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A Landmark Based Nonlinear Observer for Attitude and Position Estimation with Bias Compensation ⋆

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Abstract: This paper addresses the problem of estimating position and attitude of a rigid body based on landmark coordinate readings and biased velocity measurements. Using a Lyapunov function conveniently defined by the landmark measurement error, a nonlinear observer on SE(3) is derived. The resulting position, attitude, and biases estimation errors are shown to converge exponentially fast to the desired equilibrium points. The observer terms are explicit functions of the landmark measurements and velocity readings, exploiting the sensor information directly. Simulation results for trajectories described by time-varying linear and angular velocities are presented to illustrate the stability and convergence properties of the observer, supporting the application of the algorithm to autonomous air vehicles and other robotic platforms.

1. INTRODUCTION

Landmark based navigation is recognized as a promising strategy for providing aerial vehicles with accurate position and attitude information during the critical take-off, landing, and hover stages. Among a wide diversity of suitable estimation techniques, nonlinear observers stand out as an exciting approach often endowed with stability results [Crassidis et al., 2007] formulated rigorously in non-Euclidean spaces. Recent advances in this research topic motivate the growing interest in the inclusion of non-ideal sensor effects that have long been accounted for in filtering estimation techniques.

The problem of formulating a stabilizing feedback law in non-Euclidean spaces, such as SO(3) and SE(3), has been addressed in several references, namely [Malisoff et al., 2006, Chaturvedi and McClamroch, 2006, Fragopoulos and Innocenti, 2004, Bhat and Bernstein, 2000], where the analysis of the topological properties of the SE(3) manifold provides important guidelines for the design of observers. Nonlinear attitude observers motivated by aerospace applications are found in [Salcudean, 1991, Thienel and Sanner, 2003], yielding global exponential convergence of the attitude estimates in the Euler quaternion representation. A nonlinear complementary filter in SE(3) is proposed in [Baldwin et al., 2007], with almost global exponential convergence and tested in a Vario Benzin-Acrobatic model scale helicopter.

The observers proposed in [Baldwin et al., 2007, Thienel and Sanner, 2003, Salcudean, 1991] assume that an explicit quaternion/rotation matrix attitude measurement is available, obtained by batch processing sensor measurements such as landmarks, image based features, and vector readings. In alternative, if the landmark measurements are exploited directly in the observer, it is possible to analyze the influence of the sensor characteristics and landmark configuration in estimation results.

In this work, the position and attitude of a rigid body are estimated by exploiting landmark readings and biased velocity measurements directly. The proposed observer yields exponential stability of the position and attitude estimation errors, in the presence of biased angular and linear velocity readings. As discussed in [Vasconcelos et al., 2007], the attitude and position estimation problems can be decoupled assuming perfect angular velocity measurements. However, coupling occurs in the presence of angular velocity bias, which influences the position feedback law, and the attitude and position estimation problems must be addressed together. The stability of the origin is derived, based on recent results for parameterized linear time-varying systems presented in [Loría and Panteley, 2002], that also bring about exponential convergence rate bounds. To the best of the authors' knowledge, no prior work on nonlinear observers has addressed the problem of position estimation in the presence of biased angular velocity readings.

The paper is organized as follows. The position and attitude estimation problem is presented in Section 2. Section 3 introduces the analytical tools adopted in the derivation of the observer. A convenient landmark-based coordinate transformation and Lyapunov function are defined, and the necessary and sufficient landmark configuration for attitude determination is discussed. The main contribution of this work is detailed in Section 4, where the exponential convergence of the estimation errors to the desired equilibrium points is presented. It is also shown that the feedback law is an explicit function of the sensor readings and observer estimates. Section 5 illustrates the properties of the observer in simulation. Conclusions and directions of future developments are presented in Section 6.

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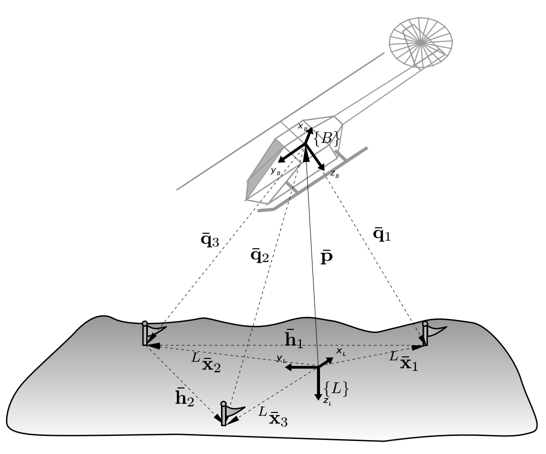

Fig. 1. Airborne Landmark Based Navigation

NOMENCLATURE

The notation adopted is fairly standard. The set of $n \times m$ matrices with real entries is denoted by $\mathrm{M}(n, m)$ and $\mathrm{M}(n) := \mathrm{M}(n, n)$. The sets of skew-symmetric, orthogonal, and special orthogonal matrices are respectively denoted by $\mathrm{K}(n) := \{\mathbf{K} \in \mathrm{M}(n) : \mathbf{K} = -\mathbf{K}'\}$, $\mathrm{O}(n) := \{\mathbf{U} \in \mathrm{M}(n) : \mathbf{U}'\mathbf{U} = \mathbf{I}\}$, and $\mathrm{SO}(n) := \{\mathbf{R} \in \mathrm{O}(n) : \det(\mathbf{R}) = 1\}$. The special Euclidean group is given by the product space of $\mathrm{SO}(n)$ with $\mathbb{R}^n$, $\mathrm{SE}(n) := \mathrm{SO}(n) \times \mathbb{R}^n$ [Murray et al., 1994]. The n-dimensional sphere and unitary ball are described by $\mathrm{S}(n) := \{\mathbf{x} \in \mathbb{R}^{n+1} : \mathbf{x}'\mathbf{x} = 1\}$ and $\mathrm{B}(n) := \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x}'\mathbf{x} \leq 1\}$, respectively.

2. PROBLEM FORMULATION

Landmark based navigation, illustrated in Fig. 1, can be summarized as the process of determining attitude and position of a rigid body using landmark observations and velocity measurements, given by sensors installed onboard the autonomous platform. The rigid body kinematics are described by

$$\dot{\bar{\mathcal{R}}} = \bar{\mathcal{R}} \left[\bar{\boldsymbol{\omega}}\times\right], \qquad \dot{\bar{\mathbf{p}}} = \bar{\mathbf{v}} - \left[\bar{\boldsymbol{\omega}}\times\right] \bar{\mathbf{p}},$$

where $\bar{\mathcal{R}}$ is the shorthand notation for the rotation matrix ${}^L_B\mathbf{R}$ from body frame $\{B\}$ to local frame $\{L\}$ coordinates, $\bar{\boldsymbol{\omega}}$ and $\bar{\mathbf{v}}$ are the body angular and linear velocities, respectively, expressed in $\{B\}$, $\bar{\mathbf{p}}$ is the position of the rigid body with respect to $\{L\}$ expressed in $\{B\}$, and $[\mathbf{a}\times]$ is the skew symmetric matrix defined by the vector $\mathbf{a} \in \mathbb{R}^3$ such that $[\mathbf{a}\times]\,\mathbf{b} = \mathbf{a} \times \mathbf{b}$, $\mathbf{b} \in \mathbb{R}^3$. Without loss of generality, the local frame is defined by translating the Earth frame $\{E\}$ to the landmarks' centroid, as depicted in Fig. 1.

The body angular and linear velocities are measured by a rate gyro sensor triad and a Doppler sensor, respectively

$$\boldsymbol{\omega}_r = \bar{\boldsymbol{\omega}}, \quad \mathbf{v}_r = \bar{\mathbf{v}}. \tag{1}$$

The landmark measurements $\mathbf{q}_{r\,i}$, illustrated in Fig. 1, are obtained by on-board sensors that are able to track terrain characteristics, such as CCD cameras or ladars, so that

$$\mathbf{q}_{r\,i} = \bar{\mathbf{q}}_i = \bar{\mathcal{R}}'\,{}^L\bar{\mathbf{x}}_i - \bar{\mathbf{p}}, \tag{2}$$

where ${}^L\bar{\mathbf{x}}_i$ represents the coordinates of landmark i in the local frame $\{L\}$. The concatenation of (2) is expressed in matrix form as $\bar{\mathbf{Q}} = \bar{\mathcal{R}}'\mathbf{X} - \bar{\mathbf{p}}\mathbf{1}_n'$ where $\bar{\mathbf{Q}} := [\bar{\mathbf{q}}_1 \ \ldots \ \bar{\mathbf{q}}_n]$, $\mathbf{X} := \left[{}^L\bar{\mathbf{x}}_1 \ \ldots \ {}^L\bar{\mathbf{x}}_n\right]$, $\bar{\mathbf{Q}}, \mathbf{X} \in \mathrm{M}(3, n)$ and $\mathbf{1}_n := [1 \ \ldots \ 1]'$, $\mathbf{1}_n \in \mathbb{R}^n$. By the definition of frame $\{L\}$, the landmarks' centroid is at the origin of the coordinate frame

$$\sum_{i=1}^{n} {}^L\bar{\mathbf{x}}_i = \mathbf{X}\mathbf{1}_n = 0. \tag{3}$$

The proposed observer which estimates the position and attitude of the rigid body takes the form

$$\dot{\hat{\mathcal{R}}} = \hat{\mathcal{R}} \left[\hat{\boldsymbol{\omega}}\times\right], \quad \dot{\hat{\mathbf{p}}} = \hat{\mathbf{v}} - \left[\hat{\boldsymbol{\omega}}\times\right] \hat{\mathbf{p}},$$

where $\hat{\boldsymbol{\omega}}$ and $\hat{\mathbf{v}}$ are feedback terms that, by construction, compensate for the estimation errors.

The position and attitude errors are defined as $\tilde{\mathbf{p}} := \hat{\mathbf{p}} - \bar{\mathbf{p}}$ and $\tilde{\mathcal{R}} := \hat{\mathcal{R}}\bar{\mathcal{R}}'$, respectively. The Euler angle-axis parameterization of the rotation error matrix $\tilde{\mathcal{R}}$ is described by the rotation vector $\boldsymbol{\lambda} \in \mathrm{S}(2)$ and by the rotation angle $\theta \in [0 \ \pi]$, yielding the formulation [Murray et al., 1994] $\tilde{\mathcal{R}} = \mathrm{rot}(\theta, \boldsymbol{\lambda}) := \cos(\theta)\mathbf{I} + \sin(\theta)\,[\boldsymbol{\lambda}\times] + (1 - \cos(\theta))\boldsymbol{\lambda}\boldsymbol{\lambda}'$. While the observer results are formulated directly in the SE(3) manifold, the rotation angle θ is adopted to characterize some of the convergence properties of the observer.

The attitude and position error kinematics are given by

$$\dot{\tilde{\mathcal{R}}} = \tilde{\mathcal{R}} \left[\bar{\mathcal{R}}(\hat{\boldsymbol{\omega}} - \bar{\boldsymbol{\omega}})\times\right], \tag{4a}$$

$$\dot{\tilde{\mathbf{p}}} = (\hat{\mathbf{v}} - \bar{\mathbf{v}}) - \left[\bar{\boldsymbol{\omega}}\times\right] \tilde{\mathbf{p}} + \left[\hat{\mathbf{p}}\times\right] (\hat{\boldsymbol{\omega}} - \bar{\boldsymbol{\omega}}), \tag{4b}$$

respectively. The attitude and position observers are obtained by defining $\hat{\boldsymbol{\omega}}$ and $\hat{\mathbf{v}}$ as functions of the velocity readings (1) and landmark observations (2), so that the closed loop estimation errors converge asymptotically to the origin, i.e. $\tilde{\mathcal{R}} \to \mathbf{I}, \tilde{\mathbf{p}} \to 0$ as $t \to \infty$.

3. OBSERVER CONFIGURATION

The attitude feedback law and the analysis of the resulting observer are derived resorting to Lyapunov's stability theory. Based on a judiciously chosen landmark transformation, which will be presented shortly, a candidate Lyapunov function is defined to synthesize the attitude and position observer.

To exploit the landmark readings information, vector and position measurements are constructed from a linear combination of (2), producing respectively

$${}^B\bar{\mathbf{u}}_j := \sum_{i=1}^{n-1} a_{ij}(\bar{\mathbf{q}}_{i+1} - \bar{\mathbf{q}}_i), \quad {}^B\bar{\mathbf{u}}_n := -\frac{1}{n}\sum_{i=1}^{n} \bar{\mathbf{q}}_i, \tag{5}$$

where $j = 1, \ldots, n-1$. To express these transformations in matrix form, define $\mathbf{D}_X := \begin{bmatrix} \mathbf{0}_{1\times n-1} \\ \mathbf{I}_{n-1} \end{bmatrix} - \begin{bmatrix} \mathbf{I}_{n-1} \\ \mathbf{0}_{1\times n-1} \end{bmatrix}$, $\mathbf{d}_p := -\frac{1}{n}\mathbf{1}_n$, and $\mathbf{A}_X := [a_{ij}] \in \mathrm{M}(n-1)$, which is assumed to be invertible. Using $\mathbf{1}_n'\mathbf{D}_X = 0$, the transformation (5) can be written as

$${}^B\bar{\mathbf{U}}_X = \bar{\mathbf{Q}}\mathbf{D}_X\mathbf{A}_X = \bar{\mathcal{R}}'\mathbf{U}_X, \quad {}^B\bar{\mathbf{u}}_n = \bar{\mathbf{Q}}\mathbf{d}_p = \bar{\mathbf{p}} \tag{6}$$

where ${}^B\bar{\mathbf{U}}_X := \left[{}^B\bar{\mathbf{u}}_1 \ \ldots \ {}^B\bar{\mathbf{u}}_{n-1}\right] \in \mathrm{M}(3, n-1)$, and $\mathbf{U}_X := \mathbf{X}\mathbf{D}_X\mathbf{A}_X \in \mathrm{M}(3, n-1)$ is known. The estimates of the transformed landmarks (6) are described by

$${}^B\hat{\mathbf{U}}_X = \hat{\mathcal{R}}'\mathbf{U}_X, \quad {}^B\hat{\mathbf{u}}_n = \hat{\mathbf{p}}. \tag{7}$$

where the columns of ${}^B\hat{\mathbf{U}}_X$ and $\mathbf{U}_X$ are denoted as ${}^B\hat{\mathbf{u}}_i$ and ${}^L\bar{\mathbf{u}}_i$, respectively.

The candidate Lyapunov function is defined by the estimation error of the transformed vectors

$$V = \tfrac{1}{2}\sum_{i=1}^{n}\|{}^B\hat{\mathbf{u}}_i - {}^B\bar{\mathbf{u}}_i\|^2. \tag{8}$$

It is straightforward to rewrite V as a linear combination of distinct attitude and position components $V = V_{\mathcal{R}} + V_p$. The attitude component is given by

$$\begin{aligned} V_{\mathcal{R}} &= \tfrac{1}{2}\sum_{i=1}^{n-1}\|{}^B\hat{\mathbf{u}}_i - {}^B\bar{\mathbf{u}}_i\|^2 = \operatorname{tr}\left[(\mathbf{I}-\tilde{\mathcal{R}})\mathbf{U}_X\mathbf{U}_X'\right] \\ &= \frac{1}{4}\|\mathbf{I}-\tilde{\mathcal{R}}\|^2\boldsymbol{\lambda}'\mathbf{P}\boldsymbol{\lambda} = (1-\cos(\theta))\,\boldsymbol{\lambda}'\mathbf{P}\boldsymbol{\lambda}, \end{aligned} \tag{9}$$

where $\mathbf{P} := \operatorname{tr}(\mathbf{U}_X\mathbf{U}_X')\mathbf{I} - \mathbf{U}_X\mathbf{U}_X' \in \mathrm{M}(3)$, and the position component described by $V_p = \frac{1}{2}\|{}^B\hat{\mathbf{u}}_n - {}^B\bar{\mathbf{u}}_n\|^2 = \frac{1}{2}\tilde{\mathbf{p}}'\tilde{\mathbf{p}}$. Taking the time derivatives produces

$$\dot{V}_{\mathcal{R}} = \left[\mathbf{U}_X\mathbf{U}_X'\tilde{\mathcal{R}} - \tilde{\mathcal{R}}'\mathbf{U}_X\mathbf{U}_X'\otimes\right]'\bar{\mathcal{R}}(\hat{\boldsymbol{\omega}}-\bar{\boldsymbol{\omega}}), \tag{10a}$$

$$\dot{V}_p = \tilde{\mathbf{p}}'([\hat{\mathbf{p}}\times](\hat{\boldsymbol{\omega}}-\bar{\boldsymbol{\omega}}) + (\hat{\mathbf{v}}-\bar{\mathbf{v}})), \tag{10b}$$

where $\otimes$ is the unskew operator such that $[[\mathbf{a}\times]\otimes] = \mathbf{a}, \mathbf{a} \in \mathbb{R}^3$.

3.1 Vector Measurement Configuration

The Lyapunov function $V_{\mathcal{R}}$ measures the error of the transformed landmarks, given by ${}^B\hat{\mathbf{u}}_i - {}^B\bar{\mathbf{u}}_i$, $i = 1,\ldots,n-1$. The landmark configuration under which zero observation error $V_{\mathcal{R}} = 0$ is equivalent to correct attitude estimation $\tilde{\mathcal{R}} = \mathbf{I}$ verifies the following assumption:

Assumption 1. (Landmark Configuration). The landmarks are not all collinear, that is, $\operatorname{rank}(\mathbf{X}) \geq 2$.

Lemma 2. The Lyapunov function $V_{\mathcal{R}}$ has a unique global minimum (at $\tilde{\mathcal{R}} = \mathbf{I}$) if and only if Assumption 1 is verified.

Proof. By [Vasconcelos et al., 2007, Lemma 1], $V_{\mathcal{R}} > 0$ if and only if $\operatorname{rank}(\mathbf{U}_X) \geq 2$. The equality $\operatorname{rank}(\mathbf{U}_X) = \operatorname{rank}([\mathbf{U}_X\ 0]) = \operatorname{rank}(\mathbf{X}[\mathbf{D}_X\ \mathbf{1}_n]\left[\begin{smallmatrix}\mathbf{A}_X & 0\\ 0 & 1\end{smallmatrix}\right]) = \operatorname{rank}(\mathbf{X})$ given that $\mathbf{A}_X$ and $[\mathbf{D}_X\ \mathbf{1}_n]$ are nonsingular, completes the proof. □

The necessity of Assumption 1 is illustrated by considering all ${}^L\bar{\mathbf{x}}_i$ are collinear, which is equivalent to all ${}^L\bar{\mathbf{u}}_i$ collinear, and hence any $\tilde{\mathcal{R}} = \operatorname{rot}(\theta, {}^L\bar{\mathbf{u}}_i/\|{}^L\bar{\mathbf{u}}_i\|)$ satisfies $V_{\mathcal{R}} = 0$.

3.2 Vector Measurement Directionality

The derivation of a stabilizing feedback law for attitude estimation relies on analyzing the level sets wherein $V_{\mathcal{R}} \leq c$ holds. For c large enough, the corresponding level sets contain multiple critical points due to the nonuniform directionality of $\mathbf{P}$, see [Vasconcelos et al., 2007, Lemma 2] for a motivation. Interestingly enough, the directionality of matrix $\mathbf{P}$ can be made uniform by construction, using the transformation $\mathbf{A}_X$.

Proposition 3. Let $\mathbf{H} := \mathbf{X}\mathbf{D}_X$ be full rank, there is a nonsingular $\mathbf{A}_X \in \mathrm{M}(n)$ such that $\mathbf{U}_X\mathbf{U}_X' = \mathbf{I}$.

Proof. Take the SVD decomposition of $\mathbf{H} = \mathbf{U}\mathbf{S}\mathbf{V}'$ where $\mathbf{U} \in \mathrm{O}(3)$, $\mathbf{V} \in \mathrm{O}(n)$, $\mathbf{S} = \left[\operatorname{diag}(s_1,s_2,s_3)\ \mathbf{0}_{3\times(n-3)}\right] \in \mathrm{M}(3,n)$, and $s_1 > s_2 > s_3 > 0$ are the singular values of $\mathbf{H}$. Any $\mathbf{A}_X$ given by $\mathbf{A}_X = \mathbf{V}\left[\begin{smallmatrix}\operatorname{diag}(s_1^{-1},s_2^{-1},s_3^{-1}) & \mathbf{0}_{3\times(n-3)}\\ \mathbf{0}_{(n-3)\times 3} & \mathbf{B}\end{smallmatrix}\right]\mathbf{V}_A'$, where $\mathbf{B} \in \mathrm{M}(n-3)$ is nonsingular and $\mathbf{V}_A \in \mathrm{O}(n)$, produces $\mathbf{U}_X\mathbf{U}_X' = \mathbf{H}\mathbf{A}_X\mathbf{A}_X'\mathbf{H} = \mathbf{U}\mathbf{V}_A'\mathbf{V}_A\mathbf{U}' = \mathbf{I}$. □

Using the transformation $\mathbf{A}_X$ defined in Proposition 3, the Lyapunov function (9) is expressed by

$$V_{\mathcal{R}} = \tfrac{1}{2}\|\mathbf{I}-\tilde{\mathcal{R}}\|^2 = 2(1-\cos(\theta)),$$

$$\dot{V}_{\mathcal{R}} = -\left[\tilde{\mathcal{R}}-\tilde{\mathcal{R}}'\otimes\right]'\bar{\mathcal{R}}'(\hat{\boldsymbol{\omega}}-\bar{\boldsymbol{\omega}}) = -2\sin(\theta)\boldsymbol{\lambda}'\bar{\mathcal{R}}'(\hat{\boldsymbol{\omega}}-\bar{\boldsymbol{\omega}}).$$

Apparently, the case $\operatorname{rank}(\mathbf{X}) = 2$ does not satisfy the conditions of Proposition 3, given that $\operatorname{rank}(\mathbf{H}) = \operatorname{rank}(\mathbf{X})$. However, by taking two linearly independent columns of $\mathbf{H}$, ${}^L\mathbf{h}_i$ and ${}^L\mathbf{h}_j$, a full rank matrix is defined as $\mathbf{H}_a = \left[\mathbf{H}\ {}^L\mathbf{h}_i \times {}^L\mathbf{h}_j\right]$. The cross product is commutable with coordinate transformations, $(\mathcal{R}'{}^L\mathbf{h}_i)\times(\mathcal{R}'{}^L\mathbf{h}_j) = \mathcal{R}'({}^L\mathbf{h}_i \times {}^L\mathbf{h}_j)$, hence a modified observer can be rederived, without loss of generality, by replacing $\mathbf{H}$ with $\mathbf{H}_a$.

4. OBSERVER SYNTHESIS

In this section, the feedback law for the observer is derived, and the problem of bias in the velocity sensors is addressed. The observer for the case of unbiased velocity measurements, which is based on previous work by the authors [Vasconcelos et al., 2007], is presented first to motivate the derivation of the attitude and position feedback laws, and expose topological limitations to global stabilization.

4.1 Unbiased Velocity Measurements

For the case of unbiased velocity readings, the Lyapunov function decoupling $V = V_{\mathcal{R}} + V_p$ allows for the attitude and position estimation problems to be addressed separately. The attitude observer (4a) is derived using the Lyapunov function $V_{\mathcal{R}}$, and the position observer (4b) is derived using V_p.

Attitude Feedback Law Under Assumption 1 and given the Lyapunov function derivatives along the system trajectories (10a), consider the following feedback law

$$\hat{\boldsymbol{\omega}} = \bar{\boldsymbol{\omega}} - K_\omega\mathbf{s}_\omega, \tag{11}$$

where the feedback term is given by

$$\mathbf{s}_\omega := \bar{\mathcal{R}}'\left[\tilde{\mathcal{R}}-\tilde{\mathcal{R}}'\otimes\right] = 2\sin(\theta)\bar{\mathcal{R}}'\boldsymbol{\lambda}, \tag{12}$$

and $K_\omega > 0$ is a positive scalar. The attitude feedback yields the autonomous attitude error system

$$\dot{\tilde{\mathcal{R}}} = K_\omega\tilde{\mathcal{R}}(\tilde{\mathcal{R}}'-\tilde{\mathcal{R}}), \tag{13}$$

and a negative semi-definite time derivative $\dot{V}_{\mathcal{R}} = -K_\omega\mathbf{s}_\omega'\mathbf{s}_\omega = -4K_\omega\sin^2(\theta) \leq 0$, so it is immediate that the attitude feedback law produces a Lyapunov function that decreases along the system trajectories. Under Assumption 1, the set of points where $\dot{V}_{\mathcal{R}} = 0$ is given by

$$C_{\mathcal{R}} = \{\tilde{\mathcal{R}} \in \mathrm{SO}(3) : \tilde{\mathcal{R}} = \mathbf{I} \vee \tilde{\mathcal{R}} = \operatorname{rot}(\pi,\boldsymbol{\lambda}), \boldsymbol{\lambda} \in \mathrm{S}(2)\}.$$

By direct substitution in the closed loop system (13), the subset $\{\tilde{\mathcal{R}} \in \mathrm{SO}(3) : \tilde{\mathcal{R}} = \operatorname{rot}(\pi,\lambda)\}$ is invariant, hence any initial condition with $\theta = \pi$ does not converge to the desired equilibrium point $\theta = 0$. This issue is a consequence of the known limitations to global stability on SO(3) discussed in [Bhat and Bernstein, 2000], for details on the present estimation problem see [Vasconcelos et al., 2007] and references therein. However, the set $\theta = \pi$ has zero measure and, as shown in the next theorem, every other initial condition converges exponentially fast to the origin yielding almost global convergence. The proof is obtained by adaptation of the [Vasconcelos et al., 2007, Theorem 3], and omitted due to space constraints.

Theorem 4. The closed-loop system (13) has an exponentially stable point at $\tilde{\mathcal{R}} = \mathbf{I}$. For any initial condition $\tilde{\mathcal{R}}(t_0)$ in the region of attraction $R_A = \{\tilde{\mathcal{R}} \in \mathrm{SO}(3) : \tilde{\mathcal{R}} = \mathrm{rot}(\theta, \boldsymbol{\lambda}), |\theta| < \pi, \boldsymbol{\lambda} \in \mathrm{S}(2)\}$ the trajectory satisfies

$$\|\tilde{\mathcal{R}}(t) - \mathbf{I}\| \leq k_{\mathcal{R}} \|\tilde{\mathcal{R}}(t_0) - \mathbf{I}\| e^{-\frac{1}{2}\gamma_{\mathcal{R}}(t-t_0)}, \qquad (14)$$

where $k_{\mathcal{R}} = 1$ and $\gamma_{\mathcal{R}} = 2K_\omega(1 + \cos(\theta(t_0)))$.

Position Feedback Law The position feedback law for the system (4b) is defined as

$$\hat{\mathbf{v}} = \bar{\mathbf{v}} + ([\bar{\boldsymbol{\omega}}\times] - K_v \mathbf{I})\mathbf{s}_v - [\hat{\mathbf{p}}\times](\hat{\boldsymbol{\omega}} - \bar{\boldsymbol{\omega}}), \qquad (15)$$

where the feedback term is defined as $\mathbf{s}_v := \tilde{\mathbf{p}}$, and K_v is a positive scalar. The position feedback law produces a closed loop linear time-invariant system $\dot{\tilde{\mathbf{p}}} = -K_v \tilde{\mathbf{p}}$, and exponential stability of the origin is immediate.

4.2 Biased Velocity Measurements

This section presents the derivation of an exponentially stabilizing observer for attitude and position estimation in the presence of angular and linear velocity biases

$$\boldsymbol{\omega}_r = \bar{\boldsymbol{\omega}} + \bar{\mathbf{b}}_\omega, \qquad \mathbf{v}_r = \bar{\mathbf{v}} + \bar{\mathbf{b}}_v \qquad (16)$$

where the nominal biases are considered constant, i.e. $\dot{\bar{\mathbf{b}}}_\omega = \mathbf{0}$, $\dot{\bar{\mathbf{b}}}_v = \mathbf{0}$. The proposed Lyapunov function (8) is augmented to account for the effect of the angular and linear velocity bias

$$\begin{aligned} V_b =& 2\gamma_\theta(1 - \cos(\theta)) + \frac{\gamma_p}{2}\|\tilde{\mathbf{p}}\|^2 + \frac{\gamma_{b_\omega}}{2}\|\tilde{\mathbf{b}}_\omega\|^2 + \frac{\gamma_{b_v}}{2}\|\tilde{\mathbf{b}}_v\|^2, \\ \dot{V}_b =& \gamma_p \mathbf{s}_v' \left([\hat{\mathbf{p}}\times](\hat{\boldsymbol{\omega}} - \bar{\boldsymbol{\omega}}) + (\hat{\mathbf{v}} - \bar{\mathbf{v}}) - [\bar{\boldsymbol{\omega}}\times]\tilde{\mathbf{p}}\right) \\ &+ \gamma_\theta \mathbf{s}_\omega'(\hat{\boldsymbol{\omega}} - \bar{\boldsymbol{\omega}}) + \gamma_{b_\omega}\tilde{\mathbf{b}}_\omega' \dot{\tilde{\mathbf{b}}}_\omega + \gamma_{b_v}\tilde{\mathbf{b}}_v' \dot{\tilde{\mathbf{b}}}_v \end{aligned} \qquad (17)$$

where $\tilde{\mathbf{b}}_\omega = \hat{\mathbf{b}}_\omega - \bar{\mathbf{b}}_\omega$, $\tilde{\mathbf{b}}_v = \hat{\mathbf{b}}_v - \bar{\mathbf{b}}_v$ are the bias compensation errors, $\hat{\mathbf{b}}_\omega$, $\hat{\mathbf{b}}_v$ are the estimated biases, and γ_θ, γ_p, γ_{b_ω} and γ_{b_v} are positive scalars. Under Assumption 1 and given Lemma 2, the Lyapunov function V_b has an unique global minimum at $(\tilde{\mathbf{p}}, \tilde{\mathcal{R}}, \tilde{\mathbf{b}}_\omega, \mathbf{b}_v) = (0, \mathbf{I}, 0, 0)$.

The feedback laws for the angular and linear velocities are given by

$$\begin{aligned} \hat{\boldsymbol{\omega}} &= (\bar{\boldsymbol{\omega}} + \bar{\mathbf{b}}_\omega - \hat{\mathbf{b}}_\omega) - K_\omega \mathbf{s}_\omega = (\bar{\boldsymbol{\omega}} - \tilde{\mathbf{b}}_\omega) - K_\omega \mathbf{s}_\omega, \qquad (18)\\ \hat{\mathbf{v}} &= \mathbf{v}_r - \hat{\mathbf{b}}_v + \left(\left[\boldsymbol{\omega}_r - \hat{\mathbf{b}}_\omega\times\right] - K_v\mathbf{I}\right)\mathbf{s}_v - [\hat{\mathbf{p}}\times](\hat{\boldsymbol{\omega}} - (\boldsymbol{\omega}_r - \hat{\mathbf{b}}_\omega)) \\ &= \bar{\mathbf{v}} - \tilde{\mathbf{b}}_v + \left(\left[\bar{\boldsymbol{\omega}} - \tilde{\mathbf{b}}_\omega\times\right] - K_v\mathbf{I}\right)\mathbf{s}_v + K_\omega[\hat{\mathbf{p}}\times]\mathbf{s}_\omega. \end{aligned}$$

which are obtained by adding bias compensation terms to the feedback laws (11) and (15), respectively. The augmented Lyapunov time derivative is described by $\dot{V}_b = -\gamma_p K_v \|\mathbf{s}_v\|^2 - \gamma_\theta K_\omega \|\mathbf{s}_\omega\|^2 + (\gamma_p[\hat{\mathbf{p}}\times]\tilde{\mathbf{p}} - \gamma_\theta \mathbf{s}_\omega + \gamma_{b_\omega}\dot{\tilde{\mathbf{b}}}_\omega)'\tilde{\mathbf{b}}_\omega + (\gamma_{b_v}\dot{\tilde{\mathbf{b}}}_v - \gamma_p\tilde{\mathbf{p}})'\tilde{\mathbf{b}}_v$. The bias estimates satisfy $\dot{\hat{\mathbf{b}}}_\omega = \dot{\tilde{\mathbf{b}}}_\omega$, $\dot{\hat{\mathbf{b}}}_v = \dot{\tilde{\mathbf{b}}}_v$, and the bias feedback laws are defined as

$$\dot{\hat{\mathbf{b}}}_\omega = \frac{1}{\gamma_{b_\omega}}(\gamma_\theta \mathbf{s}_\omega - \gamma_p[\hat{\mathbf{p}}\times]\tilde{\mathbf{p}}), \qquad \dot{\hat{\mathbf{b}}}_v = \frac{\gamma_p}{\gamma_{b_v}}\tilde{\mathbf{p}}. \qquad (19)$$

The closed loop system can be written as

$$\begin{aligned} \dot{\tilde{\mathbf{p}}} &= -[\bar{\mathbf{p}}\times]\tilde{\mathbf{b}}_\omega - K_v\tilde{\mathbf{p}} - \tilde{\mathbf{b}}_v \\ \dot{\tilde{\mathcal{R}}} &= -K_\omega\tilde{\mathcal{R}}(\tilde{\mathcal{R}} - \tilde{\mathcal{R}}') - \tilde{\mathcal{R}}\left[\bar{\mathcal{R}}\tilde{\mathbf{b}}_\omega\times\right] \\ \dot{\tilde{\mathbf{b}}}_\omega &= \frac{\gamma_\theta}{\gamma_{b_\omega}}\bar{\mathcal{R}}\left[\tilde{\mathcal{R}} - \tilde{\mathcal{R}}'\otimes\right] - \frac{\gamma_p}{\gamma_{b_\omega}}[\bar{\mathbf{p}}\times]\tilde{\mathbf{p}}, \quad \dot{\tilde{\mathbf{b}}}_v = \frac{\gamma_p}{\gamma_{b_v}}\tilde{\mathbf{p}} \end{aligned} \qquad (20)$$

which are nonautonomous and the Lyapunov function time derivative is described by $\dot{V}_b = -\gamma_p K_v \mathbf{s}_v'\mathbf{s}_v - \gamma_\theta K_\omega \mathbf{s}_\omega'\mathbf{s}_\omega$.

Let $\mathbf{x}_b := (\tilde{\mathbf{p}}, \tilde{\mathcal{R}}, \tilde{\mathbf{b}}_\omega, \tilde{\mathbf{b}}_p)$ and $D_b := \mathbb{R}^3 \times \mathrm{SO}(3) \times \mathbb{R}^3 \times \mathbb{R}^3$, the set of points where $\dot{V}_{b_\omega} = 0$ is $C_b = \{\mathbf{x}_b \in D_b : (\tilde{\mathbf{p}}, \tilde{\mathbf{b}}_\omega, \tilde{\mathbf{b}}_p) = (\mathbf{0}, \mathbf{0}, \mathbf{0}), \tilde{\mathcal{R}} \in C_{\mathcal{R}}\}$. As discussed in Section 4.1, global asymptotic stability of the origin is precluded by topological limitations associated with the estimation error $\tilde{\mathcal{R}} = \mathrm{rot}(\pi, \boldsymbol{\lambda})$. In the next proposition, the boundedness of the estimation errors is shown and used to provide sufficient conditions for excluding convergence to the equilibrium points satisfying $\tilde{\mathcal{R}} = \mathrm{rot}(\pi, \boldsymbol{\lambda})$.

Lemma 5. The estimation errors $(\tilde{\mathbf{p}}, \tilde{\mathcal{R}}, \tilde{\mathbf{b}}_\omega, \tilde{\mathbf{b}}_p)$ are bounded. For any initial condition such that

$$\frac{\gamma_{b_v}\|\tilde{\mathbf{b}}_v(t_0)\|^2 + \gamma_p\|\tilde{\mathbf{p}}(t_0)\|^2 + \gamma_{b_\omega}\|\tilde{\mathbf{b}}_\omega(t_0)\|^2}{4\gamma_\theta(1 + \cos(\theta(t_0)))} < 1, \qquad (21)$$

the attitude error is bounded by $\theta(t) \leq \theta_{\max} < \pi$ for all $t \geq t_0$.

Proof. Define the set $\Omega_\rho = \{\mathbf{x}_b \in \mathrm{D}_b : V_b \leq \rho\}$. The Lyapunov function (17) consists of the weighted distance of the state to the origin, so $\exists_\alpha \|\mathbf{x}_b\|^2 \leq \alpha V_b$ and the set Ω_ρ is compact. The Lyapunov function decreases along the system trajectories, $\dot{V}_b \leq 0$, so any trajectory starting in Ω_ρ will remain in Ω_ρ. Consequently, $\forall_{t \geq t_0} \|\mathbf{x}_b(t)\|^2 \leq \alpha V_b(\mathbf{x}(t_0))$ and the state is bounded.

The gain condition (21) is equivalent to $V_b(\mathbf{x}_b(t_0)) < 4\gamma_\theta$. The invariance of Ω_ρ implies that $V_b(\mathbf{x}_b(t)) \leq V_b(\mathbf{x}_b(t_0))$ which implies that $2\gamma_\theta(1 - \cos(\theta(t))) \leq V_b(\mathbf{x}_b(t_0)) < 4\gamma_\theta$ and consequently $\exists_{\theta_{\max}} \theta(t) \leq \theta_{\max} < \pi$ for all $t \geq t_0$. □

Adopting the analysis tools for parameterized LTV systems [Loría and Panteley, 2002], the system (20), in the form $\dot{\mathbf{x}}_b = f(t, \mathbf{x}_b)\mathbf{x}_b$, is rewritten as $\dot{\mathbf{x}}_\star = \mathbf{A}(\lambda, t)\mathbf{x}_\star$. In this formulation, the parameter $\lambda \in D_b \times \mathbb{R}$ is associated with the initial conditions of the nonlinear system and the solutions of both systems are identical wherever the initial conditions of both systems coincide, $\mathbf{x}_\star(t_0) = \mathbf{x}(t_0)$, and the parameter satisfies $\lambda = (t_0, \mathbf{x}(t_0))$. Sufficient conditions for uniform exponential stability of the parameterized LTV system, and thus of the nonlinear system, are derived in [Loría and Panteley, 2002]. Using these results, exponential convergence of the estimation errors in the presence of biased velocity measurements is shown.

Theorem 6. Let $\gamma_{b_v} = \gamma_{b_\omega}$ and assume that $\bar{\mathbf{p}}$, $\bar{\mathbf{v}}$ and $\bar{\boldsymbol{\omega}}$ are bounded. For any initial condition that satisfies (21), the position, attitude and bias estimation errors converge exponentially fast to the stable equilibrium point $(\tilde{\mathbf{p}}, \tilde{\mathcal{R}}, \tilde{\mathbf{b}}_\omega, \tilde{\mathbf{b}}_v) = (0, \mathbf{I}, 0, 0)$.

Proof. The stability of (20) is obtained by a change of coordinates, using an attitude representation similar to that proposed in [Thienel and Sanner, 2003]. Let the attitude error vector be given by $\tilde{\mathbf{q}}_q = \sin(\frac{\theta}{2})\boldsymbol{\lambda}$, the closed loop kinematics are described by

$$\dot{\tilde{\mathbf{p}}} = -[\bar{\mathbf{p}}\times]\tilde{\mathbf{b}}_\omega - K_v\tilde{\mathbf{p}} - \tilde{\mathbf{b}}_v, \quad \dot{\tilde{\mathbf{q}}}_q = \tfrac{1}{2}\mathbf{Q}(\tilde{\mathbf{q}})(-\bar{\mathcal{R}}\tilde{\mathbf{b}}_\omega - 4K_\omega\tilde{\mathbf{q}}_q\tilde{q}_s)$$

$$\dot{\tilde{\mathbf{b}}}_\omega = 4\frac{\gamma_\theta}{\gamma_{b_\omega}}\bar{\mathcal{R}}'\mathbf{Q}'(\tilde{\mathbf{q}})\tilde{\mathbf{q}}_q - \frac{\gamma_p}{\gamma_{b_\omega}}[\bar{\mathbf{p}}\times]\tilde{\mathbf{p}}, \quad \dot{\tilde{\mathbf{b}}}_v = \frac{\gamma_p}{\gamma_{b_v}}\tilde{\mathbf{p}} \tag{22}$$

where $\mathbf{Q}(\tilde{\mathbf{q}}) := \tilde{q}_s\mathbf{I} + [\tilde{\mathbf{q}}_q\times]$, $\tilde{\mathbf{q}} = \begin{bmatrix}\tilde{\mathbf{q}}_q' & \tilde{q}_s\end{bmatrix}'$, $\tilde{q}_s = \cos(\frac{\theta}{2})$ and $\dot{\tilde{q}}_s = 2K_\omega\tilde{\mathbf{q}}_q'\tilde{\mathbf{q}}_q\tilde{q}_s - \frac{1}{2}\mathbf{q}_q'\tilde{\mathbf{b}}_\omega$. The vector $\tilde{\mathbf{q}}$ is the well known Euler quaternion representation [Murray et al., 1994]. Using $\|\tilde{\mathbf{q}}_q\|^2 = \frac{1}{8}\|\tilde{\mathcal{R}} - \mathbf{I}\|^2$, the Lyapunov function in quaternion coordinates is described by $V_b = 4\gamma_\theta\|\tilde{\mathbf{q}}_q\|^2 + \frac{\gamma_p}{2}\|\tilde{\mathbf{p}}\|^2 + \frac{\gamma_{b_\omega}}{2}\|\tilde{\mathbf{b}}_\omega\|^2 + \frac{\gamma_{b_v}}{2}\|\tilde{\mathbf{b}}_v\|^2$.

Let $\mathbf{x}_q := (\tilde{\mathbf{p}}, \tilde{\mathbf{q}}_q, \tilde{\mathbf{b}}_\omega, \tilde{\mathbf{b}}_v)$, $\mathbf{x}_q \in D_q$, and $D_q := \mathbb{R}^3 \times \mathrm{B}(3) \times \mathbb{R}^3 \times \mathbb{R}^3$, define the system (22) in the domain $\mathcal{D}_q = \{\mathbf{x} \in D_q : V_b \leq \gamma_\theta(4 - \varepsilon_q)\}$, $0 < \varepsilon_q < 4$. The set $\mathcal{D}_q$ is given by the interior of the Lyapunov surface, so it is positively invariant and well defined. The condition (21) implies that the initial condition is contained in the set $\mathcal{D}_q$ for ε_q small enough.

Let $\mathbf{x}_\star := (\tilde{\mathbf{p}}_\star, \tilde{\mathbf{q}}_{q\star}, \tilde{\mathbf{b}}_{\omega\star}, \tilde{\mathbf{b}}_{v\star})$, $D_q := \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3$, $\gamma_b := \gamma_{b_\omega} = \gamma_{b_v}$, and define the parameterized LTV system

$$\dot{\mathbf{x}}_\star = \begin{bmatrix} \mathcal{A}(t,\lambda) & \mathcal{B}'(t,\lambda) \\ -\mathcal{C}(t,\lambda) & \mathbf{0}_{3\times3} \end{bmatrix}\mathbf{x}_\star \tag{23}$$

where $\lambda \in \mathbb{R}_{\geq 0} \times \mathcal{D}_q$, the submatrices are described by

$$\mathcal{A}(t,\lambda) = \begin{bmatrix} -K_v\mathbf{I} & \mathbf{0}_{3\times3} \\ \mathbf{0}_{3\times3} & -2K_\omega\tilde{q}_s(t,\lambda)\mathbf{Q}(\tilde{\mathbf{q}}(t,\lambda)) \end{bmatrix},$$

$$\mathcal{B}(t,\lambda) = \begin{bmatrix} [\bar{\mathbf{p}}\times] & -\frac{\bar{\mathcal{R}}'\mathbf{Q}'(\tilde{\mathbf{q}}(t,\lambda))}{2} \\ -\mathbf{I} & \mathbf{0}_{3\times3} \end{bmatrix}, \mathcal{C}(t,\lambda) = \frac{\mathcal{B}(t,\lambda)}{\gamma_b}\begin{bmatrix} \gamma_p\mathbf{I} & 0 \\ 0 & 8\gamma_\theta\mathbf{I} \end{bmatrix},$$

and the quaternion $\tilde{\mathbf{q}}(t,\lambda)$ represents the solution of (22) with initial condition $\lambda = (t_0, \tilde{\mathbf{p}}(t_0), \tilde{\mathbf{q}}_q(t_0), \tilde{\mathbf{b}}_\omega(t_0), \tilde{\mathbf{b}}_v(t_0))$. By the boundedness of $\bar{\mathbf{p}}$, the matrices $\mathcal{A}(t,\lambda)$, $\mathcal{B}(t,\lambda)$ and $\mathcal{C}(t,\lambda)$ are bounded, and the system is well defined [Khalil, 1996, p. 626]. If the parameterized LTV (23) is λ-UGES, then the nonlinear system (22) is uniformly exponentially stable in the domain $\mathcal{D}_q$, [Loría and Panteley, 2002, p.14-15]. The parameterized LTV system verifies the assumptions of [Loría and Panteley, 2002, Theorem 1]:

1) Given the boundedness of $\bar{\mathbf{p}}$, $\bar{\mathbf{v}}$ and $\bar{\boldsymbol{\omega}}$, $\dot{\bar{\mathbf{p}}}$ is bounded, and the elements of $\mathcal{B}(t,\lambda)$ and $\frac{\partial\mathcal{B}(t,\lambda)}{\partial t}$, as well as the corresponding induced Euclidean norm, are bounded for all $\lambda \in \mathbb{R}_{\geq 0} \times \mathcal{D}_q$, $t \geq t_0$.

2) The positive definite matrices $P(t,\lambda) = \frac{1}{\gamma_b}\begin{bmatrix} \gamma_p\mathbf{I} & 0 \\ 0 & 8\gamma_\theta\mathbf{I} \end{bmatrix}$, $Q(t,\lambda) = \frac{1}{\gamma_b}\begin{bmatrix} 2K_v\gamma_p\mathbf{I} & 0 \\ 0 & 32\tilde{q}_s^2(t,\lambda)K_\omega\gamma_\theta\mathbf{I} \end{bmatrix}$ satisfy $P(t,\lambda)\mathcal{B}'(t,\lambda) = \mathcal{C}'(t,\lambda)$, $-Q(t,\lambda) = \mathcal{A}'(t,\lambda)P(t,\lambda) + P(t,\lambda)\mathcal{A}(t,\lambda) + \dot{P}(t,\lambda)$, $\min(C_p)\mathbf{I} \leq P(t,\lambda) \leq \max(C_p)\mathbf{I}$, $\min(C_q)\mathbf{I} \leq Q(t,\lambda) \leq \max(C_q)\mathbf{I}$ with $C_p = \frac{1}{\gamma_b}\{\gamma_p, 8\gamma_\theta\}$, $C_q = \frac{1}{\gamma_b}\{c_q, c_q\cos^2(\frac{\theta_{\max}}{2}), 2K_v\gamma_p\}$, and $c_q = 32K_\omega\gamma_\theta$.

The system (23) is λ-UGES if and only if $\mathcal{B}(t,\lambda)$ is λ-uniformly persistently exciting [Loría and Panteley, 2002]. Algebraic manipulation produces for any $\mathbf{y} \in \mathbb{R}^3$,

$$\mathcal{B}(\tau,\lambda)\mathcal{B}'(\tau,\lambda) = \begin{bmatrix} \frac{1}{4}\bar{\mathcal{R}}'\mathbf{Q}'(\tilde{\mathbf{q}})\mathbf{Q}(\tilde{\mathbf{q}})\bar{\mathcal{R}} - [\bar{\mathbf{p}}\times]^2 & -[\bar{\mathbf{p}}\times] \\ [\bar{\mathbf{p}}\times] & \mathbf{I} \end{bmatrix},$$

$$\frac{1}{4}\mathbf{y}'\bar{\mathcal{R}}'\mathbf{Q}'(\tilde{\mathbf{q}})\mathbf{Q}(\tilde{\mathbf{q}})\bar{\mathcal{R}}\mathbf{y} = \frac{1}{4}\left(\|\mathbf{y}\|^2 - (\mathbf{y}'\bar{\mathcal{R}}'\tilde{\mathbf{q}}_q)^2\right)$$

$$\geq \frac{\|\mathbf{y}\|^2}{4}\left(1 - \|\tilde{\mathbf{q}}_q\|^2\right) \geq \frac{\|\mathbf{y}\|^2}{4}\left(1 - \sin^2\left(\tfrac{1}{2}\theta_{\max}\right)\right) = \|\mathbf{y}\|^2 c_\theta$$

where $c_\theta := \frac{1}{4}\cos^2\left(\frac{1}{2}\theta_{\max}\right)$. Therefore $\mathcal{B}(\tau,\lambda)\mathcal{B}'(\tau,\lambda) \geq \mathbf{B}(\tau)$, where $\mathbf{B}(\tau) := \begin{bmatrix} c_\theta\mathbf{I} - [\bar{\mathbf{p}}\times]^2 & -[\bar{\mathbf{p}}\times] \\ [\bar{\mathbf{p}}\times] & \mathbf{I} \end{bmatrix}$. Simple but long algebraic manipulations show that the eigenvalues of $\mathbf{B}(\tau)$ are lower bounded by some $\alpha_B > 0$, independent of τ, if $\bar{\mathbf{p}}$ is bounded and $\theta_{\max} < \pi$. It follows that persistency of excitation condition is satisfied, $\mathcal{B}(\tau,\lambda)\mathcal{B}'(\tau,\lambda) \geq \alpha_B\mathbf{I}$, the parameterized LTV (23) is λ-UGES, and the nonlinear system (22) is exponentially stable in the domain $\mathcal{D}_q$. □

Given γ_p, γ_θ, γ_{b_ω}, and γ_{b_v}, any initial estimation error $\mathbf{x}_b(t_0)$ satisfying (21) converges exponentially fast to the origin. The following corollary establishes that the origin is uniform exponential stable, i.e. the convergence rate bounds are independent of $\mathbf{x}_b(t_0)$, for a bounded initial estimation error, which is a reasonable assumption for most applications.

Corollary 7. Assume that the initial estimation errors are bounded

$$\begin{aligned} \|\tilde{\mathbf{p}}(t_0)\| &\leq \tilde{p}_{0\max}, \quad \theta(t_0) \leq \theta_{0\max} < \pi \\ \|\tilde{\mathbf{b}}_\omega(t_0)\| &\leq \tilde{b}_{\omega 0\max}, \quad \|\tilde{\mathbf{b}}_v(t_0)\| \leq \tilde{b}_{v0\max}, \end{aligned} \tag{24}$$

and let $\gamma_{b_v}\tilde{\mathbf{b}}_{v0\max}^2 + \gamma_p\tilde{\mathbf{p}}_{0\max}^2 + \gamma_{b_\omega}\tilde{\mathbf{b}}_{\omega 0\max}^2 < 4\gamma_\theta(1 + \cos(\theta_{0\max}))$, $\gamma_{b_\omega} = \gamma_{b_v}$, then the equilibrium point $\mathbf{x} = (0, \mathbf{I}, 0, 0)$ is exponentially stable, uniformly in the set defined by (24).

Convergence rate bounds can be obtained by applying [Loría, 2004, Theorem 1 and Remark 2]. However, the obtained values were conservative and are omitted due to the necessity of further study.

4.3 Output Feedback Law

This section shows that the feedback laws can be expressed in terms of the landmark and velocity readings, (16) and (2) respectively.

Theorem 8. The feedback laws are explicit functions of the sensor readings and state estimates

$$\hat{\boldsymbol{\omega}} = \boldsymbol{\omega}_r - \hat{\mathbf{b}}_\omega - K_\omega\mathbf{s}_\omega, \tag{25a}$$

$$\hat{\mathbf{v}} = \mathbf{v}_r - \hat{\mathbf{b}}_v + \left(\left[\boldsymbol{\omega}_r - \hat{\mathbf{b}}_\omega\times\right] - K_v\mathbf{I}\right)\mathbf{s}_v + K_\omega[\hat{\mathbf{p}}\times]\mathbf{s}_\omega, \tag{25b}$$

$$\dot{\hat{\mathbf{b}}}_\omega = \frac{1}{\gamma_{b_\omega}}(\gamma_\theta\mathbf{s}_\omega - \gamma_p[\hat{\mathbf{p}}\times]\mathbf{s}_v), \quad \dot{\hat{\mathbf{b}}}_v = \frac{\gamma_p}{\gamma_{b_v}}\mathbf{s}_v \tag{25c}$$

$$\mathbf{s}_\omega = \sum_{i=1}^{n}(\hat{\mathcal{R}}'\mathbf{X}\mathbf{D}_X\mathbf{A}_X\mathbf{e}_i) \times (\mathbf{Q}_r\mathbf{D}_X\mathbf{A}_X\mathbf{e}_i), \mathbf{s}_v = \hat{\mathbf{p}} - \mathbf{Q}_r\mathbf{d}_p \tag{25d}$$

where $\mathbf{Q}_r := [\mathbf{q}_{r\,1} \ \cdots \ \mathbf{q}_{r\,n}]$ is the concatenation of the landmark readings and $\mathbf{e}_i$ is the unit vector where $e_i = 1$.

Proof. The expressions (25a-25c) are directly obtained from (18-19). Given (6) and $\mathbf{Q}_r = \bar{\mathbf{Q}}$ produces $\mathbf{s}_v$. Using the definition of $\tilde{\mathcal{R}}$, the feedback law (12) can be rewritten as $\mathbf{s}_\omega = \left[\bar{\mathcal{R}}'\hat{\mathcal{R}} - \hat{\mathcal{R}}'\bar{\mathcal{R}}\otimes\right]$. Using ${}^B\hat{\mathbf{U}}_X{}^B\bar{\mathbf{U}}_X' = \hat{\mathcal{R}}'\bar{\mathcal{R}}$, ${}^B\bar{\mathbf{U}}_X{}^B\hat{\mathbf{U}}_X' = \sum_{i=1}^n {}^B\bar{\mathbf{u}}_i{}^B\hat{\mathbf{u}}_i'$ and ${}^B\bar{\mathbf{u}}_i{}^B\hat{\mathbf{u}}_i' - {}^B\hat{\mathbf{u}}_i{}^B\bar{\mathbf{u}}_i' = \left[({}^B\hat{\mathbf{u}}_i \times {}^B\bar{\mathbf{u}}_i)\times\right]$, bears $\mathbf{s}_\omega = \sum_{i=1}^n({}^B\hat{\mathbf{U}}_X\mathbf{e}_i) \times ({}^B\bar{\mathbf{U}}_X\mathbf{e}_i)$. Applying (6) and (7) produces the desired results. □

5. SIMULATIONS

In this section, the simulation results for the proposed observer are presented. The position of the landmarks is described by $\mathbf{X} = \begin{bmatrix} 0 & 1/2 & -1/2 \\ 1 & -1/2 & -1/2 \\ 0 & 0 & 0 \end{bmatrix}$, which satisfies the conditions expressed in Assumption 1 and corresponds to the case discussed in Section 3.2. The feedback gains

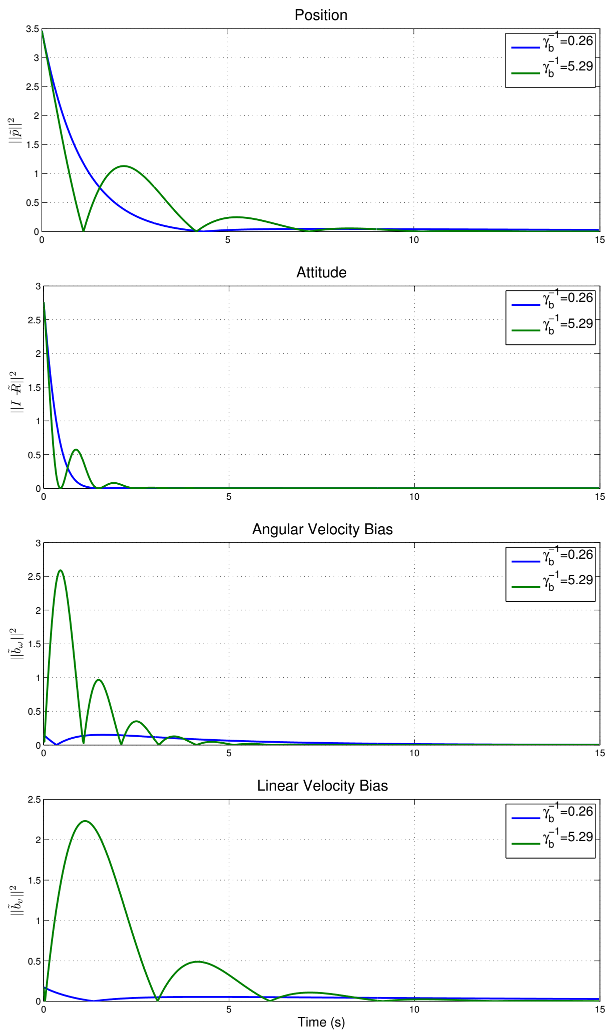

Fig. 2. Estimation Errors.

are given by $K_\omega = K_v = 1$, the values of $\frac{\gamma_p}{\gamma_\theta}$ and $\frac{\gamma_b}{\gamma_\theta}$ are computed to satisfy the condition of Corollary 7 for $\tilde{p}_{0\,\max} = 2\sqrt{3}\,\mathrm{m}$, $\theta_{0\,\max} = 90\,°$, $\tilde{b}_{\omega 0\,\max} = 5\sqrt{3}\,°/\mathrm{s}$ and $\tilde{b}_{v0\,\max} = \sqrt{3} \times 10^{-1}\,\mathrm{m/s}$, and the initial estimation errors are given by $\tilde{\mathbf{p}}(t_0) = 2\,[1\ 1\ 1]'\ \mathrm{m}$, $\theta(t_0) = 72\,°$, $\tilde{\mathbf{b}}_\omega(t_0) = 5\,[1\ 1\ 1]'\,°/s$, $\tilde{\mathbf{b}}_v = 10^{-1}\,[1\ 1\ 1]'\,°/s$. The rigid body trajectory is computed using oscillatory angular and linear velocities of 1 Hz.

The estimation errors converge to the origin as depicted in Fig. 2. The exponential convergence of the Lyapunov function (and of the estimation error) is illustrated in Fig. 3, using a logarithmic scale. The convergence of the estimation error to the origin is faster for larger bias feedback gains.

6. CONCLUSIONS

A nonlinear observer for position and attitude estimation on SE(3) was proposed, using landmark measurements and biased velocity readings. The estimation errors were shown to converge exponentially fast to the origin, by adopting recently derived stability results for parameterized linear time-varying systems. Simulation results illustrated the convergence properties of the observer for different feedback gains. Future work will focus on improving the convergence rate bounds provided by the parameterized systems' framework, and on the implementation in discrete time and testing of the algorithm on the Vario X-Treme model-scale helicopter platform.

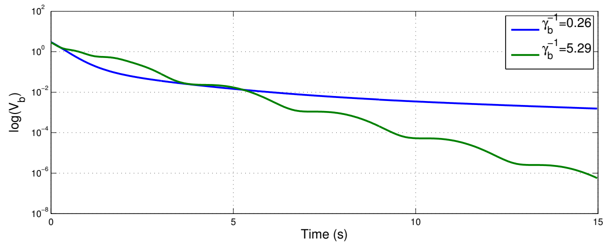

Fig. 3. Exponential Convergence of V_b.

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