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Finite-Dimensional H_∞ Filter Design for Linear Systems with State Delay ⋆

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Abstract: This paper presents the central finite-dimensional H_∞ filters for linear systems with state delay, that are suboptimal for a given threshold γ with respect to a modified Bolza-Meyer quadratic criterion including the attenuation control term with the opposite sign. In contrast to the results previously obtained for linear time delay systems, the paper reduces the original H_∞ filtering problems to H_2 (optimal mean-square) filtering problems, using the technique proposed in [1]. The paper first presents the central suboptimal H_∞ filter for linear systems with state delay, based on the optimal H_2 filter from [37], which contains a finite number of the filtering equations for any fixed filtering horizon, but this number grows unboundedly as time goes to infinity. To overcome that difficulty, the alternative central suboptimal H_∞ filter is designed for linear systems with state delay, which is based on the alternative optimal H_2 filter from [38]. Numerical simulations are conducted to verify performance of the designed central suboptimal filters for linear systems with state delay against the central suboptimal H_∞ filter available for linear systems without delays.

1. INTRODUCTION

Over the past two decades, the considerable attention has been paid to the H_∞ estimation problems for linear and nonlinear systems with and without time delays. The seminal papers in H_∞ control [1] and estimation [2–4] established a background for consistent treatment of filtering/controller problems in the H_∞-framework. A large number of results on this subject has been reported for systems in the general situation, linear or nonlinear (see ([5]–[13]). For the specific area of linear time-delay systems, the H_∞-filtering problem has also been extensively studied (see [14]–[34]). The sufficient conditions for existence of an H_∞ filter, where the filter gain matrices satisfy Riccati equations, were obtained for linear systems with state delay in [35] and with measurement delay in [36]. However, the criteria of existence and suboptimality of solution for the central H_∞ filtering problems based on the reduction of the original H_∞ problem to the induced H_2 one, similar to those obtained in [1,4] for linear systems without delay, remain yet unknown for linear systems with time delays.

This paper presents the central (see [1] for definition) finite-dimensional H_∞ filters for linear systems with state delay, that are suboptimal for a given threshold γ with respect to a modified Bolza-Meyer quadratic criterion including the attenuation control term with the opposite sign. In contrast to the results previously obtained for linear systems with state [35] or measurement delay [36], the paper reduces the original H_∞ filtering problems to H_2 (mean square) filtering problems, using the technique proposed in [1]. To the best authors' knowledge, this is the first paper which applies the reduction technique of [1] to classes of systems other than conventional LTI plants. Indeed, application of the reduction technique makes sense, since the optimal filtering equations solving the H_2 (mean square) filtering problems have been obtained for linear systems with state [37,38] or measurement [39] delays. Designing the central suboptimal H_∞ filter for linear systems with state delay presents a significant advantage in the filtering theory and practice, since (1) it enables one to address filtering problems for LTV time-delay systems, where the LMI technique is hardly applicable, (2) the obtained H_∞ filter is suboptimal, that is, optimal for any fixed γ with respect to the H_∞ noise attenuation criterion, and (3) the obtained H_∞ filter is finite-dimensional and has the same structure of the estimate and gain matrix equations as the corresponding optimal H_2 filter. Moreover, the proposed H_∞ filtering algorithms provide direct methods to calculate the minimum achievable values of the threshold γ, based on the existence properties for a bounded solution of the gain matrix equation. The corresponding calculations are made for each designed filter.

It should be commented that the proposed design of the central suboptimal H_∞ filters for linear time-delay systems with integral-quadratically bounded disturbances naturally carries over from the design of the optimal H_2 filters for linear time-delay systems with unbounded disturbances (white noises). The entire design approach creates a complete filtering algorithm of handling the linear time-delay systems with unbounded or integral-quadratically bounded disturbances optimally for all thresholds γ uniformly or for any fixed γ separately. A similar

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algorithm for linear systems without delay was developed in [1].

The paper first presents the central suboptimal H_∞ filter for linear systems with state delay, based on the optimal H_2 filter from [37], which contains a finite number of the filtering equations for any fixed filtering horizon, but this number grows unboundedly as time goes to infinity. To overcome that difficulty, the alternative central suboptimal H_∞ filter is designed for linear systems with state delay, which is based on the alternative optimal H_2 filter from [38]. The alternative filter contains only two differential equations for determining the estimate and filter gain matrix, regardless of the filtering horizon.

Numerical simulations are conducted to verify performance of the designed central suboptimal filters for linear systems with state delay against the central suboptimal H_∞ filter available for linear systems without delays [4]. The simulation results show a definite advantage in the values of the noise-output transfer function H_∞ norms in favor of the designed filters.

2. H_∞ FILTERING PROBLEM STATEMENT

Consider the following continuous-time LTV system with state delay:

$$\mathscr{S}_1: \ \dot{x}(t) = A(t)x(t-h) + B(t)\omega(t), \tag{1}$$

$$y(t) = C(t)x(t) + D(t)\omega(t), \tag{2}$$

$$z(t) = L(t)x(t), \tag{3}$$

$$x(\theta) = \varphi(\theta), \quad \forall\theta \in [t_0 - h, t_0] \tag{4}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $z(t) \in \mathbb{R}^q$ is the signal to be estimated, $y(t) \in \mathbb{R}^m$ is the measured output, $\omega(t) \in \mathscr{L}_2^p[0,\infty)$ is the disturbance input. $A(\cdot)$, $B(\cdot)$, $C(\cdot)$, $D(\cdot)$, $L(\cdot)$ are known continuous functions. $\varphi(\theta)$ is an unknown vector-valued continuous function defined on the initial interval $[t_0 - h, t_0]$. The time delay h is known.

For the system (1)–(4), the following standard conditions ([4]) are assumed:

- the pair (A,B) is stabilizable; $(\mathscr{C}_1)$
- the pair (C,A) is detectable; $(\mathscr{C}_2)$
- $D(t)B^T(t) = 0$ and $D(t)D^T(t) = I_m$. $(\mathscr{C}_3)$

Here, I_m is the identity matrix of dimension $m \times m$. As usual, the first two conditions ensure that the estimation error, provided by the designed H_∞ filter, converge to zero ([40]). The last noise orthonormality condition is technical and corresponds to the condition of independence of the standard Wiener processes (Gaussian white noises) in the stochastic filtering problems ([40]).

Now, consider a full-order $\mathscr{H}_\infty$ filter in the following form ($\mathscr{S}_2$):

$$\mathscr{S}_2: \ \dot{x}_f(t) = A(t)x_f(t-h) + K_f(t)[y(t) - C(t)x_f(t)], \tag{5}$$

$$z_f(t) = L(t)x_f(t), \tag{6}$$

where $x_f(t)$ is the filter state. The gain matrix $K_f(t)$ is to be determined.

Upon transforming the model (1)-(3) to include the states of the filter, the following filtering error system is obtained ($\mathscr{S}_3$):

$$\mathscr{S}_3: \ \dot{e}(t) = A(t)e(t-h) + B(t)\omega(t) - K_f(t)\tilde{y}(t), \tag{7}$$

$$\tilde{y}(t) = C(t)e(t) + D(t)\omega(t), \tag{8}$$

$$\tilde{z}(t) = L(t)e(t), \tag{9}$$

where $e(t) = x(t) - x_f(t)$, $\tilde{y}(t) = y(t) - C(t)x_f(t)$, and $\tilde{z}(t) = z(t) - z_f(t)$.

Therefore, the problem to be addressed is as follows: develop a robust $\mathscr{H}_\infty$ filter of the form (5)-(6) for the LTV system with state delay ($\mathscr{S}_1$), such that the following two requirements are satisfied:

(1) The resulting filtering error dynamics ($\mathscr{S}_3$) is robustly asymptotically stable in the absence of disturbances, $\omega(t) \equiv 0$;
(2) The filtering error dynamics ($\mathscr{S}_3$) ensures a noise attenuation level γ in an $\mathscr{H}_\infty$ sense. More specifically, for all nonzero $\omega(t) \in \mathscr{L}_2^p[0,\infty)$, the inequality
$$\|\tilde{z}(t)\|_2^2 < \gamma^2 \left\{ \|\omega(t)\|_2^2 + \|\varphi(\theta)\|_{2,R,[-h,0]}^2 \right\} \tag{10}$$
holds for $\mathscr{H}_\infty$ filtering problem, where
$\|f(t)\|_2^2 := \int_{t_0}^{\infty} f^T(t)f(t)dt$,
$\|\varphi(\theta)\|_{2,R,[t_0-h,t_0]}^2 = \int_{t_0-h}^{t_0} \varphi^T(\theta)R\varphi(\theta)d\theta$, R is a positive definite symmetric matrix, and γ is a given real positive scalar.

3. FINITE-DIMENSIONAL H_∞ FILTER DESIGN

The proposed design of the central H_∞ filter (see Theorem 4 in [1]) for LTV systems with state delay is based on the general result (see Theorem 3 in [1]) reducing the H_∞ controller problem to the corresponding H_2 (i.e., optimal linear-quadratic) controller problem. In this paper, only the filtering part of this result, valid for the entire controller problem, is used. Then, the optimal mean-square filter of the Kalman-Bucy type for LTV systems with state delay ([37]) is employed to obtain the desired result, which is given by the following theorem.

Theorem 1. The central H_∞ filter for the unobserved state (1) over the observations (2), ensuring the H_∞ noise attenuation condition (10) for the output estimate $z_f(t)$, is given by the equations for the state estimate $x_f(t)$ and the output estimate $z_f(t)$

$$\dot{x}_f(t) = A(t)x_f(t-h) + P(t)C^T(t)[y(t) - C(t)x_f(t)], \tag{11}$$

$$z_f(t) = L(t)x_f(t), \tag{12}$$

with the initial condition $x_f(\theta) = 0$ for $\forall\theta \in [t_0 - h, t_0]$, the equation for the filter gain matrix $P(t)$

$$dP(t) = (P_1(t)A^T(t) + A(t)P_1^T(t) + B(t)B^T(t) - \tag{13}$$

$$P(t)[C^T(t)C(t) - \gamma^{-2}L^T(t)L(t)]P(t))dt,$$

with the initial condition $P(t_0) = R^{-1}$, and the system of the equations for the complementary matrices $P_i(t)$, $i \geq 1$,

$$dP_i(t) = (A(t)P_{i-1}(t-ih) + P_{i+1}(t)A^T(t-ih))dt + \tag{14}$$

$$\frac{1}{2}(B(t)B^T(t-ih) + B(t-ih)B^T(t))dt -$$

$$\frac{1}{2}(P(t)[C^T(t)C(t-ih) - \gamma^{-2}L^T(t)L(t-ih)]P(t-ih) +$$

$$P(t-ih)[C^T(t-ih)C(t) - \gamma^{-2}L^T(t-ih)L(t)]P(t))dt,$$

with the initial conditions

$$P_i(\theta) = 0, \quad \theta \in [t_0 + (i-1)h, t_0 + ih].$$

The number of equations in (14) is equal to the integer part of the ratio T/h, where h is the state delay in (1) and T is the current filtering horizon.

Proof. First of all, note that the filtering error system (7)-(9) is already in the form used in Theorem 3 from [1]. Hence, according to Theorem 3 from [1], the H-infinity filtering part of this H_∞ controller problem would be equivalent to the H_2 (i.e., optimal mean-square) filtering problem, where the worst disturbance $w_{worst}(t) = \gamma^{-2}B^T(t)Q(t)e(t)$ is realized, and $Q(t)$ is the solution of the equation for the corresponding H_2 (optimal linear-quadratic) control gain. Therefore, the system, for which the equivalent H_2 (optimal mean-square) filtering problem is stated, takes the form

$$\begin{aligned} \mathscr{S}_4 : \dot{e}(t) &= A(t)e(t-h) + \gamma^{-2}B(t)B^T(t)Q(t)e(t) \qquad (15)\\ &- K_f(t)\tilde{y}(t), \\ \tilde{y}(t) &= C(t)e(t) + \gamma^{-2}D(t)B^T(t)Q(t)e(t), \qquad (16)\\ \tilde{z}(t) &= L(t)e(t). \qquad (17) \end{aligned}$$

As follows from Theorem 3 from [1] and Theorem 1 in [37], the H_2 (optimal mean-square) estimate equations for the error states (15) and (17) are given by

$$\begin{aligned} \mathscr{S}_5 : \ \dot{e_f}(t) &= A(t)e_f(t-h) - K_f(t)\tilde{y}(t) \qquad (18)\\ &+ P(t)C^T(t)[\tilde{y}(t) - C(t)e_f(t)], \\ \tilde{z}_f(t) &= L(t)e_f(t), \qquad (19) \end{aligned}$$

where $e_f(t)$ and $\tilde{z}_f(t)$ are the H_2 (optimal mean-square) estimates for $e(t)$ and $\tilde{z}(t)$, respectively. In the equation (18), $P(t)$ is the solution of the equation for the corresponding H_2 (optimal mean-square) filter gain, where, according to Theorem 3 from [1], the observation matrix $C(t)$ should be changed to $C(t) - \gamma^{-1}L(t)$ ($L(t)$ is the output matrix in (3)).

It should be noted that, in contrast to Theorem 3 from [1], no correction matrix $Z_\infty(t) = [I_n - \gamma^{-2}P(t)Q(t)]^{-1}$ appears in the last innovations term in the right-hand side of the equation (18), since there is no need to make the correction related to estimation of the worst disturbance $w_{worst}(t)$ in the error equation (15). Indeed, as stated in ([4]), the desired estimator must be unbiased, that is, $\tilde{z}_f(t) = 0$. Since the output error $\tilde{z}(t)$, satisfying (17), also stands in the criterion (10) and should be minimized as much as possible, the worst disturbance $w_{worst}(t)$ in the error equation (15) should be plainly rejected and, therefore, does not need to be estimated. Thus, the corresponding H_2 (optimal mean-square) filter gain would not include any correction matrix $Z_\infty(t)$. The same situation can be observed in Theorems 1–4 in [4]. However, if not the output error $\tilde{z}(t)$ but the output $z(t)$ itself would stand in the criterion (10), the correction matrix $Z_\infty(t) = [I_n - \gamma^{-2}P(t)Q(t)]^{-1}$ should be included.

Taking into account the unbiasedness of the estimator (18)-(19), it can be readily concluded that the equality $K_f(t) = P(t)C^T(t)$ must hold for the gain matrix $K_f(t)$ in (5). Thus, the filtering equations (5)-(6) take the final form (11)-(12), with the initial condition $x_f(\theta) = 0$ for $\forall\theta \in [t_0 - h, t_0]$, which corresponds to the central H_∞ filter (see Theorem 4 in [1]). It is still necessary to indicate the equations for the corresponding H_2 (optimal mean-square) filter gain matrix $P(t)$. In accordance with Theorem 1 from [37], the filter gain matrix $P(t)$ is given by the equation (13), with the initial condition $P(t_0) = R^{-1}$, which corresponds to the central H_∞ filter (see Theorems 3 and 4 in [4]). Note that the observation matrix $C(t)$ is changed to $C(t) - \gamma^{-1}L(t)$ according to Theorem 3 from [1]. Then, in view of Theorem 1 from [37], the equations (14) for complementary matrices $P_i(t)$, $i \geq 1$, should be added to obtain a closed system of the filtering equations.

It should be noted that, for every fixed t, the number of equations in (14), that should be taken into account to obtain a closed system of the filtering equations, is not equal to infinity, since the matrices $A(t)$, $B(t)$, $C(t)$, $D(t)$, and $L(t)$ are not defined for $t < t_0$. Therefore, if the current time moment t belongs to the semi-open interval $(kh, (k+1)h]$, where h is the delay value in the equation (1), the number of equations in (14) is equal to k. ■

A considerable advantage of the designed filter is a finite number of the filtering equations for any fixed filtering horizon, although the state space of the time-delay system (1) is infinite-dimensional [42].

Remark 1. The convergence properties of the obtained estimate (11) are given by the standard convergence theorem (see, for example, [40]): if in the system (1),(2) the pair $(A(t)\Psi(t-h,t), B(t))$ is uniformly completely controllable and the pair $(C(t), A(t)\Psi(t-h,t))$ is uniformly completely observable, where $\Psi(t,\tau)$ is the state transition matrix for the equation (1) (see [42] for definition of matrix Ψ), and the inequality $C^T(t)C(t) - \gamma^{-2}L^T(t)L(t) > 0$ holds, then the error of the obtained filter (11)–(14) is uniformly asymptotically stable. As usual, the uniform complete controllability condition is required for assuring non-negativeness of the matrix $P(t)$ (13) and may be omitted, if the matrix $P(t)$ is non-negative definite in view of its intrinsic properties. The uniform complete controllability and observability conditions for a linear system with delay (1) and observations (2) can be found in [42].

Remark 2. The condition $C^T(t)C(t) - \gamma^{-2}L^T(t)L(t) > 0$ assures boundedness of the filter gain matrix $P(t)$ for any finite t, and also as time goes to infinity. Apparently, if $C^T(t)C(t) - \gamma^{-2}L^T(t)L(t) < 0$, then the function $P(t)$ diverges to infinity for a finite time and the designed filter does not work. If the equality $C^T(t)C(t) - \gamma^{-2}L^T(t)L(t) = 0$ holds, then the estimation error is uniformly asymptotically stable, if the state dynamics matrix $A(t)$ itself is asymptotically stable.

Remark 3. According to the comments in Subsection V.G in [1], the obtained central H_∞ filter (11)–(14) presents a natural choice for H_∞ filter design among all admissible H_∞ filters satisfying the inequality (10) for a given threshold γ, since it does not involve any additional actuator loop (i.e., any additional external state variable) in constructing the filter gain matrix. Moreover, the obtained central H_∞ filter (11)–(14) has the suboptimality property, i.e., it minimizes the criterion

$$J = \|\tilde{z}(t)\|_2^2 - \gamma^2 \left\{ \|\omega(t)\|_2^2 + \|\varphi(\theta)\|_{2,R,[-h,0]}^2 \right\}$$

for such positive $\gamma > 0$ that the inequality $C^T(t)C(t) - \gamma^{-2}L^T(t)L(t) > 0$ holds.

Remark 4. Following the discussion in Subsection V.G in [1], note that the complementarity condition always holds for the obtained H_∞ filter (11)–(14), since the positive definiteness of the initial condition matrix R implies the positive definiteness of the filter gain matrix gain $P(t)$ as the solution of (13).

Therefore, the stability failure is the only reason why the obtained filter can stop working. Thus, the stability margin $\gamma = \sqrt{\|L^T(t)L(t)\| / \|C^T(t)C(t)\|}$ also defines the minimum possible value of γ, for which the H_∞ condition (10) can still be satisfied.

4. ALTERNATIVE FINITE-DIMENSIONAL H_∞ FILTER

Consider now another design for the central H_∞ filter for LTV systems with state delay, which is based on the alternative H_2 (optimal mean-square) filter obtained in [38]. In doing so, the system of the equations (13),(14) for determining the filter gain matrix $P(t)$, whose number grows as the filtering horizon tends to infinity, is replaced by the unique equation for $P(t)$, which includes the state transition matrix $\Psi(t,\tau)$ for the time-delay equation (1) (see [42] for the definition). The result is given by the following theorem.

Theorem 2. The alternative "central" H_∞ filter for the unobserved state (1) over the observations (2), ensuring the H_∞ noise attenuation condition (10) for the output estimate $z_f(t)$, is given by the equations (11) for the state estimate $x_f(t)$, the equation (12) for the output estimate $z_f(t)$, and the equation for the filter gain matrix $P(t)$.

$$dP(t) = A(t)(\Psi(t-h,t))P(t) + P(t)(\Psi(t-h,t))^T A^T(t) + \quad (20)$$

$$B(t)B^T(t) - P(t)[C^T(t)C(t) - \gamma^{-2}L^T(t)L(t)]P(t))dt.$$

with the initial condition $P(t_0) = R^{-1}$.

Proof. In view of Theorem 1 in [38], the alternative equation for determining the H_2 (optimal mean-square) filter gain matrix $P(t)$ in the estimate equation (11) is given by the equation (20), with the initial condition $P(t_0) = R^{-1}$, which corresponds to the central H_∞ filter (see Theorems 3 and 4 in [4]). The observation matrix $C(t)$ is changed to $C(t) - \gamma^{-1}L(t)$ according to Theorem 3 from [1].

As suggested in [38], for computational purposes, the matrix $\Psi(\tau,t)$, $\tau \le t$, can readily be calculated as a solution of the matrix equation $\Psi(\tau,t)x_1(t) = x_1(\tau)$, $\tau \le t$, for any $t,\tau \ge t_0$, where $x_1(t)$ is the solution of the homogeneous equation (1) $\dot{x}_1(t) = A(t)x_1(t-h)$, with the initial condition (4). A solution of the matrix equation for $\Psi(\tau,t)$ always exists, if $x_1(t)$ is not the zero vector. Otherwise, if $x_1(t)$ is the zero vector, the matrix $\Psi(\tau,t)$ could just be set to zero, $\Psi(\tau,t) = 0$, for any $\tau \le t$, since $x_1(\tau)$ would be equal to zero as well, regardless of the value of $\Psi(\tau,t)$. The simplest calculation method is to design $\Psi(\tau,t)$ as a diagonal matrix, $\Psi_{ij}(\tau,t) = 0$, if $i \ne j$, whose diagonal entries are defined as $\Psi_{ii}(\tau,t) = x_{1_i}(\tau)/x_{1_i}(t)$, if $x_{1_i}(t) \ne 0$, and $\Psi_{ii}(\tau,t) = 0$, otherwise, if $x_{1_i}(t) = 0$. ■

Note the designed alternative filter contains only two differential equations, the estimate equation (11) and the gain matrix equation (20), regardless of the filtering horizon. This presents a significant advantage in comparison to the preceding filter (11)-(14) consisting of a variable number of the gain matrix equations, which is specified by the ratio between the current filtering horizon and the delay value in the state equation and unboundedly grows as the filtering horizon tends to infinity. This advantage seems to be even more significant upon recalling that the state space of the time-delay system (1) is infinite-dimensional [42].

Remark 5. Since the designed alternative H_∞ filter (11),(12),(20) is based on the H_2 mean-square filter obtained in [38], which is optimal with respect to a mean square criterion, Remarks 1–4 remain true for the alternative filter also.

5. EXAMPLE

This section presents an example of designing the central H_∞ filter for a linear state with delay over linear observations and comparing it to the best H_∞ filter available for a linear state without delay, that is the filter obtained in Theorems 3 and 4 from [4].

Let the unmeasured state $x(t) = [x_1(t), x_2(t)] \in R^2$ with delay (a mechanical oscillator with a delayed force input) be given by

$$\dot{x}_1(t) = x_2(t-5), \quad (21)$$

$$\dot{x}_2(t) = -x_1(t-5) + w_1(t),$$

with an unknown initial condition $x(\theta) = \varphi(\theta)$, $\theta \in [-5,0]$, the scalar observation process satisfy the equation

$$y(t) = x_1(t) + w_2(t), \quad (22)$$

and the scalar output be represented as

$$z(t) = x_1(t). \quad (23)$$

Here, $w(t) = [w_1(t), w_2(t)]$ is an L_2^2 disturbance input. It can be readily verified that the noise orthonormality condition (see Section 2) holds for the system (21)–(23).

The filtering problem is to find the H_∞ estimate for the linear state with delay (21) over direct linear observations (22), which satisfies the noise attenuation condition (10) for a given γ, using the designed H_∞ filter (11)-(14) or the alternative H_∞ filter (11),(20). The filtering horizon is set to $T = 10$. Note that since $10 \in [1 \times 5, 2 \times 5]$, where 5 is the delay value in the state equation (21), the only first of the equations (14), along with the equations (11)–(13), should be employed.

The filtering equations (11),(13), and the first of the equations (14) take the following particular form for the system (21),(22)

$$\dot{x}_{f_1}(t) = x_{f_2}(t-5) + P_{11}(t)[y(t) - x_{f_1}(t)], \quad (24)$$

$$\dot{x}_{f_2}(t) = -x_{f_1}(t-5) + P_{12}(t)[y(t) - x_{f_1}(t)],$$

with the initial condition $x_f(\theta) = 0$, $\theta \in [-5,0]$;

$$\dot{P}_{11}(t) = 2P_{1_{12}}(t) - (1-\gamma^{-2})P_{11}^2(t), \quad (25)$$

$$\dot{P}_{12}(t) = -P_{1_{11}}(t) + P_{1_{22}}(t) - (1-\gamma^{-2})P_{11}(t)P_{12}(t),$$

$$\dot{P}_{22}(t) = 1 - 2P_{1_{21}}(t) - (1-\gamma^{-2})P_{12}^2(t),$$

with the initial condition $P(0) = R^{-1}$; and

$$\dot{P}_{1_{11}}(t) = P_{12}(t-5) + P_{2_{12}}(t) - (1-\gamma^{-2})P_{11}(t)P_{11}(t-5), \quad (26)$$

$$\dot{P}_{1_{12}}(t) = P_{22}(t-5) - P_{2_{11}}(t) - \frac{1}{2}(1-\gamma^{-2})\times$$
$$[P_{11}(t)P_{12}(t-5) + P_{12}(t)P_{11}(t-5)],$$
$$\dot{P}_{1_{21}}(t) = -P_{11}(t-5) + P_{2_{22}}(t) - \frac{1}{2}(1-\gamma^{-2})\times$$
$$[P_{11}(t)P_{12}(t-5) + P_{12}(t)P_{11}(t-5)],$$
$$\dot{P}_{1_{22}}(t) = 1 - P_{12}(t-5) - P_{2_{21}}(t) -$$

$$(1-\gamma^{-2})P_{12}(t)P_{12}(t-5),$$

with the initial condition $P_1(\theta)=0,\ \theta\in[0,5]$; finally, $P_2(\theta)=0,\ \theta\in[5,10)$.

The estimates obtained upon solving the equations (24)–(26) are compared to the conventional H_∞ filter estimates, obtained in Theorems 3 and 4 from [4], which satisfy the following equations, where the gain matrix equation is a Riccati one and the equations for matrices $P_i(t), i\geq 1$, are not employed:

$$\dot m_{f_1}(t)=m_{f_2}(t-5)+P_{11}(t)[y(t)-m_{f_1}(t)], \tag{27}$$

$$\dot m_{f_2}(t)=-m_{f_1}(t-5)+P_{12}(t)[y(t)-m_{f_1}(t)],$$

with the initial condition $m_f(\theta)=0,\ \theta\in[-5,0]$;

$$\dot P_{11}(t)=2P_{12}(t)-(1-\gamma^{-2})P_{11}^2(t), \tag{28}$$

$$\dot P_{12}(t)=-P_{11}(t)+P_{22}(t)-(1-\gamma^{-2})P_{11}(t)P_{12}(t),$$
$$\dot P_{22}(t)=1-2P_{12}(t)-(1-\gamma^{-2})P_{12}^2(t),$$

with the initial condition $P(0)=R^{-1}$.

Finally, the previously obtained estimates are compared to the alternative H_∞ filter estimates satisfying the equations (11),(20). The equation (11) for the estimate $x_f(t)$ remains the same as (24), and the gain matrix equation (20) takes the following particular form for the system (21),(22)

$$\dot P_{11}(t)=2\Psi_{22}(t-5,t)P_{12}(t)-(1-\gamma^{-2})P_{11}^2(t), \tag{29}$$

$$\dot P_{12}(t)=-\Psi_{11}(t-5,t)P_{11}(t)+\Psi_{22}(t-5,t)P_{22}(t)-$$
$$(1-\gamma^{-2})P_{11}(t)P_{12}(t),$$
$$\dot P_{22}(t)=1-2\Psi_{11}(t-5,t)P_{12}(t)-(1-\gamma^{-2})P_{12}^2(t),$$

with the initial condition $P(0)=R^{-1}$, where it is taken into account that the state transition matrix $\Psi(\tau,t)$ for the linear time-delay state (21) is calculated as a diagonal matrix according to the algorithm suggested in Section 4.

Numerical simulation results are obtained solving the systems of filtering equations (24)–(26), (27)–(28), and (24),(29). The obtained estimate values are compared to the real values of the state vector $x(t)$ in (21). For each of the three filters (24)–(26), (27)–(28), and (24),(29) and the reference system (21) involved in simulation, the following initial values are assigned: $\varphi_1(\theta)=1$, $\varphi_2(\theta)=1$, $\theta\in[-5,0]$; $R=I_2=diag[1\ 1]$. The L_2 disturbance $w(t)=[w_1(t),w_2(t)]$ is realized as $w_1(t)=1/(1+t)^2$, $w_2(t)=2/(2+t)^2$. Since $C(t)=L(t)=[1\ 0]$ in (22),(23) and the minimum achievable value of the threshold γ is equal to $\|L\|/\|C\|=1$, the value $\gamma=1.1$ is assigned for the simulations.

The following graphs are obtained: graphs of the output H_∞ estimation error $z(t)-z_f(t)$ corresponding to the estimate $x_f(t)$ satisfying the equations (24)–(26) (Fig. 1); graphs of the output H_∞ estimation error $z(t)-z_f(t)$ corresponding to the conventional estimate $m_f(t)$ satisfying the equations (27)–(28) (Fig. 2); graphs of the output H_∞ estimation error $z(t)-z_f(t)$ corresponding to the alternative estimate $x_f(t)$ satisfying the equations (24),(29) (Fig. 3). The graphs of the output estimation errors are shown in the entire simulation interval from $t_0=0$ to $T=10$. Figures 1–3 also demonstrate the dynamics of the noise-output H_∞ norms corresponding to the shown output H_∞ estimation errors in each case.

The following values of the noise-output H_∞ norm $\|T_{zw}\|^2=\|z(t)-z_f(t)\|_2^2/(\|\omega(t)\|_2^2+\|\varphi(\theta)\|_{2,R,[-h,0]}^2)$ are obtained at the final simulation time $T=10$: $\|T_{zw}\|=0.1614$ for the H_∞ estimation error $z(t)-z_f(t)$ corresponding to the estimate $x_f(t)$ satisfying the equations (24)–(26), $\|T_{zw}\|=1.46202$ for H_∞ estimation error $z(t)-z_f(t)$ corresponding to the conventional estimate $m_f(t)$ satisfying the equations (27)–(28), and $\|T_{zw}\|=0.29106$ for H_∞ estimation error $z(t)-z_f(t)$ corresponding to the alternative estimate $x_f(t)$ satisfying the equations (24),(29).

It can be concluded that the central suboptimal multi-equational H_∞ filter (24)–(26) and the central suboptimal alternative H_∞ filter (24),(29) provide reliably convergent behavior of the output estimation error, yielding very small values of the corresponding H_∞ norms, even in comparison to the assigned threshold value $\gamma=1.1$. The latter serves as an ultimate bound of the noise-output H_∞ norm as time tends to infinity. The larger value of the H_∞ norm for the alternative H_∞ filter (24),(29) appears due to MatLab discretization scheme, which poorly handles the division by numbers close to zero employed for calculating the matrix $\Psi(t-5,t)$ in (29). In contrast, the conventional central H_∞ filter (27)–(28) provides divergent behavior of the output estimation error, yielding a large value of the corresponding H_∞ norm, which exceeds the assigned threshold. Thus, the simulation results show definite advantages of the designed central suboptimal H_∞ filters for linear systems with state delay, in comparison to the previously known conventional H_∞ filter.

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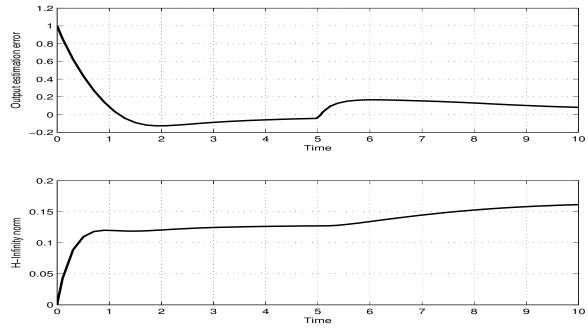

Fig. 1. **Above.** Graph of the output H_∞ estimation error $z(t) - z_f(t)$ corresponding to the estimate $x_f(t)$ satisfying the equations (24)–(26), in the simulation interval [0,10].**Below.** Graph of the noise-output H_∞ norm corresponding to the shown output H_∞ estimation error, in the simulation interval [0,10].

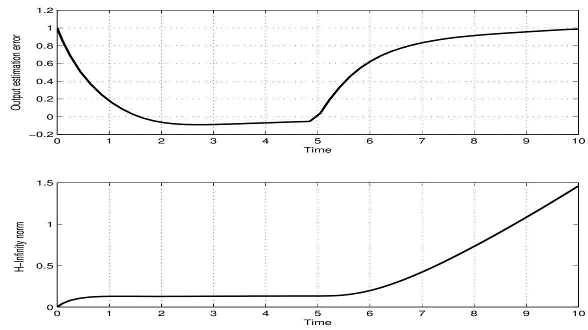

Fig. 2. **Above.** Graph of the output H_∞ estimation error $z(t) - z_f(t)$ corresponding to the estimate $x_f(t)$ satisfying the equations (27)–(28), in the simulation interval [0,10].**Below.** Graph of the noise-output H_∞ norm corresponding to the shown output H_∞ estimation error, in the simulation interval [0,10].

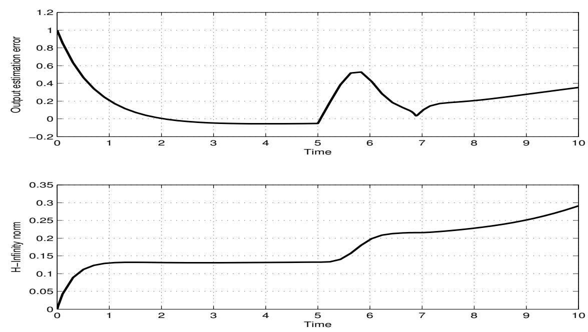

Fig. 3. **Above.** Graph of the output H_∞ estimation error $z(t) - z_f(t)$ corresponding to the estimate $x_f(t)$ satisfying the equations (24),(29), in the simulation interval [0,10].**Below.** Graph of the noise-output H_∞ norm corresponding to the shown output H_∞ estimation error, in the simulation interval [0,10].