in the confidence regions. Hence the shape of confidence regions at each frequency is rectangular, as illustrated in a simulation example in Section 4.1. □

Remark 4. (**Magnitude and phase formulation**). The procedures in the previous sections for the construction of confidence regions in terms of the real and imaginary parts of the frequency response can be easily modified to produce confidence regions for the magnitude and phase by expressing the predictor in terms of the magnitude and phase instead of (7), as remarked in Ko et al. (2007). However, the magnitude and phase at each frequency cannot be decoupled as above when calculating the empirical correlation functions. Therefore, computationally it is better to construct confidence regions for the magnitude and phase by converting the confidence regions for the real and imaginary parts by using (5). □

4. SIMULATION EXAMPLE

4.1 Two-frequency case

Suppose that the true system is given by (1) with the transfer function

$$G^0(s) = \frac{2.5}{s+2.5}. \tag{25}$$

$G^0(s)$ is of course unknown to the user and may be a system of very high order as far as the user is concerned. In order to construct confidence regions at $\Omega_1 = 1$ and $\Omega_2 = 2$ rad/sec (in this case the baseline frequency corresponds to $\Omega_0 = \Omega_1 = 1$ rad/sec), we first determine the sampling time period $T = 0.0262$ second using (13) with $i_{\max} = 2$, $P = 3$, and $S = 30$. The number of the samples within one period of the baseline sinusoid is $N_0 = 240$. By choosing $n = 4$, the total length of the samples to be used for the confidence regions is $n \cdot N_0 = 4 \times 240 = 960$.

By applying the following input signal to the system

$$u(t) = \begin{cases} \cos\Omega_1 t + \cos\Omega_2 t, & t \geqslant 0 \\ 0, & t < 0 \end{cases}$$

and gathering the output measurements $\{y(kT)\}$, we construct confidence regions for the frequency responses at the two frequencies. In order to avoid the transient phase, we wait $\ell \cdot T = 3.93$ seconds ($\ell = 150$) and then collect 960 samples of input-output data such that $N_1 = 1110$. The sampled noise $v(kT)$ is a zero-mean gaussian white noise sequence with variance of 0.16^2. This information about the noise is given for completeness of description but unknown to the user except for the fact that $v(kT)$ is a white noise sequence with symmetric distribution around zero.

The parameter vector is $\boldsymbol{\theta}^0 = [a_1^0 \;\; b_1^0 \;\; a_2^0 \;\; b_2^0]^T$ with $a_i^0 = \mathrm{Re}\{G^0(j\Omega_i)\}$ and $b_i^0 = \mathrm{Im}\{G^0(j\Omega_i)\}$. The parameters bounding the tail are $M_g = 3.8$ and $\rho = 0.8$. The predictor and prediction error are given by

$$\hat{y}_k(\boldsymbol{\theta}) = \sum_{m=1}^{2} \left[a_m \cos(\Omega_m kT) - b_m \cos(\Omega_m kT)\right]$$

$$\epsilon_k(\boldsymbol{\theta}) = y(kT) - \hat{y}_k(\boldsymbol{\theta}), \text{ for } k = 151, \ldots, 1110,$$

and we calculate

$$f^a_{r,k}(\boldsymbol{\theta}) = \epsilon_k(\boldsymbol{\theta}) \cos(\Omega_r kT), \quad f^b_{r,k}(\boldsymbol{\theta}) = \epsilon_k(\boldsymbol{\theta}) \sin(\Omega_r kT)$$

for $r = 1, 2$ and $k = 151, \ldots, 1110$.

Now in order to construct uncorrelated confidence regions for the parameters, we generate decoupling binary strings by following the steps in (**P3′**) of Section 3: we generate $n = 4$ index sets $\mathbf{K}_{1,p}$ for $p = 1, \ldots, 4$ according to (19) with $S = 30$. And then we generate $\mathbf{K}_{j,p}$ for $j = 1, \ldots, 8$ and $p = 1, \ldots, 4$ as in (20). $\mathbf{J}_p$ is then constructed as

$$\mathbf{J}_p = \left\{\mathbf{K}_{1,p}, \; \mathbf{K}_{2,p}, \ldots, \; \mathbf{K}_{8,p}\right\}, \text{ for } p = 1, \ldots, 4$$

and finally we construct

$$\mathscr{J}_1 = \left\{\mathbf{J}_1, \;\; \mathbf{J}_2, \;\; \mathbf{J}_3, \;\; \mathbf{J}_4\right\}.$$

By repeating this procedure 798 times and adding the null set $\mathscr{J}_0$, we obtain the set $\mathscr{J}$ in (23) with $M = 800$ and the corresponding binary strings $\{h_0; h_1; \ldots; h_{M-1}\}$ are given by (24).

Fig. 1 illustrates the generation of decoupling time indices: if a time index k_0 is randomly chosen in the first segment, then 7 additional time indices are chosen in the remaining 7 segments, each of which is separated by $S = 30$ samples from the other. It can be observed that these eight time indices satisfy the requirement (18) for all 4 frequencies respectively.

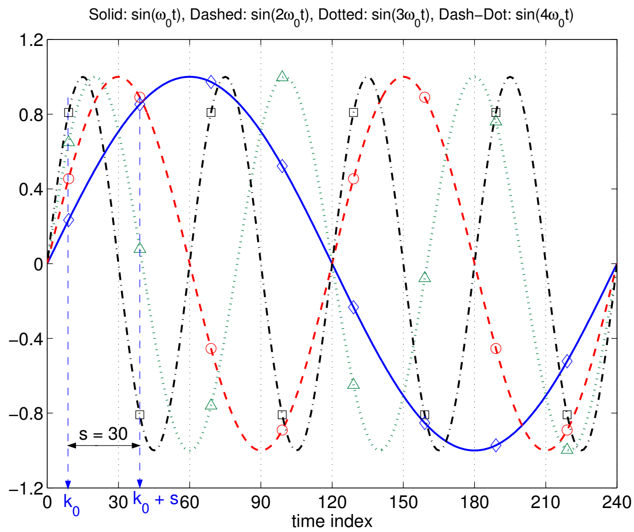

Fig. 1. Generation of a set of decoupling time indices

Now using the generated binary strings we calculate the scaled empirical correlation functions for $r = 1, 2$ and $i = 0, \ldots, 799$

$$C^a_{r,i}(\boldsymbol{\theta}) = \sum_{k=151}^{1110} h_{i,t} f^a_{r,k}(\boldsymbol{\theta}), \;\; C^b_{r,i}(\boldsymbol{\theta}) = \sum_{k=151}^{1110} h_{i,t} f^b_{r,k}(\boldsymbol{\theta}).$$

Then we construct the confidence region $\boldsymbol{\Theta}^a_r$ by discarding those values of $\boldsymbol{\theta} = [a_1 \;\; b_1 \;\; a_2 \;\; b_2]^T$ for which at most four empirical correlation functions satisfy $C^a_{r,i}(\boldsymbol{\theta}) - \Gamma^a_{r,i} < 0$ or $C^a_{r,i}(\boldsymbol{\theta}) + \Gamma^a_{r,i} > 0$. The construction for $\boldsymbol{\Theta}^b_r$ is similar. Then following Theorem 2, $\boldsymbol{\theta}^0$ belongs to the simultaneous region $\hat{\boldsymbol{\Theta}}_4 = \cap_{r=1}^2 (\boldsymbol{\Theta}^a_r \cap \boldsymbol{\Theta}^b_r)$ with probability at least $1 - 2 \cdot 2 \cdot 2 \cdot 5/800 = 0.95$ with $L = 2$ and $q = 5$.

These results are shown in Fig. 2 and Fig. 3 where the blank areas are the confidence regions at each frequency and the true values are marked with ★. The regions where at most four $C^a_{r,i}(\boldsymbol{\theta}) - \Gamma^a_{r,i}$ functions were negative are marked with □, and the regions where at most four $C^a_{r,i}(\boldsymbol{\theta}) + \Gamma^a_{r,i}$ were positive are marked with ◯. Likewise × and + represents the regions where at most four values of $C^b_{r,i}(\boldsymbol{\theta}) - \Gamma^b_{r,i}$ and $C^b_{r,i}(\boldsymbol{\theta}) + \Gamma^b_{r,i}$ were negative and positive, respectively. As we can see, each step in the construction of the confidence region excludes a particular region.

4.2 Ten-frequency case

Consider the same system as in (1) and (25) in the previous subsection. Our task is now to construct a simultaneous confidence region with 95% probability for the frequency response at ten frequencies $\Omega = 0.1, 0.2, 0.4, 0.6, 0.8, 1, 2, 4, 6, 8$ rad/sec (the baseline frequency is $\Omega_0 = 0.1$ rad/sec) from which we obtain $P = 8$. Choosing $S = 4$ requires the sampling time $T = 0.0614$ second.

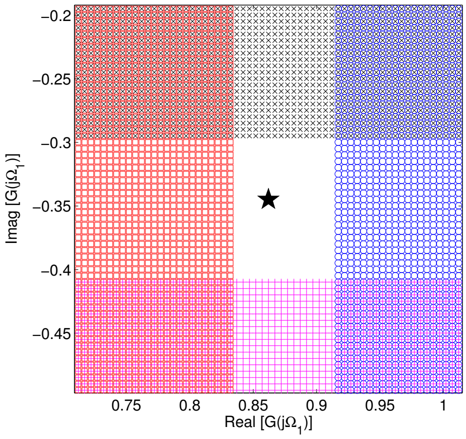

Fig. 2. Confidence region for $G^0(j\Omega_1)$

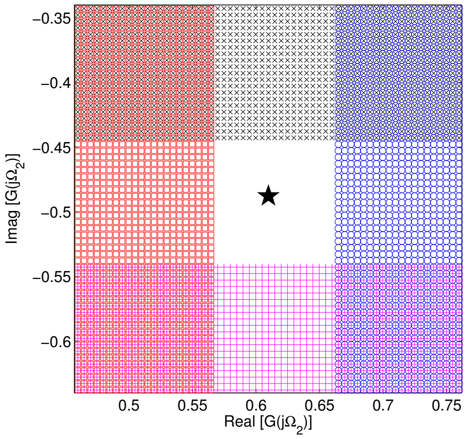

Fig. 3. Confidence region for $G^0(j\Omega_2)$

We apply the Schroeder-phased multi-sine input (Bayard 1993) with the ten frequencies. The amplitude and phases are given by

$$A_m = \sqrt{2/L}, \quad \psi_m = 2\pi \sum_{r=1}^{m} r A_r^2 / 2 \tag{26}$$

for $m = 1, \ldots, 10$. After waiting $\ell = 1000$ samples, we gather 4096 samples which corresponds to $n = 4$ periods of the baseline sinusoid, and calculate 4000 scaled empirical correlation functions after generating decoupling binary strings. Note here that the sampled noise sequence $v(kT)$ is a white noise sequence uniformly distributed on $[-0.25,\ 0.25]$ with variance of 0.0208.

Fig. 4 shows the constructed simultaneous confidence region (converted using (5)) with probability at least $1 - 2 \cdot 10 \cdot 2 \cdot 5/4000 = 0.95$ with $L = 10$, $M = 4000$, and $q = 5$.

5. CONCLUSION

In this paper, we have extended the LSCR algorithm introduced in Campi & Weyer (2005) to the problem of constructing guaranteed confidence regions of the frequency response at multiple frequencies using a finite number of input-output data points. No information about the tail of the impulse response can be obtained from a finite number of data points, and hence *a priori* information must be used in order to bound the effects of the tail. In order to reduce the amount of computations required for implementing the general algorithm, a fast numerical method with decoupling binary strings was developed, and the efficiency of the developed algorithm was demonstrated in two simulation examples with multi-sine inputs.

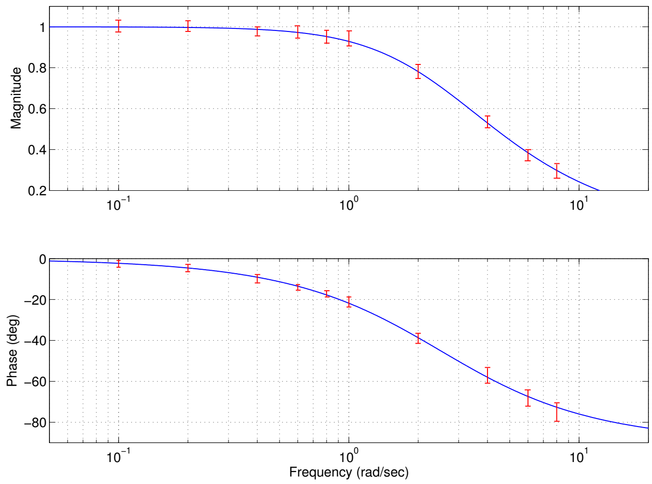

Fig. 4. True frequency response (blue line) and simultaneous 95% confidence region (red vertical lines)

REFERENCES

Bayard, D. S. (1993). Statistical plant set estimation using schroeder-phased multisinusoidal input design, *Applied Mathematics and Computation* **58**: 169–198.

Campi, M. C. & Weyer, E. (2005). Guaranteed non-asymptotic confidence regions in system identification, *Automatica* **41**(10): 1751–1764.

Campi, M. C. & Weyer, E. (2006). Identification with finitely many data points: the LSCR approach, *Proc. 14th IFAC Symposium on System Identification*, Newcastle, Australia, pp. 46–64. Available at http://bsing.ing.unibs.it/ campi/LSCRwebsite.

de Vries, D. K. & Van den Hof, P. M. J. (1995). Quantification of uncertainty in transfer function estimation: a mixed probabilistic-worst-case approach, *Automatica* **31**(4): 543–557.

Garatti, S., Campi, M. C. & Bittanti, S. (2004). Assessing the quality of identified models through the asymptotic theory - When is the result reliable?, *Automatica* **40**(8): 1319–1332.

Goodwin, G. C., Gevers, M. & Ninnes, B. (1992). Quantifying the error in estimated transfer functions with application to model order selection, *IEEE Transactions on Automatic Control* **37**(7): 913–928.

Ko, S., Weyer, E. & Campi, M. C. (2007). Non-asymptotic uncertainty assessment of frequency responses using the LSCR approach, *Proc. International Conference on Control, Automation and Systems 2007*, Seoul, Korea.

Ko, S., Weyer, E. & Campi, M. C. (2008). Non-asymptotic model quality assessment of transfer functions at multiple frequency points, *Technical report*, Department of Electrical and Electronic Engineering, The University of Melbourne.

Ljung, L. (1999). *System Identification: Theory for the User*, Prentice Hall, Upper Saddle River, NJ.