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Sampled-Data Output Feedback Control of Distributed Parameter Systems via Semi-discretization in Space ⋆

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Abstract: This paper provides sufficient conditions to stabilize a sampled-data linear distributed parameter system with finite-dimensional input and output via a family of finite-dimensional approximations that are obtained from numerical schemes. This family of finite-dimensional approximations can be exponentially stabilized by a family of output feedback controllers when the space discretization parameter h is sufficiently small. The sufficient conditions presented in this paper guarantee that the same family of output feedback controllers can exponentially stabilize the exact sampled-data linear distributed parameter system for a sufficiently small sampling period. Since the output feedback controller design is based on the family of finite-dimensional approximations which only require a standard finite-dimensional control theory, this result can simplify the design of controller for sampled-data infinite-dimensional systems when the sampling period is fast enough and those sufficient conditions are satisfied. Moreover, the analysis method is applicable to more general situations when the sampled-data state feedback controllers are designed based on finite-dimensional approximations.

1. INTRODUCTION

Linear distributed parameter systems (LDPS) arise in a range of different processes in optical telecommunications, fluid flows, thermal processes, biology, chemistry, environmental sciences, mechanical systems, and so on. LDPS are modelled by linear partial differential equations (PDE), as opposed to linear lumped parameter systems (LPS) that are modelled by linear ordinary differential equations (ODE).

There are two important practical issues that deserve special attention in the context of controller design for LDPS: (i) The designed controller has to be *finite dimensional* in order to be implementable in practice. Indeed, while infinite-dimensional controllers are theoretically very important and often arise naturally in theory, they have to be approximated by a finite-dimensional controller before implementation. (ii) Nowadays most control systems are implemented using digital technology since it is very cheap, fast, relatively easy to operate, flexible and reliable. This motivates investigation of the so called *sampled-data systems* that consist of a continuous-time plant or process controlled by a discrete-time controller as discussed in Chen and Francis (1995). The plant and the controller are interconnected via the analog-to-digital (A-D) and digital-to-analog (D-A) converters. Consequently, the designed controller needs to be time-discretized in order to be implemented using the prevalent digital technology.

Emulation based infinite-dimensional sampled-data control design for linear distributed parameter systems has been discussed in Rebarber and Townley (1998), Logemann *et al.* (2003) and references cited therein. Here, emulation method means that one first designs a continuous-time infinite-dimensional controller for the continuous-time infinite-dimensional system (ignoring sampling in time) and then time-discretize the obtained infinite-dimensional controller for digital implementation. As mentioned above, the controller has to be finite-dimensional to be implemented. That is, the space-discretization is thus also needed. However, theoretical emulation results with the consideration of space-discretization for linear distributed parameter systems are very scarce in literature.

In this paper, we take the space-discretization into account by using a family of finite-dimensional continuous-time approximations of the infinite-dimensional continuous-time system to design a family of finite-dimensional continuous-time controllers. First, Theorem 1 shows that if this family of finite-dimensional continuous-time controllers can uniformly stabilize the family of finite-dimensional continuous-time approximations, they can also stabilize the exact infinite-dimensional continuous-time system under appropriate conditions and when h is sufficiently small. Using emulation based infinite-dimensional sampled-data control design method as discussed in Logemann *et al.* (2003), it is not difficult to show that the family of finite-dimensional continuous-time approximations, after sampler and zero-order-hold, can stabilize the exact infinite-dimensional continuous-time system if the sampling T is sufficiently small.

To simplify the presentation, our focus is on linear distributed parameter systems with finite-dimensional input and output. When an output feedback controller is employed, sampling in time is needed for controllers due to finite-dimensional input and output. However, the analysis method is applicable to distributed parameter systems when both input and output are infinite-dimensional with the help of appropriate space-discretization methods.

The key question of sampled-data systems on the basis of emulation is whether the designed *finite-dimensional, continuous-time* controller can stabilize the original infinite-dimensional continuous-time plant with sampler and zero-order-hold. This question is exactly the same as the one addressed in Nešić *et al.* (1999) for sampled-data nonlinear lumped parameter systems. Motivated by results in sampled-data nonlinear lumped parameter systems, sufficient conditions are provided in this paper to guarantee that the controllers which can uniformly exponen-

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tially stabilize a family approximate finite-dimensional continuous-time linear models can also exponentially stabilize the exact *sampled-data infinite-dimensional* for any sufficiently small sampling period T. Analogous to results Nešić and Teel (2004) for (nonlinear) LPS, those sufficient conditions are closely related to the some kind of "closeness" of the trajectories between the approximate finite-dimensional continuous-time linear model and exact *infinite-dimensional* continuous-time model. It should be noted that compared with the definition of "closeness of solutions" in finite-dimensional spaces, the "closeness" of solutions in infinite-dimensional spaces are much weaker (see Definition 5). Moreover, since approximate models (in X_h, will define later) is finite-dimensional while the exact model (in X) is infinite-dimensional, in order to measure the "closeness" between two trajectories, the prolongation operator and the restriction operator are employed.

This paper is organized as follows. Section 2 presents preliminaries including the notation, problem setting, output feedback as well as stability properties. Closeness of solutions between exact continuous-time model and a family of continuous-time approximations is briefed in Section 3 followed by the conclusion. Main results are stated in Section 4. Proofs of main results are given in the Appendix.

2. PRELIMINARIES

2.1 Problem setting and notation

The set of real numbers is denoted as $\mathbb{R}$. X is a Hilbert space with a norm $\|\cdot\|_X$. $\mathcal{L}(X,Y)$ is denoted as the space of all linear bounded operators from X to Y where both X and Y are Hilbert spaces and $\mathcal{L}(X) := \mathcal{L}(X,X)$. I_X is the identity in X.

In this paper, we consider the following distributed parameter system characterized by an abstract differential equation in X:

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad x(0) = x^0 \in X, \tag{1}$$

$$y(t) = Cx(t), \tag{2}$$

where A generates a strongly continuous (or $\mathcal{C}_0$) semigroup $S(t)$ on X. The input $u \in \mathbb{R}^m$ and the output $y \in \mathbb{R}^p$ are both finite-dimensional. The systems (1) and (2) satisfy the following assumptions.

Assumption 1. The Semigroup $S(t)$ is analytic.

Remark 1. To simplify the presentation, $S(t)$ is assumed to be analytic. That is, we just consider parabolic systems. The sampled-data control for hyperbolic systems is much more complicated. For hyperbolic systems, filtering out high frequency components in space is necessary when space-discretization is considered.

Assumption 2. Linear operators B and C are bounded, i.e., $B \in \mathcal{L}(\mathbb{R}^m, X)$ and $C \in \mathcal{L}(X, \mathbb{R}^p)$. In other words, there exists M_1, strictly positive, such that

$$\|B\|_{\mathcal{L}(\mathbb{R}^m,X)} \le M_1, \|C\|_{\mathcal{L}(X,\mathbb{R}^p)} \le M_1 \tag{3}$$

Remark 2. Assumption 2 is also restrictive. How to convert systems governed by partial differential equations into the form in (1) and (2) satisfying Assumption 2 has been discussed in Curtain and Zwart (1995).

First of all, we assume that the dynamic system (1) is well-posed. The well-poseness is defined as follows.

Definition 1. The abstract differential equation (1) is well-posed, if the operator A generates a $\mathcal{C}_0$ semigroup $S(\cdot)$ and B is a bounded linear operator so that the solution of (1) is given by

$$x(t) = S(t)x_0 + \int_0^t S(t-s)Bu(s)ds \in X, \tag{4}$$

and (1) holds on X for any $t \ge 0$.

Remark 3. This paper focused on stabilization, instead of controllability. The system (1) is thus assumed to be well-posed. However, controllability problem is a typical ill-posed problem in the sense that existence, uniqueness and continuous dependence may fail simultaneously. How to deal with sampled-data controllability problem will be future work.

Remark 4. Since $S(t)$ is a C_0 semigroup and it is analytic, it has the following properties (see, (Curtain and Zwart, 1995, Theorem 2.16), (Lasiecka and Trigiani, 2000, Page 122))

$$\|S(t)\|_{\mathcal{L}(X)} \le Me^{\omega t}, \qquad \forall 0 \le t < \infty. \tag{5}$$

$$\left\|e^{-\hat{A}t}\right\|_{\mathcal{L}(X)} \le \hat{M}e^{-\hat{\omega}t}, \forall t \ge 0. \tag{6}$$

where $M > 1$ and $\omega > 0$ are positive constants, $\hat{A} = -A + \bar{\omega}I$, where $\bar{\omega} =$ fixed $> \omega$. $\hat{\omega} = \bar{\omega} - \omega - \epsilon > 0$, $\hat{M}$ is a positive constant and $\epsilon > 0$.

2.2 Zero-order-Hold Equivalent method

The system (1), (2) is assumed to be between a sampler (A/D converter) and zero-order-hold (D/A converter). The control signal is assumed to be piecewise constant, i.e.

$$u(t) = u(kT) := u(k), \forall t \in [kT, (k+1)T), k \in \mathbb{N}, \tag{7}$$

where $T > 0$ is a sampling period. Moreover, we assume that the output measurement $y(k)$, where

$$y(k) := y(kT) \tag{8}$$

is available at sampling instants. Using sampled-data control (7), the solution of the system (1) becomes

$$x(t) = S(t-kT)x(k) + \int_{kT}^t S(t-s)Bu(k)ds, \tag{9}$$

for all $t \in [kT, (k+1)T)$ and $k \in \mathbb{N}$.

2.3 Semi-discretization in space

The analytical solutions of (1) are hard to compute. Numerical methods are widely used. In this paper, we just consider space approximations via semi-discretization on space. We introduce adapted numerical discretization assumptions, inspired by (Lasiecka and Trigiani (2000); S. Labbé and E. Trélat (2006)) for semi-discretization on space. Consider a family of finite-dimensional approximating spaces $(X_h)_{0<h<h_0}$, where h is the space discretization parameter that tends to zero, $0 < h \le h_0$. I_h denotes the identity in X_h.

Assumption 3. For every $h \in (0, h_0)$, there exist mappings $R_h : X \to X_h$ and $P_h : X_h \to X$ such that the following conditions hold:

(a) For every $h \in (0, h_0)$, the following holds

$$R_h P_h = I_h. \tag{10}$$

(b) For any $\phi \in X$, we have

$$\|\{I_X - P_h R_h\}\phi\|_X \xrightarrow[h\to 0]{} 0. \quad (11)$$

Assumption 4. For every $h \in (0, h_0)$, there holds

$$R_h^* = P_h \quad (12)$$

where the adjoint operator is considered with respect to the pivot spaces X and X_h.

For every $h \in (0, h_0)$, we define the approximation operators $B_h^* : X_h \to \mathbb{R}^m$ and $C_h^* : \mathbb{R}^p \to X_h$ by

$$B_h^* = B^* P_h; C_h^* = R_h C^*. \quad (13)$$

After the numerical discretization, we have a family of finite-dimensional continuous-time linear systems in the following form

$$\dot{x}_h = A_h x_h(t) + B_h u(t), x_h(0) = R_h x^0, \quad (14)$$

$$y(t) = C_h x_h(t) \quad (15)$$

where $A_h \in \mathcal{L}(X_h)$ and $B_h \in \mathcal{L}(\mathbb{R}^m, X_h)$ and $C_h \in \mathcal{L}(X_h, \mathbb{R}^p)$. Due to (12) in Assumption 3, it is clear that $B_h = R_h B$ and $C_h = C P_h$.

Assumption 5. It is also assumed that

(a) The family of operators $e^{A_h t}$ is uniformly analytical in the sense that there exists C_θ such that

$$\|A_h^\theta e^{A_h t}\|_{\mathcal{L}(X_h)} \le C_\theta \frac{e^{(\omega+\epsilon)t}}{t^\theta}, \quad (16)$$

for all $t > 0$, $\theta \in [0, 1]$ with constant C_θ independent of h. ω is from (5) and $\epsilon > 0$.

(b) There exists a positive constant C_1 such that

$$\left\|\hat{A}^{-1} - P_h \hat{A}_h^{-1} R_h\right\|_{\mathcal{L}(X)} \le C_1 h^s, \quad (17)$$

for some $s > 0$, where s and C_1 are independent of h, $\hat{A}_h \triangleq -A_h + \omega I_h \in \mathcal{L}(X_h)$.

In the sequel, for any $h \in (0, h_0)$, the vector space X_h is endowed with the norm $\|\cdot\|_{X_h}$ defined by

$$\|x_h\|_{X_h} = \|P_h x_h\|_X. \quad (18)$$

Remark 5. With the endowed norm defined in (18), it is obvious that P_h is a linear operator satisfying

$$\|P_h\|_{\mathcal{L}(X_h, X)} = 1. \quad (19)$$

Remark 6. By using the Banach-Steinhaus Theorem (see Kreyszig (1989)), Condition (b) (or equations (11)) in Assumption 3 implies that R_h is a linear operator, i.e., there exists $M_R > 0$ such that

$$\|R_h\|_{\mathcal{L}(X, X_h)} \le M_R. \quad (20)$$

Remark 7. Remark 3.1 in S. Labbé and E. Trélat (2006) explained how general Assumptions 3–5 are, even though Assumption 5 of uniform analyticity is not standard, and has to be checked in each case as discussed in Lasiecka and Trigiani (2000).

2.4 Output feedback

In this paper, the output controller design is based on the approximate model (15). That is, our controller is to find a family of output feedback gain matrices $K_h \in \mathcal{L}(\mathbb{R}^p, \mathbb{R}^m)$,

$$u(t) = K_h y(t) = K_h C_h x_h(t), \quad (21)$$

such that the closed-loop of the approximate model (15) becomes

$$\dot{x}_h(t) = (A_h + B_h K_h C_h)\, x_h(t) = \Phi_h^a x_h(t), \quad (22)$$

is "stable" (we will define the stability property later), where $\Phi_h^a \triangleq A_h + B_h K_h C_h$.

Once the output feedback gain K_h is obtained from the approximate model, first we apply this feedback gain K_h to the exact model. Let $\Phi_h^e \triangleq A + BK_hC$, we have the following continuous-time exact model:

$$\dot{x}(t) = (A + BK_hC)x(t) = \Phi_h^e x(t),\ x(0) = x^0. \quad (23)$$

On the other hand, when the sampler and the zero-order-hold are considered, with $x(0) = x^0$, it leads to the following closed-loop system

$$x(t) = \left\{S(t - kT) + \int_{kT}^{t} S(s)BK_hC\right\} x(k)ds. \quad (24)$$

2.5 Stability properties

Exponential stability is most frequently used in the control of infinite-dimensional systems Curtain and Zwart (1995). Several exponential stability properties are defined here.

Definition 2. A family of finite-dimensional systems (14) are said to be exponentially stable uniformly in small h if there exists $h_u^* > 0$ such that for all $h \in (0, h_u^*)$, there exists a positive pair (K, λ) and $K > 1$, such that solutions of (14) satisfy

$$\|x_h(t)\|_{X_h} \le Ke^{-\lambda t}\|x_h(0)\|_{X_h}, \forall x_h(0) \in X_h. \quad (25)$$

Definition 3. The infinite-dimensional continuous system (23) is exponentially stable uniformly in h if there exists h_i^* such that for any $h \in (0, h_i^*)$, there exists a positive pair (K, λ), $K > 1$ and the solutions of the system (24) satisfy

$$\|x(t)\|_X \le Ke^{-\lambda t}\left\|x^0\right\|_X. \quad (26)$$

Definition 4. The sampled-data system (24) is exponentially stable uniformly in $[T, h]$ if there exists h^* and $T^* > 0$ such that for any $h \in (0, h^*)$ and $T \in (0, T^*)$, there exists a positive pair (K, λ), $K > 1$ and the solutions of the system (24) satisfy

$$\|x(t)\|_X \le Ke^{-\lambda t}\left\|x^0\right\|_X. \quad (27)$$

The main objective of this paper is to provide sufficient conditions that can ensure uniform exponential stability properties of the sampled-data exact model (24) from a family of output feedback gain matrices K_h that are designed to uniformly exponentially stabilize a family of continuous-time approximations (22).

It is well-known that the stability properties of the continuous model (23) indicate the stability properties of the sampled-data system (24) uniformly in sampling T under appropriate assumptions as indicated in (Logemann *et al.*, 2003, Theorem 3.1). More precisely, when A is a generator a C_0 semigroup and B is a bounded control operator, if $A + BK_hC$ generates an exponentially stable semigroup with the linear compact operator K_hC, then the sampled-data system (24) is exponentially stable uniformly in the sampling period T. To show the uniform exponential stability of the sampled-data system (24), the continuous-time linear system (23) is first shown to be uniformly stable in small h (Theorem 1). Then by showing the compactness of K_hC, Theorem 2 shows that the sampled-data system (24) is exponentially stable uniformly in sampling period T.

3. CLOSENESS OF SOLUTIONS BETWEEN APPROXIMATION MODEL AND EXACT MODEL

Sufficient conditions are needed in order to show that the continuous model (23) is uniformly stable in small h. Analogous to results in LPS (see Nešić *et al.* (1999)), these sufficient conditions that can guarantee the stability properties of the continuous model (23) from the stability properties of the approximate model (22) are closely related to the closeness of the trajectories between the exact model and the family of approximate models.

It is intuitively clear that when the trajectories from two systems are close enough, stability properties of one system can indicate "some" stability properties of the other system. To characterize the closeness of solutions, the following definitions is provided.

Definition 5. We say that the solutions of (1) and the solutions of the numerical method (14) can be made weakly close on compact time intervals if the following holds. For any $t > 0$ and any positive constant δ, there exists a positive constant $h_c^* > 0$, such that for any $h \in (0, h_c^*)$, $x^0 \in X$, we have [1]

$$\left\| t^{\theta} \left(x(t) - P_h x_h(t) \right) \right\|_X \leq \delta \left\| x^0 \right\|_X \tag{28}$$

for all $t > 0$ and $0 \leq \theta \leq 1$.

In the sequel, the following proposition is needed in our main result.

Proposition 1. (Lasiecka and Trigiani, 2000, Proposition 4.1.2.1) Assume that Assumptions 1, 3 and 5 hold. Then for $0 \leq \theta \leq 1$, there exists $C_2 > 0$ such that the following holds.

$$\left\| P_h e^{A_h t} R_h - S(t) \right\|_{\mathcal{L}(X)} \leq C_2 \frac{h^{s\theta} e^{(\omega+\epsilon)t}}{t^{\theta}} \tag{29}$$

for all $t > 0$ and $\epsilon > 0$, where ω is from (5) and s is from Assumption 5. C_2 is independent of the choice of h.

Remark 8. Proposition 1 comes from (Lasiecka and Trigiani, 2000, Proposition 4.1.2.1). But there exists a slight difference. In (Lasiecka and Trigiani, 2000, Proposition 4.1.2.1), only restriction operator R_h (Π_h in Lasiecka and Trigiani (2000)) is used. However, with the endowed norm $\|\cdot\|_{X_h}$, it is straightforward to extend Proposition 4.1.2.1 in Lasiecka and Trigiani (2000) to Proposition 1. Therefore the proof is omitted.

Remark 9. Proposition 1 indicates the "weak closeness" of solutions between (23) and approximations (22) when $K_h = 0_{p\times m}$. Indeed, given any positive pair (δ, t) and $t > 0$, for some $s > 0$, we can find that $h^* > 0$ sufficiently small such that $C_2 h^{s\theta} e^{(\omega+\epsilon)t} < \delta$ so that

$$\begin{aligned}\left\| t^{\theta} \left(x(t) - P_h x_h(t) \right) \right\|_X &= \left\| t^{\theta} \left(P_h e^{A_h t} R_h x^0 - S(t) x^0 \right) \right\|_X \\ &\leq \delta \left\| x^0 \right\|_X ,\end{aligned}$$

for all $t > 0$ and $0 \leq \theta \leq 1$. However, when $K_h \neq 0_{p\times m}$, we need to show that Φ_h^a in (22) satisfies Assumptions 5 so that (29) holds true when A is replaced by Φ_h^e and A_h is replaced by Φ_h^a.

Proposition 2. Assume that the following conditions hold

(a) Assumptions 1– 5 hold.
(b) The family of output feedback operators K_h is uniformly bounded in small h. That is, there exists $h_k > 0$ such that for all $h \in (0, h_k)$, we have $\|K_h\|_{\mathcal{L}(\mathbb{R}^p,\mathbb{R}^m)} \leq M_K$, where M_K is a positive constant.

Then the following holds

$$\left\| \left(\hat{\Phi}_h^e \right)^{-1} - P_h \left(\hat{\Phi}_h^a \right)^{-1} R_h \right\|_{\mathcal{L}(X)} \leq C h^s \tag{30}$$

for some $s > 0$ independent of h.

Proof: see Appendix. ∘

Proposition 3. Assume that all conditions in Proposition 2 hold true. If a family of closed-loop approximations (22) are exponentially stable uniformly in small h, then solutions of (23) and the solutions of the numerical approximation (22) can be made weakly close on compact time intervals.

Sketch of Proof: As a family of closed-loop approximations (22) are exponentially stable uniformly in small h, condition (a) in Assumption 5 holds. Proposition 2 shows that condition (b) in Assumption 5 holds true. The result holds true by applying Proposition 1. ∘

4. MAIN RESULTS

The main results of this paper are to show under which conditions that the sampled-data system (24) is stable if K_h is designed to uniformly stabilize a family approximate models (22) in small h. In this paper, it is first shown that the infinite-dimensional continuous system (23) is exponentially stable uniformly in small h provided that the family approximate models (22) is uniformly exponentially stable in small h and the appropriate " weak closeness" of solutions.

Theorem 1. Assume

(a) Solutions of (23) and the solutions of the numerical method (22) can be made weakly close on compact time intervals.
(b) The family of closed-loop approximation systems (22) are exponentially stable uniformly in small h

Then the system (23) is exponentially stable uniformly in small h.

Proof: see Appendix. ∘

With stability properties of the continuous model (23), we can conclude the stability of the exact sampled-data system (24) if K_hC can be shown to be bounded and compact. The second result of this paper is stated as follows.

Theorem 2. Assume that the following conditions hold

(a) Assumptions 1–5 hold true.
(b) The family of output feedback operators K_h is uniformly bounded in small h.
(c) The family of closed-loop approximation systems (22) are exponentially stable uniformly in small h

Then the exact sampled-data system (24) is exponentially stable uniformly in small $[T, h]$.

[1] We assume that the initial condition x^0 for the solutions of (1) and while the initial condition of (14) is $x_h(0) = R_h x^0$.

Sketch of Proof: Proposition 3 shows that solutions of (23) and the solutions of the numerical method (22) can be made weakly close on compact time intervals. Using Proposition 3 and Condition (c), it can be concluded that the system (23) is exponentially stable uniformly in small h. Condition (b) implies that K_hC is compact ((Kreyszig, 1989, Theorem 8.1-3)). Using (Logemann *et al.*, 2003, Theorem 3.1), the exact sampled-data system (24) is exponentially stable uniformly in small $[T, h]$ ◦

5. CONCLUSION

In this paper, sampled-data output feedback control design of distributed parameter systems is based on their finite-dimensional continuous-time approximate models. Sufficient conditions have been shown to guarantee that the controller that can uniformly stabilize a family finite-dimensional continuous-time approximations can also stabilize the exact sampled-data infinite-dimensional system when sampling period is sufficient small. Those sufficient conditions are closely related to the "closeness" of trajectories between the exact model and approximate model. Since the controller is designed on the basis of the finite-dimensional continuous-time approximations, the proposed methods simplify the design of controllers for sampled-data distributed parameter systems.

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APPENDIX

5.1 Proof of Proposition 2

Using perturbation theory (Pazy, 1983, Theorem 1.1), Φ_e^h generates analytic semigroup. There exist positive constants ω_c and λ_c, such that the solutions of (23) satisfy

$$\|x(t)\|_X \leq M_c e^{\omega_c t} \left\|x^0\right\|_X . \tag{31}$$

Moreover, we have

$$\left(\hat{\Phi}_h^e\right)^{-1} = \left(\bar{\omega} I - \Phi_h^e\right)^{-1} (\bar{\omega} I - (A + BK_hC))^{-1} = R(\bar{\omega}; \Phi_h^e)$$

where $\bar{\omega}$ = fixed $> \omega + MM_1^2 M_K$, where M_1 is from Assumption 2 and ω is from (5). Similarly, we have

$$\begin{aligned}
&\left\|\left(\hat{\Phi}_h^e\right)^{-1} - P_h \left(\hat{\Phi}_h^a\right)^{-1} R_h\right\|_{\mathcal{L}(X)} \\
&= \left\|R(\bar{\omega}; \Phi_h^e) - P_h R(\bar{\omega}; \Phi_h^a) R_h\right\|_{\mathcal{L}(X)} \\
&= \left\|[I_X - R(\bar{\omega}; A) BK_hC]^{-1} R(\bar{\omega}; A) \right. \\
&\quad \left. - P_h \left[I_h - R(\bar{\omega}; A_h) B_h K_h C_h\right]^{-1} R(\bar{\omega}; A_h) R_h\right\|_{\mathcal{L}(X)}
\end{aligned} \tag{32}$$

By adding and subtracting the following term

$$[I_X - R(\bar{\omega}; A) BK_hC]^{-1} P_h R(\bar{\omega}; A_h) R_h,$$

it yields

$$\left\|\left(\hat{\Phi}_h^e\right)^{-1} - P_h \left(\hat{\Phi}_h^a\right)^{-1} R_h\right\|_{\mathcal{L}(X)} = \|(1) + (2)\|_{\mathcal{L}(X)}, \tag{33}$$

where

$$\begin{aligned}
(1) &= (I_X - T_1)^{-1} [R(\bar{\omega}; A) - P_h R(\bar{\omega}; A_h) R_h]; \\
(2) &= \left\{I_X - T_1)^{-1} - P_h \left[I_h - R(\bar{\omega}; A_h) R_h BK_hCP_h\right]^{-1}\right\} T_3 \\
&= (2a) + (2b)
\end{aligned}$$

where $T_1 = R(\bar{\omega}; A) BK_hC$, $T_2 = P_h R(\bar{\omega}; A_h) R_h BK_hC$, $T_3 = P_h R(\bar{\omega}; A_h) R_h$ and

$$\begin{aligned}
(2a) &= \left[(I_X - T_1)^{-1} - (I_X - T_2)^{-1}\right] T_3 \\
(2b) &= \left\{I_X - T_2)^{-1} - P_h \left[I_h - R(\bar{\omega}; A_h) R_h BK_hCP_h\right]^{-1} R_h\right\} T_3
\end{aligned}$$

Using the identity

$$(I - T_1)^{-1} - (I - T_2)^{-1} = (I_X - T_1)^{-1}(T_1 - T_2)(I_X - T_2)^{-1},$$

it follows that

$$(2a) = (I_X - T_1)^{-1}(R(\bar{\omega}; A) - P_h R(\bar{\omega}; A_h) R_h)(I_X - T_2)^{-1}.$$

Noting the following facts,

$$P_h\left[I_h - R(\bar{\omega}; A_h)R_hBK_hCP_h\right]^{-1} R_hT_3$$
$$= T_3 + P_h\left[\sum_{k=1}^{\infty}\left(R(\bar{\omega}; A_h)R_hBK_hCP_h\right)^k\right] R_hT_3,$$
$$\left[I_X - P_hR(\bar{\omega}; A_h)R_hBK_hC\right]^{-1} T_3$$
$$= T_3 + \left[\sum_{k=1}^{\infty}\left(P_hR(\bar{\omega}; A_h)R_hBK_hC\right)^k\right] T_3.$$

By using Induction, it is not difficult to show that

$$P_h\left[\sum_{k=1}^{\infty}\left(R(\bar{\omega}; A_h)R_hBK_hCP_h\right)^k\right] R_hT_3$$
$$= \left[\sum_{k=1}^{\infty}\left(P_hR(\bar{\omega}; A_h)R_hBK_hC\right)^k\right] T_3, \tag{34}$$

which implies that $(2b) \equiv 0$. Denoting $T_4 = R(\bar{\omega}; A) - P_hR(\bar{\omega}; A_h)R_h$, consequently,

$$\left\|\left(\hat{\Phi}_h^e\right)^{-1} - P_h\left(\hat{\Phi}_h^a\right)^{-1} R_h\right\|_{\mathcal{L}(X)}$$
$$\le \left\|(I_X - T_1)^{-1}\right\|_{\mathcal{L}(X)} \|T_4\|_{\mathcal{L}(X)}$$
$$+ \left\|(I_X - T_1)^{-1}\right\|_{\mathcal{L}(X)} \|T_4\|_{\mathcal{L}(X)} \left\|(I_X - T_2)^{-1}\right\|_{\mathcal{L}(X)}. \tag{35}$$

Noting the boundedness of $(I_X - T_1)^{-1}$ and $(I_X - T_2)^{-1}$, using (b) in Assumption 5 yields the result. ■

5.2 Proof of Theorem 1

Let arbitrary $\delta \in (0, 1)$, $t > 1$ be given. Let h_u^*, K and λ come from condition (b) in Theorem 1. Moreover, the solutions of (23) and (22) can be made weakly close on compact time intervals $[0, t]$. Given $\frac{\delta}{2}$, we can find h_c^*, such that for any $h \in (0, \min\{h_u^*, h_c^*\})$, such that for all we have the weak closeness of solutions (see Definition 5), that is,

$$\|t\,(x(t) - P_hx_h(t))\|_X \le \frac{\delta}{2}\left\|x^0\right\|_X. \tag{36}$$

Since (22) is exponentially stable uniformly in small h, there exist positive constants K and λ such that

$$\|x_h(t)\|_{X_h} \le Ke^{-\lambda t}\left\|R_hx^0\right\|_{X_h} = Ke^{-\lambda t}\left\|P_hR_hx^0\right\|_X$$
$$\le K_1e^{-\lambda t}\left\|x^0\right\|_X, \tag{37}$$

where $K_1 = M_RK$ and M_R is from (20). In the sequel, we have

$$\left\|t^\theta x(t)\right\|_X \le \left\|t^\theta P_hx_h(t)\right\|_X + \left\|t^\theta\left(x(t) - P_hx_h(t)\right)\right\|_X$$
$$\le t^\theta\|x_h(t)\|_{X_h} + \frac{\delta}{2}\left\|x^0\right\|_X$$
$$\le t^\theta K_1e^{-\lambda t}\left\|x^0\right\|_X + \frac{\delta}{2}\left\|x^0\right\|_X. \tag{38}$$

Let $t > 1$ be such that for all $s \ge t$, we have $K_1\left(se^{-\lambda s}\right) \le \frac{\delta}{2}$[2].

[2] Such a t exists since for any $t \ge \frac{1}{\lambda}$, $te^{-\lambda t}$ monotonically decreases.

We introducing $k_i = i \cdot t$ and $k_0 = 0$. The proof consists of the following steps:

Step 1: We will show that for all $i = 1, 2, \cdots$, the following holds

$$\left\|t^\theta x(k_i)\right\|_X \le \delta^i\left\|x^0\right\|_X. \tag{39}$$

We prove it by Induction:

(1) Note that $K_1\left(te^{-\lambda t}\right) \le \frac{\delta}{2}$, $t > 1$ and $\theta \in [0, 1]$, using (38), it follows that

$$\left\|t^\theta x(t)\right\|_X \le tK_1\left(e^{-\lambda t}\right)\left\|x^0\right\|_X + \frac{\delta}{2}\left\|x^0\right\|_X$$
$$\le \delta\left\|x^0\right\|_X. \tag{40}$$

(2) Assume that when $i = n$, inequality (39) holds true. We can re-initialize the two systems at $x(k_i)$[3]. Using re-initialization and weak closeness of the solutions, noting that $K_1\left(te^{-\lambda t}\right) \le \frac{\delta}{2}$, when $i = n + 1$, it yields that

$$\left\|t^\theta x\left(k_{n+1}\right)\right\|_X$$
$$\le \left\|t^\theta P_hx_h(k_{n+1})\right\|_X$$
$$+ \left\|t^\theta\left[x\left(k_{n+1}\right) - P_hx_h\left(k_{n+1}\right)\right]\right\|_X$$
$$\le t\left(Ke^{-\lambda t}\|P_hR_hx(k_n)\|_X\right) + \frac{\delta}{2}\|x(k_n)\|_X$$
$$\le tK_1e^{-\lambda t}\|x(k_n)\|_X + \frac{\delta}{2}\|x(k_n)\|_X$$
$$\le \delta\|x(k_n)\|_X \le \delta\|tx(k_n)\|_X \le \delta^{n+1}\left\|x^0\right\|_X, \tag{41}$$

which shows (39) holds true. ○

Step 2: Since $\delta < 1$, we can bound (39) as

$$\left\|t^\theta x(k_i)\right\|_X \le e^{-\bar{\lambda}k_i}\left\|x^0\right\|_X \tag{42}$$

where $\bar{\lambda} := \frac{\ln\left(\frac{1}{\delta}\right)}{t}$. Noting that $t > 1$, it yields

$$\|x(k_i)\|_X < \left\|t^\theta x(k_i)\right\|_X \le e^{-\bar{\lambda}k_i}\left\|x^0\right\|_X. \tag{43}$$

Step 3: We now consider inter-sampling behavior. Since $A + BK_hC$ is analytic, for all $k \in [k_i, k_{i+1}]$, $i = 1, 2, \cdots$, we have

$$\|x(k)\| \le \left(M_ce^{\omega_c(k-k_i)}\right)\|x(k_i)\|_X$$

where M_c and ω_c are from (31). It follows that

$$\|x(k)\|_X \le \left(M_ce^{\omega_c(k-k_i)}\right)\|x(k_i)\|_X$$
$$\le \left(M_ce^{\omega_ct}\right)\left(e^{-\bar{\lambda}k_i}\left\|x^0\right\|_X\right)$$
$$= \left(M_ce^{\omega_ct}\right)e^{-\bar{\lambda}k}e^{\bar{\lambda}(k-k_i)}\left\|x^0\right\|_X$$
$$\le \left[\left(M_ce^{\omega_ct}\right)e^{\bar{\lambda}t}\right]e^{-\bar{\lambda}k}\left\|x^0\right\|_X$$
$$\le \bar{K}e^{-\bar{\lambda}k}\left\|x^0\right\|_X, \tag{44}$$

where $\bar{K} \triangleq \left[\left(M_ce^{\omega_ct}\right)e^{\bar{\lambda}t}\right]$, in the sequel, the proof is completed. ■

[3] The re-initialization implies that $x_h(k_i) = R_hx(k_i)$.