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Disturbance Decoupling with Preview for Two-Dimensional Systems ⋆

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Abstract: In this paper a solution is given to the exact disturbance decoupling problem (DDP) for two-dimensional (2-D) systems, whereby the control action consists of a static local state feedback and a preview function of the signal to be rejected. Importantly, stability of the closed loop is taken into account.

1. INTRODUCTION

The notion of controlled invariance for 1-D systems introduced in (Basile and Marro, 1969) is the cornerstone of the so-called geometric approach to control theory for LTI systems. The most celebrated control application of this concept is the disturbance decoupling problem (DDP), solved for the first time in (Basile and Marro, 1969). Disturbance decoupling with the extra requirement of closed-loop stability was addressed for the first time in (Wonham and Morse, 1970). Many important extensions of the classic DDP were proposed in the literature in the last thirty years. The most relevant for this paper is the so-called DDP with PID control law, (Willems, 1982, Bonilla Estrada and Malabre, 1999, Barbagli *et al.*, 2001). In the discrete-time case, this problem is also referred to as DDP with preview, since the control law is allowed to include – in addition to the standard proportional state feedback component – feedforward terms depending on 'future' values of the disturbance up to the present.

In the last two decades, many valuable results have been achieved in the attempt to develop a geometric theory for 2-D systems, (Conte and Perdon, 1988, Karamancioğlu and Lewis, 1992, Ntogramatzidis *et al.*, 2007). In particular, a geometric approach for 2-D systems was introduced in (Conte and Perdon, 1988) to treat 2-D decoupling problems of nonmeasurable and measurable disturbances, but without a guarantee of stability. In (Ntogramatzidis *et al.*, 2007), new geometric techniques for internal and external stabilisation of controlled invariant subspaces were developed. This led to a new solution for the two aforementioned decoupling problems, while achieving asymptotic stability of the closed-loop.

In this paper the DDP with preview is extended for the first time to 2-D causal systems. Its solution is carried out by recasting this problem into a full information problem. This contrivance enables the structural solvability condition to be easily stated in terms of the matrices of a suitably defined extended system. However, the stability condition must be addressed independently, and here it is captured in terms of the stability property of an output-nulling subspace of the original system.

Notation. The symbol $\mathbf{0}_n$ stands for the origin of the vector space $\mathbb{R}^n$. The $n \times m$ zero matrix is denoted by $0_{n\times m}$. Given the subspace $\mathcal{S}$, the symbol $\mathcal{S}^2$ stands for the Cartesian product $\mathcal{S} \times \mathcal{S}$.

2. PROBLEM STATEMENT

Consider a Fornasini-Marchesini (FM) model

$$\begin{aligned} x_{i+1,j+1} &= A_1\, x_{i+1,j} + A_2\, x_{i,j+1} + B_1\, u_{i+1,j} + B_2\, u_{i,j+1} \\ &\quad + H_1\, w_{i+1,j} + H_2\, w_{i,j+1}, \\ y_{i,j} &= C\, x_{i,j} + D\, u_{i,j} + G\, w_{i,j}, \end{aligned} \tag{1}$$

where for all $i,j \in \mathbb{Z}$, $x_{i,j} \in \mathbb{R}^n$ is the local state, $u_{i,j} \in \mathbb{R}^m$ is the control input, $w_{i,j} \in \mathbb{R}^d$ is a disturbance to be decoupled from the output $y_{i,j} \in \mathbb{R}^p$. The matrices appearing in (1) have sizes compatible with these signals. We identify the system $(A_1, A_2, [\,B_1\; H_1\,], [\,B_2\; H_2\,], C, [\,D\; G\,])$ with the symbol Σ. For $k \in \mathbb{Z}$, we define the separation sets $\mathbb{S}_k \triangleq \{(i,j) \in \mathbb{Z} \times \mathbb{Z} \;|\; i+j = k\}$, along with the so-called *global state* on $\mathbb{S}_k$ as $\mathcal{X}_k \triangleq \{x_{i,j} \,|\, (i,j) \in \mathbb{S}_k\}$, see (Fornasini and Marchesini, 1978). Similarly, we can define the global control $\mathcal{U}_k \triangleq \{u_{i,j} \,|\, (i,j) \in \mathbb{S}_k\}$, the global disturbance $\mathcal{W}_k \triangleq \{w_{i,j} \,|\, (i,j) \in \mathbb{S}_k\}$ and the global output $\mathcal{Y}_k \triangleq \{y_{i,j} \,|\, (i,j) \in \mathbb{S}_k\}$ on the separation sets. The boundary conditions usually associated with (1) take the form $x_{i,j} = b_{i,j}$ for $(i,j) \in \mathbb{S}_0$ for some constants $b_{i,j} \in \mathbb{R}^n$ for $(i,j) \in \mathbb{S}_0$. This uniquely defines $\mathcal{X}_k$ for all $k > 0$ given $\mathcal{U}_h$ and $\mathcal{W}_h$ for all $0 \le h < k$.

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Given a subspace $\mathcal{S}$, by a $\mathcal{S}$-valued boundary condition we intend $x_{i,j} \in \mathcal{S}$ for all $(i,j) \in \mathbb{S}_0$. By defining $||\mathcal{X}_r|| \triangleq \sup_{n\in\mathbb{Z}} ||x_{r-n,n}||$, we recall that system (1) – and therefore, with a slight abuse of nomenclature, the pair (A_1, A_2) – is asymptotically stable if, for finite $||\mathcal{X}_0||$ and with both inputs set to zero, the sequence $\{\,||\mathcal{X}_i||\,\}_{i=0}^{\infty}$ converges to zero. A simple sufficient condition that can be used to check asymptotic stability of the pair (A_1, A_2) is the one proposed in (Kar and Sigh, 2003): The pair (A_1, A_2) is asymptotically stable if two symmetric positive definite matrices P_1 and P_2 exist such that:

$$\begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} - \begin{bmatrix} A_1^\top \\ A_2^\top \end{bmatrix} (P_1 + P_2)\,[\,A_1 \;\; A_2\,] > 0. \tag{2}$$

Problem 2.1. (Disturbance decoupling with preview) Given $N, M \in \mathbb{N}$, find matrices $F \in \mathbb{R}^{m\times n}$ and $S_{k,l} \in \mathbb{R}^{m\times d}$, for $(k,l) \in [0,N]\times[0,M]$, so that the system

$$\begin{aligned} x_{i+1,j+1} &= (A_1 + B_1 F)\, x_{i+1,j} + (A_2 + B_2 F)\, x_{i,j+1} \\ &\quad + B_1\, \varphi_{i+1,j} + B_2\, \varphi_{i,j+1} + H_1\, w_{i+1,j} + H_2\, w_{i,j+1}, \\ y_{i,j} &= (C + D\,F)\, x_{i,j} + D\, \varphi_{i,j} + G\, w_{i,j}, \end{aligned} \tag{3}$$

obtained by imposing the control action

$$u_{i,j} = F\, x_{i,j} + \varphi_{i,j}, \tag{4}$$

with $\varphi_{i,j} \triangleq \sum_{k=0}^{N}\sum_{l=0}^{M} S_{k,l}\, w_{i+k,j+l}$, on the system dynamics (1), yields a global output sequence $\{\mathcal{Y}_i\}_{i=0}^{\infty}$ with elements that converge to zero for any global-state boundary condition $\mathcal{X}_0$ and any global disturbance $\{\mathcal{W}_i\}_{i=0}^{\infty}$.

By linearity, Problem 2.1 is equivalent to requiring that

- with the boundary conditions set to zero, the output generated by (3) satisfies $y_{i,j} = 0$ for all $i + j \geq 0$ and for any global disturbance $\{\mathcal{W}_i\}_{i=0}^{\infty}$;
- the pair $(A_1 + B_1 F, A_2 + B_2 F)$ be asymptotically stable, to ensure dissipation of the effect of non-zero boundary conditions on the output.

3. GEOMETRIC BACKGROUND FOR 2-D SYSTEMS

We now introduce some preliminaries for 2-D systems, which are taken from (Ntogramatzidis *et al.*, 2007). We begin by considering the autonomous FM system

$$x_{i+1,j+1} = A_1\, x_{i+1,j} + A_2\, x_{i,j+1}. \tag{5}$$

The subspace $\mathcal{J}$ of $\mathbb{R}^n$ is (A_1, A_2)-invariant if $A_1\,\mathcal{J} \subseteq \mathcal{J}$ and $A_2\,\mathcal{J} \subseteq \mathcal{J}$. If $\mathcal{J}$ is (A_1, A_2)-invariant, by choosing a nonsingular matrix $T = [\,T_1 \; T_2\,] \in \mathbb{R}^{n\times n}$ where the columns of T_1 span $\mathcal{J}$, we find that (5) can be written in the new coordinates described as $\begin{bmatrix} x'_{i,j} \\ x''_{i,j} \end{bmatrix} = T^{-1}\, x_{i,j}$:

$$\begin{aligned} x'_{i+1,j+1} = A_1^{(1,1)}\, x'_{i+1,j} + A_1^{(1,2)}\, x''_{i+1,j} \\ + A_2^{(1,1)}\, x'_{i,j+1} + A_2^{(1,2)}\, x''_{i,j+1}, \end{aligned} \tag{6}$$

$$x''_{i+1,j+1} = A_1^{(2,2)}\, x''_{i+1,j} + A_2^{(2,2)}\, x''_{i,j+1}. \tag{7}$$

Given an (A_1, A_2)-invariant subspace $\mathcal{J}$ for (5), any $\mathcal{J}$-valued boundary condition gives rise to a local state trajectory such that $x_{i,j} \in \mathcal{J}$ for all $i + j \geq 0$. Asymptotic stability of (5) can be "split" into two parts with respect to the invariant subspace $\mathcal{J}$. The (A_1, A_2)-invariant subspace $\mathcal{J}$ is said to be *inner stable* if $(A_1^{(1,1)}, A_2^{(1,1)})$ is asymptotically stable and *outer stable* if $(A_1^{(2,2)}, A_2^{(2,2)})$ is asymptotically stable.

Now, consider the nonautonomous FM system

$$x_{i+1,j+1} = A_1\, x_{i+1,j} + A_2\, x_{i,j+1} + B_1\, u_{i+1,j} + B_2\, u_{i,j+1} \tag{8}$$

$$y_{i,j} = C\, x_{i,j} + D\, u_{i,j}. \tag{9}$$

The boundary conditions associated with (8-9) can still be assigned by specifying the global state over $\mathbb{S}_0$. The subspace $\mathcal{V} \subseteq \mathbb{R}^n$ is output-nulling for (8-9) if $\begin{bmatrix} A_1 \\ A_2 \\ C \end{bmatrix} \mathcal{V} \subseteq (\mathcal{V}^2 \times \mathbf{0}_p) + \mathrm{im} \begin{bmatrix} B_1 \\ B_2 \\ D \end{bmatrix}$, (Conte and Perdon, 1988). Let $\mathcal{V}$ be a subspace of $\mathbb{R}^n$ and let V be a basis matrix of $\mathcal{V}$. The following are equivalent, (Ntogramatzidis *et al.*, 2007):

- The subspace $\mathcal{V}$ is output-nulling for (8-9);
- There exist X and Ω such that
$$\begin{bmatrix} A_1 \\ A_2 \\ C \end{bmatrix} V = \begin{bmatrix} V & 0 \\ 0 & V \\ 0 & 0 \end{bmatrix} X + \begin{bmatrix} B_1 \\ B_2 \\ D \end{bmatrix} \Omega. \tag{10}$$
- There exist F and X such that
$$\begin{bmatrix} A_1 + B_1 F \\ A_2 + B_2 F \\ C + D\,F \end{bmatrix} V = \begin{bmatrix} V & 0 \\ 0 & V \\ 0 & 0 \end{bmatrix} X, \tag{11}$$

Any F such that (11) holds for some X is called a *friend* of $\mathcal{V}$. Given a $\mathcal{V}$-valued boundary condition for (8-9), a control action $u_{i,j} = F\, x_{i,j}$, where F satisfies (11), is such that $x_{i,j} \in \mathcal{V}$ and $y_{i,j} = 0$ for all $i + j \geq 0$. The output-nulling subspace $\mathcal{V}$ is said to be *inner stabilisable* (resp. *outer stabilisable*) if a friend F exists such that $\mathcal{V}$ is an inner stable (resp. outer stable) $(A_1 + B_1 F, A_2 + B_2 F)$-invariant subspace. The set of friends of $\mathcal{V}$ are parameterised as the solutions of the linear equation $\Omega = -F\,V$, where Ω satisfies (10) for some matrix X. In particular, the solutions of $\Omega = -F\,V$ can be written as $F = F_\Omega + \Lambda$, with $F_\Omega \triangleq -\Omega\,(V^\top V)^{-1} V^\top$, where Ω satisfies (10) for some X and Λ is any matrix of suitable size such that $\Lambda V = 0$, see (Ntogramatzidis *et al.*, 2007). Writing the local state equation of the autonomous system obtained by applying $u_{i,j} = F\, x_{i,j}$, with $F = F_\Omega + \Lambda$, to (8) in a new basis given $T = [\,T_1 \; T_2\,]$ with $\mathrm{im}\, T_1 = \mathcal{V}$, yields

$$\begin{bmatrix} x'_{i+1,j+1} \\ x''_{i+1,j+1} \end{bmatrix} = \begin{bmatrix} M_1^{1,1} & M_1^{1,2} \\ 0 & M_1^{2,2} \end{bmatrix} \begin{bmatrix} x'_{i+1,j} \\ x''_{i+1,j} \end{bmatrix} + \begin{bmatrix} M_2^{1,1} & M_2^{1,2} \\ 0 & M_2^{2,2} \end{bmatrix} \begin{bmatrix} x'_{i,j+1} \\ x''_{i,j+1} \end{bmatrix},$$

where $M_i \triangleq A_i + B_i\, F$. In (Ntogramatzidis *et al.*, 2007) it is shown in that the pair $(M_1^{1,1}, M_2^{1,1})$ only depends on F_Ω while the pair $(M_1^{2,2}, M_2^{2,2})$ only depends on Λ. Therefore, we can independently choose F_Ω and Λ, so that – if $\mathcal{V}$ is inner stabilisable – F_Ω stabilises the pair $(M_1^{1,1}, M_2^{1,1})$ and – if $\mathcal{V}$ is outer stabilisable – Λ stabilises $(M_1^{2,2}, M_2^{2,2})$. By using the stability criterion (2) established in (Kar and Sigh, 2003), two procedures are derived in (Ntogramatzidis *et al.*, 2007) for the inner and outer stabilisation of output-nulling subspaces. In particular, it is shown that the inner stabilisation of the controlled invariant subspace $\mathcal{V}$ requires the solution of a simple LMI; the outer stabilisation requires the solution of a bilinear matrix inequality. For its solution, different techniques may be employed. For example, in (Ntogramatzidis *et al.*, 2007) the so-called *sequential linear programming matrix method* (SLPMM) developed in (Leibfritz, 2001) is exploited for this purpose.

We end this section by recalling that, as in the 1-D case, the set of output-nulling subspaces of (8-9) is closed under subspace addition, and the largest output-nulling subspace is denoted by $\mathcal{V}^\star$. This subspace can be computed in finite terms as the $(n-1)$-th term of the monotonic sequence $\mathcal{V}_0 = \mathbb{R}^n$ and $\mathcal{V}_i = \begin{bmatrix} A_1 \\ A_2 \\ C \end{bmatrix}^{-1} \left((\mathcal{V}_{i-1}^2 \times \mathbf{0}_p) + \operatorname{im} \begin{bmatrix} B_1 \\ B_2 \\ D \end{bmatrix} \right)$ for $i > 0$, see Theorem 2 in (Ntogramatzidis *et al.*, 2007).

4. SOLUTION OF PROBLEM 2.1

In the 1-D case, the solution of the decoupling problem with preview can be expressed in terms of output-nulling and input-containing subspaces of the original system, (Willems, 1982, Bonilla Estrada and Malabre, 1999, Barbagli *et al.*, 2001). In the 2-D case, this does not seem to be possible.We now analyse the possibility of solving Problem 2.1 by turning it into a decoupling problem of measurable input signals. In fact, the input $w_{i,j}$ in (1) can be thought of as being generated by a 2-D system Δ, whose input is $\hat{w}_{i,j} \triangleq w_{i+N,j+M}$ and whose output is $w_{i,j}$. See Figure 1, where the system ruled by (1) is denoted by Σ, the system Δ is simply a shift by N and M of the signal indexes i and j, respectively, and $\widehat{\Sigma}$ denotes the extended

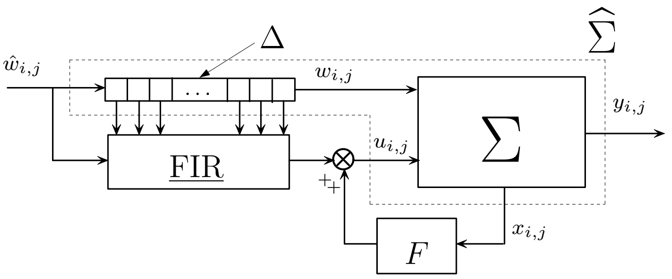

Fig. 1. Block diagram of the compensation scheme.

system obtained by the series connection of Σ and Δ. Let

$$\begin{aligned} \xi_{i+1,j+1} &= A_1^\Delta \xi_{i+1,j} + A_2^\Delta \xi_{i,j+1} + B_1^\Delta \hat{w}_{i+1,j} + B_2^\Delta \hat{w}_{i,j+1}, \\ y_{i,j}^\Delta &= C^\Delta \xi_{i,j} = \hat{w}_{i-N,j-M} = w_{i,j}, \end{aligned}$$

denote a FM realisation for the (N,M)-shift system Δ, whose local state is denoted by ξ. The DDP with preaction can be turned into a measurable signal decoupling problem, where the plant is given by the series connection $\widehat{\Sigma}$, with input $\left[\begin{smallmatrix} u \\ \hat{w} \end{smallmatrix}\right]$, output y and local state $z_{i,j} \triangleq \left[\begin{smallmatrix} x \\ \xi \end{smallmatrix}\right]$. The corresponding system matrices are $\widehat{A}_k = \left[\begin{smallmatrix} A_k & H_k\, C^\Delta \\ 0 & A_k^\Delta \end{smallmatrix}\right]$, $\widehat{B}_k = \left[\begin{smallmatrix} B_k \\ 0 \end{smallmatrix}\right]$, $\widehat{H}_k = \left[\begin{smallmatrix} 0 \\ B_k^\Delta \end{smallmatrix}\right]$ for $k = 1,2$, $\widehat{C} = [\,C \quad G\,C^\Delta\,]$, $\widehat{D} = D$, $\widehat{G} = 0$. Problem 2.1 can be recast as a decoupling problem of the measurable signal $\hat{w}_{i,j}$. In fact, suppose we are able to decouple the signal $\hat{w}_{i,j}$ from the output $y_{i,j}$ by means of the control

$$u_{i,j} = \widehat{F} \begin{bmatrix} x_{i,j} \\ \xi_{i,j} \end{bmatrix} + S\, \hat{w}_{i,j}. \tag{12}$$

By partitioning $\widehat{F} = [\, F_x \;\; F_\xi \,]$, conformably with $\left[\begin{smallmatrix} x \\ \xi \end{smallmatrix}\right]$, the feedback matrix F of the original system can be taken to be equal to F_x. To find the matrices $S_{k,l}$ in (4), we can compare the solution of the measurable signal decoupling problem (12) with the input structure (4) imposed for our problem. In other words, the matrices $S_{k,l}$ can be derived by matching (4) with (12). For this to be possible, a particular FM realisation is required for the system Δ. Another problem is how to accomodate the stability requirement. This corresponds to a requirement that F_x stabilises Σ, which is quite different to requiring that $\widehat{F} = [\, F_x \;\; F_\xi \,]$ stabilises $\widehat{\Sigma}$. For the moment, we concentrate on achieving decoupling and constructing an appropriate realisation for Δ.

The 2-D decoupling problem with full information, solved in (Conte and Perdon, 1988, Ntogramatzidis *et al.*, 2007), can be stated for system $\widehat{\Sigma}$ as follows. Find $\widehat{F}$ and S such that $u_{i,j} = \widehat{F}\, z_{i,j} + S\, \hat{w}_{i,j}$ decouples $\hat{w}$ from y. This problem is solvable if

$$\operatorname{im} \begin{bmatrix} \widehat{H}_1 \\ \widehat{H}_2 \\ \widehat{G} \end{bmatrix} \subseteq (\widehat{\mathcal{V}}^\star \times \widehat{\mathcal{V}}^\star \times \mathbf{0}_p) + \operatorname{im} \begin{bmatrix} \widehat{B}_1 \\ \widehat{B}_2 \\ \widehat{D} \end{bmatrix} \tag{13}$$

holds, where $\widehat{\mathcal{V}}^\star$ is the largest output-nulling of the system $(\widehat{A}_1, \widehat{A}_2, \widehat{B}_1, \widehat{B}_2, \widehat{C}, \widehat{D})$. This solvability condition is contructive. In fact, if (13) is satisfied, there exist matrices Φ_1, Φ_2 and Ψ such that (Ntogramatzidis *et al.*, 2007)

$$\begin{bmatrix} \widehat{H}_1 \\ \widehat{H}_2 \\ \widehat{G} \end{bmatrix} = \begin{bmatrix} \widehat{V} & 0 \\ 0 & \widehat{V} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \Phi_1 \\ \Phi_2 \end{bmatrix} + \begin{bmatrix} \widehat{B}_1 \\ \widehat{B}_2 \\ \widehat{D} \end{bmatrix} \Psi, \tag{14}$$

where $\widehat{V}$ is a basis matrix for $\widehat{\mathcal{V}}^\star$. If we take any friend $\widehat{F}$ of $\widehat{\mathcal{V}}^\star$, the input $u_{i,j} = \widehat{F}\, z_{i,j} + S\, \hat{w}_{i,j}$ achieves exact decoupling without stability. Indeed, by substituting this control input in (1) we obtain

$$\begin{aligned} z_{i+1,j+1} &= (\widehat{A}_1 + \widehat{B}_1 \widehat{F})\, z_{i+1,j} + (\widehat{A}_2 + \widehat{B}_2 \widehat{F})\, z_{i,j+1} \\ &\quad + \widehat{V}\, \Phi_1\, \hat{w}_{i+1,j} + \widehat{V}\, \Phi_2\, \hat{w}_{i,j+1}, \\ y_{i,j} &= (\widehat{C} + \widehat{D}\widehat{F})\, z_{i,j}, \end{aligned}$$

which is clearly disturbance decoupled, since, given any $\widehat{\mathcal{V}}^\star$-valued boundary condition over the separation set $\mathbb{S}_0$, we get $z_{i,j} \in \widehat{\mathcal{V}}^\star$ and $y_{i,j} = 0$ for all i,j such that $i+j \geq 0$.

In order to find the matrices $S_{k,l}$, the expressions (4) and (12) must be matched. To this end, it suffices to ensure that the local state $\xi_{i,j}$ of Δ incorporates the values of the disturbance w for indexes in the rectangle $\mathbb{B}_{i,j} \triangleq \{(k,l) \in \mathbb{Z} \times \mathbb{Z} \,|\, i \leq k \leq N, j \leq l \leq M\}$, excluding $w_{i+N,j+M}$, which can be directly used as an input of the compensator. This is achieved by finding a realisation for Δ of order $q \triangleq d[(N+1)(M+1)-1]$, where d is the dimension of the disturbance, so that its local state is given by the values of w on $\mathbb{B}_{i,j} \setminus \{(i+N, j+M)\}$. A realisation meeting this requirement is given as follows. For $P \in \mathbb{N}$, define

$$N'_P = \begin{bmatrix} 0 & 0 & \dots & 0 & 0 \\ I_d & 0 & \dots & 0 & 0 \\ 0 & I_d & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & I_d & 0 \end{bmatrix}, \; N''_P = \begin{bmatrix} I_d & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 \end{bmatrix}, \; V_P = \begin{bmatrix} I_d \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

where $N'_P, N''_P \in \mathbb{R}^{P\cdot d \times P \cdot d}$ and $V_P \in \mathbb{R}^{P\cdot d\times d}$. The matrices

$$A_1^{\Delta} = \begin{bmatrix} N'_M & 0 & 0 & \dots & 0 \\ 0 & N'_{M+1} & 0 & \dots & 0 \\ 0 & 0 & N'_{M+1} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & N'_{M+1} \end{bmatrix}, \qquad B_1^{\Delta} = \begin{bmatrix} V_M \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

$$A_2^{\Delta} = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & N''_{M+1} & 0 & \dots & 0 & 0 \\ 0 & 0 & N''_{M+1} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & N''_{M+1} & 0 \end{bmatrix}, \qquad B_2^{\Delta} = \begin{bmatrix} 0 \\ V_{M+1} \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

$$C^{\Delta} = [\, 0 \; 0 \; \dots \; 0 \; I_d \,]$$

are a realisation of Δ. If the decoupling problem of $\hat{w}_{i,j}$ is solvable for $\widehat{\Sigma}$, a control function having the structure (12) can be found to achieve perfect decoupling. By partitioning $F_\xi = \left[\, F_\xi^1 \; F_\xi^2 \; F_\xi^3 \; \dots \; F_\xi^{NM+N+M} \,\right]$ conformably with $\xi = \begin{bmatrix} \xi^1 \\ \vdots \\ \xi^{NM+N+M} \end{bmatrix}$, by comparing (4) with (12), it follows that $F = F_x$, $S_{N,M} = S$, $S_{N-1,M} = F_\xi^{M+1}$, $S_{N,M-1} = F_\xi^1$, $S_{N-1,M-1} = F_\xi^{M+2}$, $S_{N-2,M-1} = F_\xi^{2M+3}$, …, solve Problem 2.1.

Now we turn our attention to the stability requirement. Requiring that $\widehat{\mathcal{V}}^\star$ is inner and outer stabilisable, as one might expect at first sight due to the analogy with the measurable signal decoupling problem, is not correct in this case, since F_ξ cannot be used to stabilise Σ. The stability condition required for the solution of Problem 2.1 can be stated in terms of the stabilisability of the largest output-nulling subspace $\mathcal{V}^\star$ of the system $(A_1, A_2, B_1, B_2, C, D)$. We first present the following lemma, where the relation between $\mathcal{V}^\star$ and $\widehat{\mathcal{V}}^\star$ is established.

Lemma 4.1. The following identity holds:

$$\widehat{\mathcal{V}}^\star \cap \operatorname{im} \begin{bmatrix} I_n \\ 0_{q\times n} \end{bmatrix} = \mathcal{V}^\star \times \mathbf{0}_q. \tag{15}$$

Proof: First, we show that the subspace on the left-hand side of (15) contains that on the right-hand side, i.e., $\widehat{\mathcal{V}}^\star \supseteq \mathcal{V}^\star \times \mathbf{0}_q$. Consider the two sequences of subspaces $\{\mathcal{V}_i\}_{i\in\mathbb{N}}$ and $\{\widehat{\mathcal{V}}_i\}_{i\in\mathbb{N}}$ converging respectively to $\mathcal{V}^\star$ and to $\widehat{\mathcal{V}}^\star$. By induction, suppose that $\widehat{\mathcal{V}}_{i-1} \supseteq \mathcal{V}_{i-1} \times \mathbf{0}_q$. Take $\left[\begin{smallmatrix} x \\ 0 \end{smallmatrix}\right] \in \mathcal{V}_i \times \mathbf{0}_q$. Since $x \in \mathcal{V}_i$ we find that there exist $\xi_1, \xi_2 \in \mathcal{V}_{i-1}$ and $\omega \in \mathbb{R}^m$ such that $\left[\begin{smallmatrix} A_1 \\ A_2 \\ C \end{smallmatrix}\right] x = \left[\begin{smallmatrix} \xi_1 \\ \xi_2 \\ 0 \end{smallmatrix}\right] + \left[\begin{smallmatrix} B_1 \\ B_2 \\ D \end{smallmatrix}\right] \omega$. Now, we show that $\left[\begin{smallmatrix} x \\ 0 \end{smallmatrix}\right] \in \widehat{\mathcal{V}}_i$. In fact

$$\begin{bmatrix} A_1 & H_1 C^{\Delta} \\ 0 & A_1^{\Delta} \\ \hline A_2 & H_2 C^{\Delta} \\ 0 & A_2^{\Delta} \\ \hline C & G C^{\Delta} \end{bmatrix} \begin{bmatrix} x \\ 0 \end{bmatrix} = \begin{bmatrix} A_1 x \\ 0 \\ \hline A_2 x \\ 0 \\ \hline C x \end{bmatrix} = \begin{bmatrix} \xi_1 + B_1 \omega \\ 0 \\ \hline \xi_2 + B_2 \omega \\ 0 \\ \hline D u \end{bmatrix}$$

lies in $(\mathcal{V}_{i-1} \times \mathbf{0}_q)^2 \times \mathbf{0}_p + \operatorname{im} \begin{bmatrix} \widehat{B}_1 \\ \widehat{B}_2 \\ \widehat{D} \end{bmatrix}$. From the inductive assumption $\mathcal{V}_{i-1} \times \mathbf{0}_q \subseteq \widehat{\mathcal{V}}_{i-1}$ it follows that $\begin{bmatrix} \widehat{A}_1 \\ \widehat{A}_2 \\ \widehat{C} \end{bmatrix} \left[\begin{smallmatrix} x \\ 0 \end{smallmatrix}\right] \subseteq (\widehat{\mathcal{V}}_{i-1}^2 \times \mathbf{0}_p) + \operatorname{im} \begin{bmatrix} \widehat{B}_1 \\ \widehat{B}_2 \\ \widehat{D} \end{bmatrix}$, and hence $\left[\begin{smallmatrix} x \\ 0 \end{smallmatrix}\right] \in \widehat{\mathcal{V}}_i$. Now, we prove the opposite inclusion. Let

$$\widehat{\mathcal{V}}_{i-1} \cap \operatorname{im} \begin{bmatrix} I_n \\ 0_{q\times n} \end{bmatrix} \subseteq \mathcal{V}_{i-1} \times \mathbf{0}_q. \tag{16}$$

To prove that the same is true for i, let $\left[\begin{smallmatrix} x \\ 0 \end{smallmatrix}\right] \in \widehat{\mathcal{V}}_i \cap \operatorname{im} \begin{bmatrix} I_n \\ 0_{q\times n} \end{bmatrix}$, so that

$$\begin{bmatrix} A_1 & H_1 C^{\Delta} \\ 0 & A_1^{\Delta} \\ \hline A_2 & H_2 C^{\Delta} \\ 0 & A_2^{\Delta} \\ \hline C & G C^{\Delta} \end{bmatrix} \begin{bmatrix} x \\ 0 \end{bmatrix} \in (\widehat{\mathcal{V}}_{i-1}^2 \times \mathbf{0}_p) + \operatorname{im} \begin{bmatrix} B_1 \\ 0 \\ \hline B_2 \\ 0 \\ \hline D \end{bmatrix}.$$

On the other hand, by (16) we have

$$\left(\widehat{\mathcal{V}}_{i-1} \cap \operatorname{im} \begin{bmatrix} I_n \\ 0_{q\times n} \end{bmatrix} \right)^2 + \operatorname{im} \begin{bmatrix} \widehat{B}_1 \\ \widehat{B}_2 \\ \widehat{D} \end{bmatrix} \subseteq (\mathcal{V}_{i-1} \times \mathbf{0}_p)^2 + \operatorname{im} \begin{bmatrix} \widehat{B}_1 \\ \widehat{B}_2 \\ \widehat{D} \end{bmatrix},$$

which leads to

$$\begin{bmatrix} A_1 & H_1 C^{\Delta} \\ 0 & A_1^{\Delta} \\ \hline A_2 & H_2 C^{\Delta} \\ 0 & A_2^{\Delta} \\ \hline C & G C^{\Delta} \end{bmatrix} \begin{bmatrix} x \\ 0 \end{bmatrix} = \begin{bmatrix} A_1 x \\ 0 \\ \hline A_2 x \\ 0 \\ \hline C x \end{bmatrix} \in (\mathcal{V}_{i-1} \times \mathbf{0}_q)^2 \times \mathbf{0}_p + \operatorname{im} \begin{bmatrix} B_1 \\ 0 \\ \hline B_2 \\ 0 \\ \hline D \end{bmatrix}.$$

This in turn implies $\begin{bmatrix} A_1 \\ A_2 \\ C \end{bmatrix} x \in (\mathcal{V}_{i-1}^2 \times \mathbf{0}_p) + \operatorname{im} \begin{bmatrix} B_1 \\ B_2 \\ D \end{bmatrix}$ and so $x \in \mathcal{V}_i$, so that $\left[\begin{smallmatrix} x \\ 0 \end{smallmatrix}\right] \in \mathcal{V}_i \times \mathbf{0}_q$. ■

Armed with Lemma 4.1, we can now provide a complete solution to Problem 2.1.

Theorem 4.1. Problem 2.1 is solvable if

(i) $\operatorname{im} \begin{bmatrix} \widehat{H}_1 \\ \widehat{H}_2 \end{bmatrix} \subseteq (\widehat{\mathcal{V}}^\star \times \widehat{\mathcal{V}}^\star) + \begin{bmatrix} \widehat{B}_1 \\ \widehat{B}_2 \end{bmatrix} \ker D$;

(ii) $\mathcal{V}^\star$ is inner and outer stabilisable.

Proof: First, observe that the structural condition *(i)* is just a simplified way of writing (13), due to the fact that $\widehat{G}$ is zero. Now we show *(ii)*. By virtue of Lemma 4.1, it follows that $\widehat{\mathcal{V}}^\star$ can be written as

$$\widehat{\mathcal{V}}^\star = \operatorname{im} \begin{bmatrix} V & V_2 \\ 0 & V_3 \end{bmatrix}, \tag{17}$$

where V is a basis matrix of $\mathcal{V}^\star$ and V_3 is of full column-rank. Since $\widehat{\mathcal{V}}^\star$ is output-nulling for $\widehat{\Sigma}$, any friend $\widehat{F} =$

$[\,F_x \;\; F_\xi\,]$ of $\widehat{\mathcal{V}}^\star$ is such that the matrix associated with the internal dynamics on $\widehat{\mathcal{V}}^\star$ satisfies

$$\left[\begin{array}{cc} A_1+B_1\,F_x & H_1\,C^\Delta+B_1\,F_\xi \\ 0 & A_1^\Delta \\ \hline A_2+B_2\,F_x & H_2\,C^\Delta+B_2\,F_\xi \\ 0 & A_2^\Delta \\ \hline C+D\,F_x & G\,C^\Delta+D\,F_\xi \end{array}\right]\begin{bmatrix} V & V_2 \\ 0 & V_3 \end{bmatrix}$$

$$= \left[\begin{array}{cc|cc} V & V_2 & 0 & 0 \\ 0 & V_3 & 0 & 0 \\ \hline 0 & 0 & V & V_2 \\ 0 & 0 & 0 & V_3 \\ \hline 0 & 0 & 0 & 0 \end{array}\right]\left[\begin{array}{cc} X_{11} & X_{12} \\ \hline X_{21} & X_{22} \\ \hline X_{31} & X_{32} \\ \hline X_{41} & X_{42} \end{array}\right]. \tag{18}$$

From (18) we find the two identities $V_3\,X_{21}=0$ and $V_3\,X_{41}=0$, which lead to $X_{21}=0$ and to $X_{41}=0$ since V_3 is full column-rank. From the identities $A_1^\Delta\,V_3=V_3\,X_{22}$ and $A_2^\Delta\,V_3=V_3\,X_{42}$, which follow from (18), we find that $\operatorname{im} V_3$ is an (A_1^Δ,A_2^Δ)-invariant subspace. Let us now write (10) for the output-nulling $\widehat{\mathcal{V}}^\star$ in the partitioned form

$$\left[\begin{array}{cc} A_1 & H_1\,C^\Delta \\ 0 & A_1^\Delta \\ \hline A_2 & H_2\,C^\Delta \\ 0 & A_2^\Delta \\ \hline C & G\,C^\Delta \end{array}\right]\begin{bmatrix} V & V_2 \\ 0 & V_3 \end{bmatrix} =$$

$$\left[\begin{array}{cc|cc} V & V_2 & 0 & 0 \\ 0 & V_3 & 0 & 0 \\ \hline 0 & 0 & V & V_2 \\ 0 & 0 & 0 & V_3 \\ \hline 0 & 0 & 0 & 0 \end{array}\right]\left[\begin{array}{cc} X_{11} & X_{12} \\ 0 & X_{22} \\ \hline X_{31} & X_{32} \\ 0 & X_{42} \end{array}\right]+\left[\begin{array}{c} B_1 \\ 0 \\ \hline B_2 \\ 0 \\ \hline D \end{array}\right][\,\Omega_1 \;\; \Omega_2\,].$$

Since $\mathcal{V}^\star$ is inner and outer stabilisable, we can find a friend F_x of $\mathcal{V}^\star$ such that $(A_1+B_1\,F_x)\,V=V\,\bar{X}_1$, $(A_2+B_2\,F_x)\,V=V\,\bar{X}_2$ and $(C+D\,F_x)\,V=0$ for some $\bar{X}_1$ and $\bar{X}_2$, and the pair $(A_1{+}B_1\,F_x,A_2{+}B_2\,F_x)$ is asymptotically stable; i.e., F_x internally and externally stabilises $\mathcal{V}^\star$. It follows, in particular, that the pair $(\bar{X}_1,\bar{X}_2)$ is asymptotically stable. Such a matrix F_x is associated with another matrix Ω for which $\begin{bmatrix} A_1 \\ A_2 \\ C \end{bmatrix} V = \begin{bmatrix} V & 0 \\ 0 & V \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \bar{X}_1 \\ \bar{X}_2 \end{bmatrix}+\begin{bmatrix} B_1 \\ B_2 \\ D \end{bmatrix}\Omega$ with $\Omega=-F\,V$. Take $\Omega_1=\Omega$, $X_{11}=\bar{X}_1$ and $X_{31}=\bar{X}_2$ in (18), and compute X_{12}, X_{22}, X_{32}, X_{42}, Ω_2 by

$$\left[\begin{array}{c} X_{12} \\ \hline X_{22} \\ \hline X_{32} \\ \hline X_{42} \\ \hline \Omega_2 \end{array}\right] = \left[\begin{array}{cc|cc|c} V & V_2 & 0 & 0 & B_1 \\ 0 & V_3 & 0 & 0 & 0 \\ \hline 0 & 0 & V & V_2 & B_2 \\ 0 & 0 & 0 & V_3 & 0 \\ \hline 0 & 0 & 0 & 0 & D \end{array}\right]^\dagger \left[\begin{array}{c} A_1\,V_2+H_1\,C^\Delta V_3 \\ A_1^\Delta V_3 \\ \hline A_2\,V_2+H_2\,C^\Delta V_3 \\ A_2^\Delta\,V_3 \\ \hline C\,V_2+G\,C^\Delta\,V_3 \end{array}\right] \tag{19}$$

Now, the friend $\widehat{F}=[\,F_x \;\; F_\xi\,]$ of $\widehat{\mathcal{V}}^\star$ can be computed as a solution of the equation $[\,\Omega_1 \;\; \Omega_2\,]=-[\,F_x \;\; F_\xi\,]\begin{bmatrix} V & V_2 \\ 0 & V_3 \end{bmatrix}=-[\,F_x\,V \;\; F_x\,V_2+F_\xi\,V_3\,]$. The first component F_x of $\widehat{F}$ satisfies $\Omega_1=-F_x\,V$, so that it stabilises $\mathcal{V}^\star$ internally and externally. The second component F_ξ can be computed as

$$F_\xi=-(\Omega_2+F_x\,V_2)(V_3^\top V_3)^{-1}V_3^\top. \tag{20}$$

■

In Theorem 4.1, the structural condition is given in terms of $\widehat{\mathcal{V}}^\star$, while the stability condition is expressed in terms of the inner and outer stabilisability of $\mathcal{V}^\star$.

Example 4.1. Let (8-9) be defined over $\mathbb{N}\times\mathbb{N}$ with

$$A_1=\begin{bmatrix} -0.03 & 0 & 0.04 & 0 & 0 \\ 0 & -0.02 & 0.1 & 0 & 0 \\ 0 & 0 & -0.07 & 0 & 0 \\ 0 & 0 & -0.02 & 0.06 & 0.05 \\ -0.1 & -0.09 & 0.08 & 0.03 & 0.03 \end{bmatrix},\; B_1=\begin{bmatrix} 0 & -9 \\ 0 & 5 \\ 0 & 0 \\ 0 & 0 \\ 0 & -5 \end{bmatrix},\; H_1=\begin{bmatrix} -5 \\ 8 \\ 0 \\ -2 \\ -4 \end{bmatrix},$$

$$A_2=\begin{bmatrix} -0.12 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.04 & 0.08 & 0.1 \\ 0 & 0 & 0.04 & 0.18 & 0.08 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.06 & -0.16 \end{bmatrix},\; B_2=\begin{bmatrix} 0 & 0 \\ 0 & -9 \\ 1 & -8 \\ 0 & 0 \\ 0 & 5 \end{bmatrix},\; H_2=\begin{bmatrix} 0 \\ -9 \\ 5 \\ 0 \\ 0 \end{bmatrix},$$

$$C=\begin{bmatrix} 0 & 0 & 0 & -3 & 0 \end{bmatrix},\quad D=\begin{bmatrix} 0 & 0 \end{bmatrix},\quad G=0.$$

The conditions associated with this system are random assignments of the local state over the region $(\mathbb{N}\times\{0\})\cup(\{0\}\times\mathbb{N})$. By computing of the largest output-nulling subspace $\mathcal{V}^\star$ of the system (A_1,A_2,B_1,B_2,C,D) we get

$$\mathcal{V}^\star=\operatorname{im} V,\qquad \text{where}\qquad V=\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 5 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

A simple check shows that the structural condition $\operatorname{im}\begin{bmatrix} H_1 \\ H_2 \\ G \end{bmatrix}\subseteq(\mathcal{V}^\star\times\mathcal{V}^\star\times\mathbf{0}_p)+\operatorname{im}\begin{bmatrix} B_1 \\ B_2 \\ D \end{bmatrix}$ for the solution of the measurable signal decoupling problem is not satisfied in this case. As such, the decoupling problem cannot be solved using the techniques described herein for the control structure $u_{i,j}=F\,x_{i,j}+S\,w_{i,j}$, nor any other existing geometric technique for 2-D systems. Suppose now that the control law is allowed to be in the form (4) with $N=2$ and $M=1$, so that $u_{i,j}=F\,x_{i,j}+S_{0,0}\,w_{i,j}+S_{0,1}\,w_{i,j+1}+S_{1,0}\,w_{i+1,j}+S_{1,1}\,w_{i+1,j+1}+S_{2,0}\,w_{i+2,j}+S_{2,1}\,w_{i+2,j+1}$. First, we find a stabilising friend F of $\mathcal{V}^\star$. In this case the null-space of $W=\begin{bmatrix} V & 0 & B_1 \\ 0 & V & B_2 \\ 0 & 0 & D \end{bmatrix}$ is zero. As such, the unique solution (X_1,X_2) to $\begin{bmatrix} X_1 \\ X_2 \\ \Omega \end{bmatrix}=W^\dagger\begin{bmatrix} A_1 \\ A_2 \\ C \end{bmatrix}V$ must satisfy condition (2), and is such that the pair (X_1,X_2) is asymptotically stable, i.e., it satisfies (2), so that F stabilises $\mathcal{V}^\star$ internally. With this choice we find

$$F=-\Omega(V^\top V)^{-1}V^\top=\begin{bmatrix} -0.410 & -0.369 & 0.2241 & 0 & 0.0897 \\ -0.020 & -0.0180 & 0.0207 & 0 & 0.0083 \end{bmatrix}.$$

A direct check shows that the pair $(A_1{+}B_1\,F,A_2{+}B_2\,F)$ is asymptotically stable as it satisfies (2), so that F stabilises $\mathcal{V}^\star$ externally as well. A model for system Δ is

$$A_1^\Delta=\begin{bmatrix} 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&1&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&1&0 \end{bmatrix},\; A_2^\Delta=\begin{bmatrix} 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&1&0&0&0 \\ 0&0&0&0&0 \end{bmatrix},\; B_1^\Delta=\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},\; B_2^\Delta=\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$C^\Delta=[\,0\;\,0\;\,0\;\,0\;\,1\,].$$

A basis matrix for the subspace $\widehat{\mathcal{V}}^\star$ can be expressed as in (17), where

$$V_2 = \begin{bmatrix} 0 & 0\,0\,0\,0 \\ 0 & 0\,0\,0\,0 \\ 0 & 0\,0\,0\,0 \\ 0 & 0\,0\,0\,0 \\ 0.9997 & 0\,0\,0\,0 \end{bmatrix} \text{ and } V_3 = \begin{bmatrix} 0 & 1\;0 & 0 & 0 \\ 0 & 0\;1 & 0 & 0 \\ 0 & 0\;0 & -1 & 0 \\ 0 & 0\;0 & 0 & -1 \\ 0.0250 & 0\;0 & 0 & 0 \end{bmatrix}.$$

A direct check shows that conditions *(i-ii)* in Theorem 4.1 are satisfied, so that an input function in the form (4) with $N = 2$ and $M = 1$ exists such that the overall system is disturbance decoupled from the input $\hat{w}$ to the output y. Our aim now is to find the matrices $S_{k,l}$, $(k,l) \in [0,2] \times [0,1]$ to be employed for the synthesis of the FIR system. Let us exploit (19) for the computation of X_{12}, X_{22}, X_{32}, X_{42} and Ω_2, so that (20) can be used to compute F_ξ:

$$F_\xi = \begin{bmatrix} 0 & 0 & 0 & -164 & -39.2662 \\ 0 & 0 & 0 & -8 & -0.8910 \end{bmatrix}$$

As a result, the gain matrices of the FIR system are $S_{0,0} = \begin{bmatrix} -39.26624 \\ -0.8910 \end{bmatrix}$, $S_{0,1} = \begin{bmatrix} -164 \\ -8 \end{bmatrix}$, $S_{1,0} = S_{1,1} = S_{2,0} = 0$, while matrix $S_{2,1}$ can be computed by solving equation (14) written with respect to $\widehat{\Sigma}$ in Ψ and by taking $S_{2,1} = -\Psi$. In this case, $S_{2,1} = 0$. It follows that the input $u_{i,j} = F\,x_{i,j} + S_{0,0}\,w_{i,j} + S_{0,1}\,w_{i,j+1}$ solves the DDP. Clearly, the same result would have been found by choosing $N = 0$ and $M = 1$. This example shows that the possibility of enriching the control law (4) with the previewed terms $\varphi_{i,j}$ enlarges the possibilities of decoupling exactly the disturbance input w from the output y.

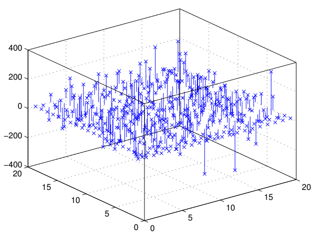

Fig. 2. Disturbance w in the bounded frame $[0,20] \times [0,20]$.

Let Σ be subject to the randomly generated input w depicted within the interval $[0,20] \times [0,20]$ in Figure 2 and with randomly generated boundary conditions for Σ. Asymptotic stability of the closed-loop guarantees that the output approaches zero as the index (i,j) evolves away from the axes, see Figure 3.

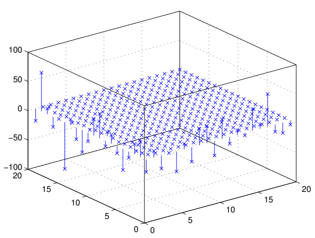

Fig. 3. Output $y_{i,j}$ in the interval $[0,20] \times [0,20]$ for nonzero boundary conditions.

In order to see that as the index (i,j) evolves away from the axis the output $y_{i,j}$ decreases in an exponential fashion, the first figure in Figure 4, shows the base 10 logarithm of $|y_{i,i}|$ for $i \in [0,20]$.

If on the other hand we assume zero boundary conditions, the disturbance signal w in Figure 2 leads to the output y depicted in the second figure in Figure 4, which shows perfect decoupling (to within numerical noise).

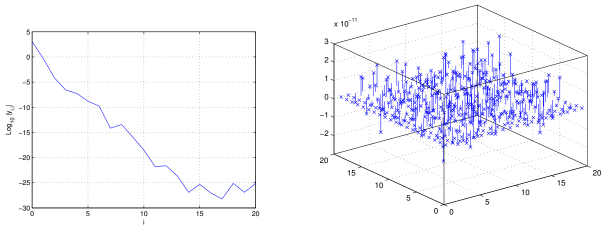

Fig. 4. Logarithm of the output $|y_{i,i}|$ for $i \in [0,20]$. Output $y_{i,j}$ obtained with boundary conditions at zero. Note that $|y_{i,j}| \sim 10^{-11}$.

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