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Robust Stackelberg Equilibrium for a multi-scenario two players linear affine-quadratic differential game

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Abstract: This paper presents the formulation of a new concept dealing with a Robust Stackelberg equilibrium for a multi scenario or mutiple models of a linear affine-quadratic game. The game dynamic is given by a family of N different possible differential equations (Multi-Model representation) with no information about the trajectory which is realized. The robust Stackelberg strategy for each player must confront with all possible models simultaneously. The problem consists in the designing of min-max strategies for each player which guarantee an equilibrium for the worst case scenario. Based on the Robust Maximum Principle, the general conditions for a game to be in Robust Stackelberg Equilibrium are also presented. A numerical procedure for resolving the case of LQ differential game is designed.

1. INTRODUCTION

Beginning from the seminal works of Chen and Cruz [1972], Simman and Cruz [1973a] and Simman and Cruz [1973b] the Stackelberg solution for an open loop information structure in a two person differential games has been well established (see also Basar and Olsder, [1982], B. Tolwinski [1983]). This concept of a solution for a game, introduced in 1934 by the economist H. Von Stackelberg H. Von Stackelberg [1952], is suitable when one of the players is forced to subordinate to an authority that must announces his decision first, before play his own strategy, or when one of the players chooses to play the game passively, that is, composes just his optimal reaction (solve an one-player optimization problem) given that the other player has announced his strategy. As it is shown in the above publications this kind of problems are still tractable using results of optimal control theory. All of them tackle this problem when the model of the considered dynamics is exactly known.

In the case of one player optimization problems the robust version of the traditional Maximum Principle referred to as *Robust Maximum Principle* (RMP) (see Boltyanski and Poznyak [1999] and Poznyak *et al* [2002]) allows to design an "optimal policy" of a min-max type for a multi-plant or multi-scenario problem, where each possible scenario is seen as possible parametric realization of the dynamic equation. The one player version of the RMP is based on the concept of min-max control problem where the operation of the maximization is taken over a set of uncertaintyand the operation of the minimization is taken over set of strategies. For a game with multi-participants, choosing Nash strategies, some similar concept of Multi-scenarios has been used to exemplify the discrepancies of the players in information sets, models, cost functions or even different amount of information that the players could hold of a large scale system, Khalil [1980], Khalil and Kokotovic [1978], V. Saksena *et al* [1983], and recently, applying the Robust Maximum Principle there was derived a type of Robust Nash equilibrium Jimenez-Lizarraga and Poznyak [2007] for a multi-scenario game parametrized by a parameter belonging to a given *finite* parametric set and the problem was formulated as a *min-max* problem of the game.

The purpose of this paper is to develop the RMP for a family of two person Multi-Plant game (none treated before) when one player is considered as the leader and the second one as the follower. We formulate the problem for both leader and follower as min-max problem, that to the best of our knowledge has never been considered before, but presents a great interest because of its high spread applications. The focus is, as in Jimenez-Lizarraga and Poznyak [2007], the designing of strategies that should provide a *"Robust Stackelberg equilibrium"* being applied to all scenarios (or models) of the game simultaneously. It is shown that the resulting robust strategies for each player again appears as a mixture (with the weights fulfilling the Stackelberg condition) of the controls which are the Stackelberg strategies for each fixed parameter value. This technique permits to transform the formulation of the game with a leader from a Banach space (where it was initially formulated) into a finite dimensional space where the strategies to be found are the preference weights of each scenario in the weighted sum of the individual optimal controls. The problem of finding such equilibrium weights is solve by the implementation of a special numerical procedure which extends the method given in Jimenez-Lizarraga and Poznyak [2007]. A numerical example illustrate the effectiveness of this approach.

2. STANDARD STACKELBERG EQUILIBRIUM

We begin with a briefly review of the concept for Stackelberg equilibrium. The basic idea of a leader-follower strategy for a static two-person game seems to be very simple. Consider two players Player 1 and Player. The cost function associated with the players are

$$\begin{array}{ll} J^1\left(u^1,u^2\right) & \text{for Player 1} \\ J^2\left(u^1,u^2\right) & \text{for Player 2} \end{array}$$

Both players want to minimize their criteria that naturally provokes a conflict situation. Defining the sets U^1 and U^2 for the admissible strategies for the player 1 and player 2. The resolution of this problem is given by the following equilibrium concept. Choose the player 1 as the ***leader*** and player 2 as the ***follower***. The set of strategies are said to be in a *Stackelberg equilibria with Player 1 as the leader and Player 2 as the follower* if:

$$J^1\left(u^{1*},u^{2*}\left(u^{1*}\right)\right) \le J^1\left(u^1,u^{2^o}\left(u^1\right)\right) \tag{1}$$

and

$$J^2\left(u^1,u^{2^o}\left(u^1\right)\right) = \min_{u^2\in U^2} J^2\left(u^1,u^2\right) \tag{2}$$

and $u^{2^o} = u^{2^o}\left(u^1\right)$ is the optimal policy of the player 2 for a given strategy of the leader, and as is usual $u^{2*}\left(u^{1*}\right) = u^{2^o}\left(u^{1*}\right)$. This means that being player 1 as the leader, he must advance his strategy to play first and because the player 2 want to minimize his functional J^2 then u^{2^o} is the "optimal reaction" (OR) of player 2 for the minimization of J^2 given u^1. If Player 1 chooses any other strategy u^1, then Player 2 will choose a strategy u^{2^o} that minimizes J^2, but the resulting cost for Player 1 may be greater than or equal to that when the Stackelberg strategy with Player 1 as the leader is used.

3. MULTI-MODEL TWO PLAYERS GAME

Consider the following two players multi-model differential game:

$$\dot{x}^\alpha = f^\alpha\left(x^\alpha,u^1,u^2,t\right) \tag{3}$$

where $x^\alpha \in \Re^n$ is the state vector of the game at time $t\in[t_0,T]$, $u^j \in \Re^{m_j}$ $(j=1,2)$ are the control strategies of each player at time t and α is the entire index from a finite set $\mathcal{A} := \{1,2,..,M\}$ describing each possible α-model of the dynamics game (3), M is the number of possible model.

Let the *individual aim performance* $h^{i,\alpha}$ of each player $(i=1,2)$ for each α-model (scenario) be given by

$$h^{i,\alpha} := h_0^i\left(x^\alpha(T)\right) + \int_{t=t_0}^{T} g_i^\alpha\left(x^\alpha,u^1,u^2,t\right)dt \quad i=1,2 \tag{4}$$

The *worst-case* (with respect to a possible scenario) cost functional F^i for each player under fixed admissible strategies $u^1\in U^1$ and $u^2\in U^2$ is defined by

$$F^i\left(u^1,u^2\right) := \max_{\alpha\in\mathcal{A}} h^{i,\alpha}\left(u^1,u^2\right) \tag{5}$$

3.1 Robust Stackelberg Equilibrium

The set of strategies are said to be in a *Robust Stackelberg equilibria with Player 1 as the leader and Player 2 as the follower* if: for any admissible strategy $(u^1,u^2)\in U^1\times U^2$ the next inequalities hold

$$F^1\left(u^{1*},u^{2*}\right) \le F^1\left(u^1,u^{2^o}\left(u^1\right)\right),\ u^{2*} := u^{2^o}\left(u^{1*}\right) \tag{6}$$

and

$$F^2\left(u^1,u^{2^o}\left(u^1\right)\right) = \min_{u^2\in U^2} F^2\left(u^1,u^2\right) \tag{7}$$

Note here that the main difference with the Standard Stackelberg equilibrium is the max operation taken over all possible scenarios.

3.2 Open-Loop Robust Stackelberg Strategies for Multi-Model Differential Games

Proceeding first outlining the robust optimal reaction of the follower, let us represent the robust optimal control problem for the follower following the standard procedure in Optimal Control Theory. For each possible scenario $\alpha\in A$ introduce the extended variables for the follower $\bar{x}^{\alpha,f} = \left(x_1^\alpha,...,x_n^\alpha,x_{n+1}^{\alpha,f}\right)$ defined in $\Re^{n+1}$ and the last component $x_{n+1}^{\alpha,f}$ given by

$$x_{n+1}^{\alpha,f} = \int_{t_0=0}^{t} g_2^\alpha\left(x^\alpha,u^1,u^2,\tau\right)d\tau$$

or, in the differential form,

$$\dot{x}_{n+1}^{\alpha,f}(t) = g_2^\alpha\left(x^\alpha,u^1,u^2,t\right),\ x_{n+1}^{\alpha,f}(t_0) = 0 \tag{8}$$

Now the initial individual aim performance for the follower (4) can be represented in the *Mayer form* (without an integral term):

$$h^{2,\alpha} = h_0^{2,\alpha}\left(x^\alpha(T)\right) + x_{n+1}^{\alpha,f}(T) \tag{9}$$

Notice that $h_0^2\left(x^\alpha\right)$ does not depend on the last coordenate $x_{n+1}^{\alpha,f}$, that is, $\dfrac{\partial}{\partial x_{n+1}^{\alpha,f}} h_0^{2,\alpha}\left(x^\alpha\right) = 0$. Define also the new extended conjugate vector-variable by

$$\begin{array}{l} \psi^\alpha := \left(\psi_1^\alpha,...,\psi_n^\alpha\right) \in \Re^n \\ \bar{\psi}^\alpha := \left(\psi^\alpha,...,\psi_n^\alpha,\psi_{n+1}^\alpha\right) \in \Re^{n+1} \end{array} \tag{10}$$

satisfying

$$\dot{\psi}_j^\alpha = -\sum_{k=1}^{n} \frac{\partial f_k^\alpha\left(x^\alpha,u^1,u^2,t\right)}{\partial x_j^\alpha}\psi_k^\alpha - \frac{\partial g_2^\alpha\left(x^\alpha,u^1,u^2,t\right)}{\partial x_{n+1}^\alpha}\psi_{n+1}^\alpha \quad j=1,...,n+1 \tag{11}$$

with some terminal condition

$$\psi_j^\alpha(T) = b_j^\alpha \quad 0\le t\le T,\ \ \alpha\in\mathcal{A}\ \ (j=1,..,n+1) \tag{12}$$

where the vector b^α will be defined below. For the "*super-extended*" vectors for the follower defined by

$$\begin{array}{l} \bar{x}^{\diamond,f} := \left(x_1^1,...,x_{n+1}^{1,f};...;x_1^M,...,x_{n+1}^{M,f}\right)^\intercal \\ \bar{\psi}^{\diamond} := \left(\psi_1^1,...,\psi_{n+1}^1;...;\psi_1^M,...,\psi_{n+1}^M\right)^\intercal \\ \bar{f}^{\diamond,f} := \left(\bar{f}^{1\intercal},..\bar{f}^{M\intercal}\right)^\intercal = \left(f_1^1,..,f_n^1,g_2^1,..,f_n^M,..,g_2^M\right)^\intercal \\ \bar{f}^{\alpha,f} = \left(f_1^\alpha,...,f_n^\alpha,g_2^\alpha\right)^\intercal \in \Re^{n+1} \end{array}$$

these vectors represent the complete family of trajectories including the $n+1$ state variables for the follower. Using the previous vectors, following Boltyanski and Poznyak [1999], define the next *"generalized"* Hamiltonian function for the follower:

$$\mathcal{H}_2^{\diamondsuit}\left(\bar{\psi}^{\diamondsuit},\ \bar{x}^{\diamondsuit,f},u^1,u^2,t\right):=\left\langle\bar{\psi}^{\diamondsuit},\bar{f}^{\diamondsuit,f}\left(\bar{x}^{\diamondsuit,f},u^1,u^2,t\right)\right\rangle=$$
$$\sum_{\alpha\in A}\mathcal{H}_2^{\alpha}\left(\bar{\psi}^{\alpha},\bar{x}^{\alpha,f},u^1,u^2,t\right)=$$
$$\sum_{\alpha\in A}\left[\sum_{j=1}^{n}\left(\psi_j^{\alpha}f_j^{\alpha}\left(x^{\alpha},u^1,u^2,t\right)\right)+\psi_{n+1}^{\alpha}g_2^{\alpha}\left(x^{\alpha,f},u^1,u^2,t\right)\right] \tag{13}$$

the direct (3), (8) and conjugate (11) ODE equations may be represented shortly in the standard Hamiltonian form as

$$\begin{aligned}\frac{d}{dt}\bar{x}^{\diamondsuit,f}&=\frac{\partial\mathcal{H}_2^{\diamondsuit}\left(\bar{\psi}^{\diamondsuit},\ \bar{x}^{\diamondsuit,f},u^1,u^2,t\right)}{\partial\bar{\psi}^{\diamondsuit}}\\ \frac{d}{dt}\bar{\psi}^{\diamondsuit}&=-\frac{\partial\mathcal{H}_2^{\diamondsuit}\left(\bar{\psi}^{\diamondsuit},\ \bar{x}^{\diamondsuit,f},u^1,u^2,t\right)}{\partial\bar{x}^{\diamondsuit,f}}\end{aligned} \tag{14}$$

Again, as it follows from the definition (7), and as it is shown in Chen and Cruz [1972], Simman and Cruz [1973b], the optimal reaction u^{2^o} of the follower to a *fixed* leader strategy u^1 of the leader, is solved as a standard optimal control problem, for the presented Multi-Model problem the solution is given as in Poznyak *et al* [2002], the follower should solve the following *robust optimal control problem*:

$$\begin{aligned}&\max_{\alpha\in A}h^{2,\alpha}(u^{1*},u^2\left(u^1\right))\rightarrow\min_{u^2\in U_{adm}^2}\\ &U_{adm}^i:=\left\{u^i\mid(u^{1*},u^{2^o}\left(u^{1*}\right))\in U_{adm}^i\right\}\end{aligned} \tag{15}$$

that provides the robust Stackelberg strategy for the follower. Following the result of Poznyak *et al* [2002] the necessary condition for a robust optimality of a strategy for the follower must fulfill the next condition:

- (**the maximality condition**) The control strategies for the follower $u^{2^o}(t)\in U_{adm}^2$ $(t\in[0,T])$ satisfies
$$\mathcal{H}_2^{\diamondsuit}\left(\bar{\psi}^{\diamondsuit},\ \bar{x}^{\diamondsuit,f},u^1,u^{2^o},t\right)\geq\mathcal{H}_2^{\diamondsuit}\left(\bar{\psi}^{\diamondsuit},\ \bar{x}^{\diamondsuit,f},u^1,u^2,t\right)$$
- or, equivalently,
$$u^{2^o}\in\text{Arg}\max_{u^2\in U_{adm}^2}\mathcal{H}_2^{\diamondsuit}\left(\bar{\psi}^{\diamondsuit},\ \bar{x}^{\diamondsuit,f},u^1,u^2,t\right) \tag{16}$$
- (**the complementary slackness condition**) For every $\alpha\in\mathcal{A}$ next conditions hold
$$\mu^2(\alpha)\left(h^{2,\alpha}-F^{2*}\right)=0 \tag{17}$$
- (**the transversality condition (Follower)**) For every $\alpha\in\mathcal{A}$ and every $i=1,...,N$
$$\begin{aligned}&b^{\alpha}+\mu^2(\alpha)\,\text{grad}h^{2,\alpha}\left(x^{\alpha*}(T)\right)=0\\ &\psi^{\alpha}(T)=b^{\alpha},\quad\psi_{n+1}^{\alpha}(T)+\mu^2(\alpha)=0\end{aligned} \tag{18}$$

To derive the robust optimal condition for the leader, using the same Mayer representation as before, the functional for the leader is representated as:

$$h^{1,\alpha}=h_0^{1,\alpha}\left(x^{\alpha}(T)\right)+x_{n+1}^{\alpha,l}(T) \tag{19}$$

where

$$x_{n+1}^{\alpha,l}=\int_{t_0=0}^{t}g_1^{\alpha}\left(x^{\alpha},u^1,u^2,\tau\right)d\tau$$

Now consider the next hamiltonian representation for the leader:

$$\mathcal{H}_1^{\diamondsuit}\left(\bar{\psi}^{\diamondsuit},\bar{\lambda}_1^{\diamondsuit},\lambda_2^{\diamondsuit},\lambda_3^{\diamondsuit},\bar{x}^{\diamondsuit,l},u^1,u^2,t\right):=$$
$$\left\langle\bar{\lambda}_1^{\diamondsuit},\bar{f}^{\diamondsuit,l}\left(\bar{x}^{\diamondsuit,l},u^1,u^2,t\right)\right\rangle-\bar{\lambda}_2^{\diamondsuit\intercal}\frac{\partial\mathcal{H}_2^{\diamondsuit}}{\partial\bar{x}^{\diamondsuit,f}}+\lambda_3^{\diamondsuit\intercal}\frac{\partial\mathcal{H}_2^{\diamondsuit}}{\partial u^2}=$$
$$\sum_{\alpha\in A}\mathcal{H}_1^{\alpha}\left(\bar{\psi}^{\diamondsuit},\bar{\lambda}_1^{\diamondsuit},\lambda_2^{\diamondsuit},\lambda_3^{\diamondsuit},\bar{x}^{\diamondsuit,l},u^1,u^2,t\right)=$$
$$\sum_{\alpha\in A}\left[\sum_{j=1}^{n}\left(\lambda_{1,j}^{\alpha}f_j^{\alpha}\left(x^{\alpha},u^1,u^2,t\right)\right)+\lambda_{1,n+1}^{\alpha}g_1^{\alpha}\left(x^{\alpha,f},u^1,u^2,t\right)-\lambda_2^{\alpha\intercal}\frac{\partial\mathcal{H}_2^{\alpha}}{\partial\bar{x}^{\alpha,f}}+\lambda_3^{\alpha\intercal}\frac{\partial\mathcal{H}_2^{\alpha}}{\partial u^2}\right]$$

where the extended vector are defined as:

$$\begin{aligned}\bar{x}^{\diamondsuit,l}&:=\left(x_1^1,...,x_{n+1}^{1,f};...;x_1^M,...,x_{n+1}^{M,l}\right)^{\intercal}\\ \bar{\lambda}_1^{\diamondsuit}&:=\left(\lambda_{1,1}^1,...,\lambda_{1,n+1}^1;...;\lambda_{1,1}^M,...,\lambda_{1,n+1}^{M,l}\right)^{\intercal}\\ \bar{\lambda}_2^{\diamondsuit}&:=\left(\lambda_{2,1}^1,...,\lambda_{2,n+1}^1;...;\lambda_{2,1}^M,...,\lambda_{2,n+1}^{M,l}\right)^{\intercal}\\ \bar{f}^{\diamondsuit,l}&:=\left(\bar{f}^{1\intercal},..\bar{f}^{M\intercal}\right)^{\intercal}=\left(f_1^1,..,f_1^1,g_1^1;..;f_n^M,..,g_1^M\right)^{\intercal}\\ \bar{f}^{\alpha,l}&=\left(f_1^{\alpha},...,f_n^{\alpha},g_1^{\alpha}\right)^{\intercal}\in\Re^{n+1}\end{aligned}$$

with the adjoint variables satisfying

$$\begin{aligned}\frac{d}{dt}\bar{\lambda}_1^{\diamondsuit}&=-\frac{\partial\mathcal{H}_1^{\diamondsuit}\left(\bar{\psi}^{\diamondsuit},\bar{\lambda}_1^{\diamondsuit},\lambda_2^{\diamondsuit},\lambda_3^{\diamondsuit},\bar{x}^{\diamondsuit,l},u^1,u^2,t\right)}{\partial\bar{x}^{\diamondsuit,l}}\\ \frac{d}{dt}\bar{\lambda}_2^{\diamondsuit}&=-\frac{\partial\mathcal{H}_1^{\diamondsuit}\left(\bar{\psi}^{\diamondsuit},\bar{\lambda}_1^{\diamondsuit},\lambda_2^{\diamondsuit},\lambda_3^{\diamondsuit},\bar{x}^{\diamondsuit,l},u^1,u^2,t\right)}{\partial\bar{\psi}^{\diamondsuit}}\\ \bar{\lambda}_2^{\diamondsuit}(0)&=0\end{aligned}$$

The application of the result in Poznyak *et al* [2002], yields to the next conditions that must be satisfy by the leader

- (**the maximality condition**) The control strategies for the leader $u^{1*}(t)\in U_{adm}^1$ $(t\in[0,T])$ satisfies
$$\mathcal{H}_1^{\diamondsuit}\left(\bar{\psi}^{\diamondsuit},\bar{\lambda}_1^{\diamondsuit},\lambda_2^{\diamondsuit},\lambda_3^{\diamondsuit},\bar{x}^{\diamondsuit,l},u^{1*},u^2,t\right)\geq\mathcal{H}_1^{\diamondsuit}\left(\bar{\psi}^{\diamondsuit},\bar{\lambda}_1^{\diamondsuit},\lambda_2^{\diamondsuit},\lambda_3^{\diamondsuit},\bar{x}^{\diamondsuit,l},u^1,u^2,t\right)$$
or, equivalently,
$$u^{1*}\in Arg\max_{u^1\in U^1}\mathcal{H}_1^{\diamondsuit}\left(\bar{\psi}^{\diamondsuit},\bar{\lambda}_1^{\diamondsuit},\lambda_2^{\diamondsuit},\lambda_3^{\diamondsuit},\bar{x}^{\diamondsuit,l},u^1,u^2,t\right) \tag{20}$$
- (**the complementary slackness condition**) For every $\alpha\in\mathcal{A}$ next conditions hold
$$\mu^1(\alpha)\left(h^{1,\alpha}-F^{1*}\right)=0 \tag{21}$$
- (**the transversality condition (Leader)**) For every $\alpha\in\mathcal{A}$ and every $i=1,...,N$
$$\begin{aligned}&c^{\alpha}+\mu^1(\alpha)\left[\text{grad}h^{1,\alpha}\left(x^{\alpha*}(T)\right)-\frac{\partial^2}{\left(\partial x^{\alpha*}\right)^2}h^{2,\alpha}\left(x^{\alpha*}(T)\right)\lambda_2^{\alpha}(T)\right]=0;\ \lambda_1^{\alpha}(T)=c^{\alpha},\\ &\lambda_{1,n+1}^{\alpha}(T)+\mu^1(\alpha)=0\end{aligned} \tag{22}$$
additionaly the leader must satisfy the next relation:
$$\frac{\partial\mathcal{H}_1^{\diamondsuit}\left(\bar{\psi}^{\diamondsuit},\bar{\lambda}_1^{\diamondsuit},\lambda_2^{\diamondsuit},\lambda_3^{\diamondsuit},\bar{x}^{\diamondsuit,l},u^1,u^2,t\right)}{\partial u^2}=0 \tag{23}$$

3.3 Open-Loop Robust Stackelberg Strategies in linear affine-quadratic Differential Games

Consider the next two-player *linear affine-quadratic* multimodel game:

$$\begin{array}{c}\dot{x}^{\alpha}(t){=}A^{\alpha}(t)x^{\alpha}(t){+}\sum\limits_{j=1}^{2}B^{j,\alpha}(t)u^{j}(t){+}d(t)\\ x^{\alpha}(t)\in\Re^{n},\ x(0)=x_0,\ u^{j}(t)\in\Re^{m_j},\ \ j=1,2;\\ t\in[0,T],\ T<\infty;\ \alpha\in\mathcal{A}:=\{1,2,...,M\}\end{array}\tag{24}$$

Suppose that the *individual aim performance* $h^{i,\alpha}$ of each i-player ($i=1,2$) for each α-model (scenario) is given by

$$\begin{array}{c}h^{i,\alpha}\left(u^1,u^2\right)=\frac{1}{2}x^{\alpha\intercal}(T)\,Q_f^i x^{\alpha}(T)+\\ \frac{1}{2}\int\limits_{t=0}^{T}\left[x^{\alpha\intercal}Q^i x^{\alpha}+\sum\limits_{j=1}^{2}u^{j\intercal}R^{ij}u^j\right]dt\end{array}\tag{25}$$

As we mentioned before the robust optimal control problem for each player is formulated as (15). In what follows we illustrate how the construction given above is applied for the case of two players leader follower LQ multi-model differential games. The Hamiltonians for the leader and follower LQ are:

$$\begin{array}{c}\mathcal{H}_2^{\Diamond}=\sum\limits_{\alpha\in A}\left[\frac{1}{2}\psi_{n+1}^{\alpha\intercal}\left(x^{\alpha\intercal}Q^2x^{\alpha}+\sum\limits_{j=1}^{2}u^{j\intercal}R^{2j}u^j\right)+\right.\\ \left.\psi^{\alpha\intercal}\left(A^{\alpha}x^{\alpha}+B^{\alpha,1}u^1+B^{\alpha,2}u^2+d\right)\right]\\ \mathcal{H}_1^{\Diamond}=\sum\limits_{\alpha\in A}\left[\frac{1}{2}\lambda_{1,n+1}^{\alpha\intercal}\left(x^{\alpha\intercal}Q^1x^{\alpha}+\sum\limits_{j=1}^{2}u^{j\intercal}R^{1j}u^j\right)+\right.\\ \lambda_1^{\alpha\intercal}\left(A^{\alpha}x^{\alpha}+B^{\alpha,1}u^1+B^{\alpha,2}u^2+d\right)\text{-}\lambda_2^{\alpha\intercal}\left(Q^2x^{\alpha}+A^{\alpha,\intercal}\psi^{\alpha}\right)+\\ \left.\left(R^{22}u^2+\psi^{\alpha\intercal}B^{\alpha,2}\right)\lambda_3^{\alpha}\right]\end{array}\tag{26}$$

The conditions (16)-(23) for the LQ multimodel plant are as follows:

$$\begin{array}{c}\dot{\psi}^{\alpha}\text{=-}\frac{\partial}{\partial\bar{x}^{\alpha}}\mathcal{H}_2^{\alpha}\text{=-}A^{\alpha\intercal}\psi^{\alpha}\text{-}\psi_{n+1}^{\alpha\intercal}Q^2x^{\alpha};\dot{\psi}_{n+1}^{\alpha}(t){=}0\\ \psi^{\alpha}(T)\text{=-}\mu^2(\alpha)\,\mathrm{grad}\left[x^{\alpha}(T)\,Q_f^2x^{\alpha}(T)+x_{n+1}^{\alpha,f}(T)\right]=\\ -\mu^2(\alpha)\,Q_f^2x^{\alpha}(T);\quad \psi_{n+1}^{\alpha}(T)=-\mu^2(\alpha)\\ \dot{\lambda}_1^{\alpha}=-\frac{\partial}{\partial\bar{x}^{\alpha}}\mathcal{H}_1^{\alpha}\text{=-}A^{\alpha\intercal}\lambda_1^{\alpha}\text{-}\lambda_{1,n+1}^{\alpha\intercal}Q^1x^{\alpha}{+}Q^2\lambda_2^{\alpha}\\ \dot{\lambda}_{1,n+1}^{\alpha}(t)=0\\ \lambda_1^{\alpha}(T)=\mu^1(\alpha)\left[Q_f^1x^{\alpha}(T)-Q_f^2\lambda_2^{\alpha}(T)\right]\\ \lambda_{1,n+1}^{\alpha\intercal}(T)=-\mu^1(\alpha)\end{array}\tag{27}$$

$$\dot{\lambda}_2^{\alpha}=-\frac{\partial}{\partial\bar{\psi}^{\alpha}}\mathcal{H}_1^{\alpha}=-A^{\alpha}\lambda_2^{\alpha}+B^{\alpha,2}\lambda_3^{\alpha};\qquad \lambda_2^{\alpha}(0)=0$$

the robust stackelberg strategies for the leader satisfies:

$$u^{1*}(t)=\left(\sum_{\alpha\in\mathcal{A}}\mu^1(\alpha)\right)^{-1}\left(R^{11}\right)^{-1}\sum_{\alpha\in A}B^{\alpha,1\intercal}\lambda_1^{\alpha}\tag{28}$$

and for the follower:

$$u^{2*}(t)=\left(\sum_{\alpha\in\mathcal{A}}\mu^2(\alpha)\right)^{-1}\left(R^{22}\right)^{-1}\sum_{\alpha\in A}B^{\alpha,2\intercal}\psi^{\alpha}\tag{29}$$

and

$$\begin{array}{c}\frac{\partial\mathcal{H}^1}{\partial u^2}=\sum\limits_{\alpha\in\mathcal{A}}\left[\text{-}\left(\mu^1(\alpha)\right)R^{12}u^{2*}(t)+\lambda_1^{\alpha\intercal}B^{\alpha,2}{+}R^{22}\lambda_3^{\alpha}\right]\\ =\sum\limits_{\alpha\in A}\left[-\mu^1(\alpha)\left(\mu^2(\alpha)\right)^{-1}R^{12}\left(R^{22}\right)^{-1}B^{\alpha,2\intercal}\psi^{\alpha}\right.\\ \left.+\lambda_1^{\alpha\intercal}B^{\alpha,2}{+}R^{22}\lambda_3^{\alpha}\right]=0\end{array}\tag{30}$$

finally:

$$\begin{array}{c}\sum\limits_{\alpha\in A}\lambda_3^{\alpha}=\sum\limits_{\alpha\in A}\left(R^{22}\right)^{-1}\times\\ \left[\mu^1(\alpha)\left(\mu^2(\alpha)\right)^{-1}R^{12}\left(R^{22}\right)^{-1}B^{\alpha,2\intercal}\psi^{\alpha}-B^{\alpha,2\intercal}\lambda_1^{\alpha}\right]\end{array}\tag{31}$$

Since at least one index $\alpha\in\mathcal{A}$ is active we have: $\sum\limits_{\alpha\in\mathcal{A}}\mu^i(\alpha)>0$. Introducing the *normalized* adjoint variables with as

$$\tilde{\psi}_n^{\alpha}(t){=}\begin{cases}\psi_n^{\alpha}(t)\left(\mu^2\right)^{-1}(\alpha) & \text{if } \mu^2(\alpha)>0\\ 0 & \text{if } \mu^2(\alpha)=0\end{cases}$$

$$\tilde{\lambda}_{k,n}^{\alpha}(t){=}\begin{cases}\lambda_{k,n}^{\alpha}(t)\left(\mu^1\right)^{-1}(\alpha) & \text{if } \mu^1(\alpha)>0\\ 0 & \text{if } \mu^1(\alpha)=0\end{cases}$$

$$k=1,2,3$$

we get

$$\begin{array}{c}\dot{\tilde{\psi}}^{\alpha}(t)=-A^{\alpha\intercal}\tilde{\psi}^{\alpha}-\tilde{\psi}_{n+1}^{\alpha\intercal}Q^2x^{\alpha}\\ \dot{\tilde{\psi}}_{n+1}^{\alpha}(t)=0,\end{array}$$

with the corresponding transversality conditions given by

$$\begin{array}{c}\tilde{\psi}^{\alpha}(T)=-Q_f^2x^{\alpha}(T),\ \tilde{\psi}_{n+1}^{\alpha}(T)=-1\\ \dot{\tilde{\lambda}}_1^{\alpha}=-A^{\alpha\intercal}\tilde{\lambda}_1^{\alpha}-\tilde{\lambda}_{1,n+1}^{\alpha\intercal}Q^1x^{\alpha}+Q^2\tilde{\lambda}_2^{\alpha}\\ \tilde{\lambda}_1^{\alpha}(T)=Q_f^1x^{\alpha}(T)-Q_f^2\tilde{\lambda}_2^{\alpha}(T);\qquad \tilde{\lambda}_{1,n+1}^{\alpha\intercal}(T)=-1\\ \dot{\tilde{\lambda}}_2^{\alpha}=-A^{\alpha}\tilde{\lambda}_2^{\alpha}+B^{\alpha,2}\tilde{\lambda}_3^{\alpha};\qquad \tilde{\lambda}_2^{\alpha}(0)=0\end{array}$$

the robust stackelberg strategies becomes:

$$\begin{array}{c}u^{1*}(t)=\left(\sum\limits_{\alpha\in A}\left(\mu^1\right)\right)^{-1}\left(R^{11}\right)^{-1}\sum\limits_{\alpha\in A}\left(\mu^1\right)B^{\alpha,2\intercal}\tilde{\lambda}_1^{\alpha}=\\ -\left(R^{11}\right)^{-1}\sum\limits_{\alpha\in A}\nu^{\alpha,1}B^{\alpha,1\intercal}\tilde{\lambda}_1^{\alpha}\end{array}\tag{32}$$

for the leader and for the follower:

$$\begin{array}{c}u^{2*}(t)=\left(\sum\limits_{\alpha\in\mathcal{A}}\mu^2(\alpha)\right)^{-1}\left(R^{22}\right)^{-1}\sum\limits_{\alpha\in A}\left(\mu^2\right)B^{\alpha,2\intercal}\tilde{\psi}^{\alpha}=\\ -\left(R^{22}\right)^{-1}\sum\limits_{\alpha\in A}\nu^{\alpha,2}B^{\alpha,2\intercal}\tilde{\psi}^{\alpha}\end{array}\tag{33}$$

where the vectors $\boldsymbol{\nu}^i:=\left(\boldsymbol{\nu}_1^{\alpha,i},...,\boldsymbol{\nu}_M^{\alpha,i}\right)^{\intercal}$ ($i=1,2$) belongs to the simplex:

$$S^{i,M}{:=}\left\{\boldsymbol{\nu}^i\in R^{M=|\mathcal{A}|}:\boldsymbol{\nu}^{\alpha,i}{=}\mu^i(\alpha)\,(\sum_{\alpha\in\mathcal{A}}\mu^i(\alpha))^{-1}\geq 0,\ \sum_{\alpha=1}^{N}\boldsymbol{\nu}^{\alpha,i}=1\right\}\tag{34}$$

Finally:

$$\sum_{\alpha\in A}\tilde{\lambda}_3^{\alpha}{=}\sum_{\alpha\in A}\left(R^{22}\right)^{-1}\left[R^{12}\left(R^{22}\right)^{-1}B^{\alpha,2\intercal}\tilde{\psi}^{\alpha}\text{-}B^{\alpha,2\intercal}\tilde{\lambda}_1^{\alpha}\right]\tag{35}$$

As one can see from (32) and (33), the Robust Optimal Control for both leader an follower is a mixture of the control actions optimal for each independent index $\alpha\in\mathcal{A}$.

3.4 Extended form for the game

Now consider the next representation for the game

$$\mathbf{A}:=\begin{bmatrix} A^1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & A^M \end{bmatrix}, \mathbf{Q}^i:=\begin{bmatrix} Q^i & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & Q^i \end{bmatrix}$$
$$\mathbf{Q}_f^i:=\begin{bmatrix} Q_f^i & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & Q_f^i \end{bmatrix}, \Gamma^i:=\begin{bmatrix} \nu^{1,i}I & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \nu^{M,i}I \end{bmatrix} \quad (36)$$
$$\mathbf{B}^{i,\intercal} := \begin{bmatrix} B^{i,1\intercal} & \cdots & B^{i,M\intercal} \end{bmatrix}, \quad I \in \Re^{n\times n} \; i=1,2$$

In the extended form we obtain the general dynamics given by (bold stand for extended vectors and matrices):

$$\begin{gathered} \dot{\mathbf{x}}=\mathbf{A}\mathbf{x}+\mathbf{B}^1u^1+\mathbf{B}^2u^2+\mathbf{d},\ \mathbf{x}^\intercal(0)=\left(x^{1\intercal}(0),..,x^{M\intercal}(0)\right) \\ \dot{\boldsymbol{\psi}}=-\mathbf{A}^\intercal\boldsymbol{\psi}+\mathbf{Q}^2\mathbf{x},\ \psi(T)=-\mathbf{Q}_f^2\mathbf{x}(T) \\ \dot{\boldsymbol{\lambda}}_1=-\mathbf{A}^\intercal\boldsymbol{\lambda}_1+\mathbf{Q}^1\mathbf{x}+\mathbf{Q}^2\boldsymbol{\lambda}_2,\ \boldsymbol{\lambda}_1(T)=\mathbf{Q}_f^1\mathbf{x}(T)-\mathbf{Q}_f^2\boldsymbol{\lambda}_2(T) \\ \dot{\boldsymbol{\lambda}}_2=\mathbf{A}\boldsymbol{\lambda}_2-\mathbf{B}^2\boldsymbol{\lambda}_3 \quad \boldsymbol{\lambda}_2(0)=0 \\ \boldsymbol{\lambda}_3=\left(R^{22}\right)^{-1}\left[R^{12}\left(R^{22}\right)^{-1}\mathbf{B}^{2,\intercal}\boldsymbol{\psi}-\mathbf{B}^{2,\intercal}\boldsymbol{\lambda}_1\right] \end{gathered} \quad (37)$$

$$\begin{aligned} u^1 &= -\left(R^{11}\right)^{-1}\mathbf{B}^{1,\intercal}\Gamma^1\boldsymbol{\lambda}_1 \\ u^2 &= -\left(R^{22}\right)^{-1}\mathbf{B}^{2,\intercal}\Gamma^2\boldsymbol{\psi} \end{aligned} \quad (38)$$

where:

$$\begin{gathered} \mathbf{x}^\intercal:=\left(x_1^{1,\intercal},...,x_n^{1,\intercal};...;x_1^{M,\intercal},...,x_n^{M,\intercal}\right)\in R^{1\times nM} \\ \boldsymbol{\psi}^\intercal:=\left(\tilde{\psi}_1^{1,\intercal},...,\tilde{\psi}_n^{1,\intercal},...,\tilde{\psi}_1^{M,\intercal},...,\tilde{\psi}_n^{M,\intercal}\right)\in R^{1\times nM} \\ \boldsymbol{\lambda}_k^\intercal:=\left(\tilde{\lambda}_{k,1}^{1,\intercal},...,\tilde{\lambda}_{k,n}^{1,\intercal},...,\tilde{\lambda}_{k,1}^{M,\intercal},...,\tilde{\lambda}_{k,n}^{M,\intercal}\right)\in R^{1\times nM}, \\ \mathbf{d}^\intercal:=\left(d^{1,\intercal},...,d^{M,\intercal}\right); \quad k=1,2,3 ; \end{gathered}$$

Theorem 1. If for the two person linear quadratic differential game (3) with the following restrictions to the matrices:$R^{ii} > 0$, $R^{12} \geq 0$, $\mathbf{Q}^i \geq 0$ and $\mathbf{Q}_f^i \geq 0$ $(i = 1, 2)$ there exists a solution set of the following parametrized coupled differential equation (Abou-Khandil *et al* [2003]):

$$\begin{gathered} \dot{\mathbf{P}}_{\nu^{2*}}\left(\mathbf{A}\text{-}\mathbf{B}^1R^{11^{-1}}\mathbf{B}^{1\intercal}\boldsymbol{\Lambda}_{1,\nu^{1*}}\text{-}\mathbf{B}^2R^{22^{-1}}\mathbf{B}^{2\intercal}\mathbf{P}_{\nu^{2*}}\right)+ \\ \mathbf{P}_{\nu^{2*}}+\mathbf{A}^\intercal\mathbf{P}_{\nu^{2*}}+\Gamma^{2*}\mathbf{Q}^2=0; \\ \mathbf{P}_{\nu^{2*}}(T)=\Gamma^{2*}\mathbf{Q}_f^2;\mathbf{P}_{\nu^{2*}}=\mathbf{P}_{\nu^{2*}}^\intercal\in\Re^{nM\times nM} \\ \dot{\boldsymbol{\Lambda}}_{1,\nu^{1*}}\left(\mathbf{A}\text{-}\mathbf{B}^1R^{11^{-1}}\mathbf{B}^{1\intercal}\boldsymbol{\Lambda}_{1,\nu^{1*}}\text{-}\mathbf{B}^2R^{22^{-1}}\mathbf{B}^{2\intercal}\mathbf{P}_{\nu^{2*}}\right)+ \\ \boldsymbol{\Lambda}_{1,\nu^{1*}}+\mathbf{A}^\intercal\boldsymbol{\Lambda}_{1,\nu^{1*}}+\Gamma^{1*}\mathbf{Q}^1\text{-}\Gamma^{1*}\mathbf{Q}^2\boldsymbol{\Lambda}_{2,\nu^{1*}}=0; \\ \boldsymbol{\Lambda}_{1,\nu^{1*}}(T)=\Gamma^{1*}\left[\mathbf{Q}_f^1\text{-}\mathbf{Q}_f^2\boldsymbol{\Lambda}_{2,\nu^{1*}}(T)\right] \\ \dot{\boldsymbol{\Lambda}}_{2,\nu^{1*}}\left(\mathbf{A}\text{-}\mathbf{B}^1R^{11^{-1}}\mathbf{B}^{1\intercal}\boldsymbol{\Lambda}_{1,\nu^{1*}}\text{-}\mathbf{B}^2R^{22^{-1}}\mathbf{B}^{2\intercal}\mathbf{P}_{\nu^{2*}}\right) \\ +\boldsymbol{\Lambda}_{2,\nu^{1*}}\text{-}\mathbf{A}\boldsymbol{\Lambda}_{2,\nu^{1*}}+\Gamma^{1*}\mathbf{B}^2R^{22^{-1}}R^{12}R^{22^{-1}}\mathbf{B}^{2\intercal}\mathbf{P}_{\nu^{2*}}\text{-} \\ \Gamma^{1*}\mathbf{B}^2R^{22^{-1}}\mathbf{B}^{2\intercal}\boldsymbol{\Lambda}_{1,\nu^{1*}}=0; \quad \Gamma^{1*}\boldsymbol{\Lambda}_{2,\nu^{1*}}(0)=0 \end{gathered} \quad (39)$$

then the open-loop robust stackelberg equilibruim strategies with *Player 1* acting as the leader are:

$$\begin{aligned} u^1 &= -\left(R^{11}\right)^{-1}\mathbf{B}^{1,\intercal}\left(\boldsymbol{\Lambda}_{1,\nu^{1*}}\mathbf{x}+\mathbf{l}_{1,\nu^{1*}}\right) \\ u^2 &= -\left(R^{22}\right)^{-1}\mathbf{B}^{2,\intercal}\left(\mathbf{P}_{\nu^{2*}}\mathbf{x}+\mathbf{p}_{\nu^{2*}}\right) \end{aligned} \quad (40)$$

where $\mathbf{l}_{1,\nu^{1*}}$, $\mathbf{p}_{\nu^{2*}}$ are the "*shifting equations*" given by the solution of the next coupled linear equations :

$$\begin{gathered} \dot{\mathbf{p}}_{\nu^{2*}}+\mathbf{P}_{\nu^{2*}}\left(\mathbf{d}\text{-}\mathbf{B}^1R^{11^{-1}}\mathbf{B}^{1\intercal}\mathbf{l}_{1,\nu^{1*}}\text{-}\mathbf{B}^2R^{22^{-1}}\mathbf{B}^{2\intercal}\mathbf{p}_{\nu^{2*}}\right) \\ +\mathbf{A}^\intercal\mathbf{p}_{\nu^{2*}}=0; \quad \mathbf{p}(T)=0 \\ \dot{\mathbf{l}}_{1,\nu^{1*}}+\boldsymbol{\Lambda}_{1,\nu^{1*}}\left(\mathbf{d}\text{-}\mathbf{B}^1R^{11^{-1}}\mathbf{B}^{1\intercal}\mathbf{l}_{1,\nu^{1*}}\text{-}\mathbf{B}^2R^{22^{-1}}\mathbf{B}^{2\intercal}\mathbf{p}_{\nu^{2*}}\right) \\ +\mathbf{A}^\intercal\mathbf{l}_{1,\nu^{1*}}+\Gamma^{1*}\mathbf{Q}^2\mathbf{l}_{2,\nu^{1*}}=0; \quad \mathbf{l}_{1,\nu^{1*}}(T)=0 \\ \boldsymbol{\Lambda}_{2,\nu^{1*}}\left(\mathbf{d}\text{-}\mathbf{B}^1R^{11^{-1}}\mathbf{B}^{1\intercal}\mathbf{l}_{1,\nu^{1*}}\text{-}\mathbf{B}^2R^{22^{-1}}\mathbf{B}^{2\intercal}\mathbf{p}_{\nu^{2*}}\right)+ \\ \dot{\mathbf{l}}_{2,\nu^{1*}}-\mathbf{A}\mathbf{l}_{2,\nu^{1*}}+\Gamma^{1*}\mathbf{B}^2R^{22^{-1}}R^{12}R^{22^{-1}}\mathbf{B}^{2\intercal}\mathbf{p}_{\nu^{2*}} \\ -\Gamma^{1*}\mathbf{B}^2R^{22^{-1}}\mathbf{B}^{2\intercal}\mathbf{l}_{1,\nu^{1*}}=0; \quad \mathbf{l}_{2,\nu^{1*}}(0)=0 \end{gathered} \quad (41)$$

define the open-loop robust Stackelberg equilibrium solution if the matrix Γ^{i*} $(i = 1, 2)$ in (36) contains the vectors ν^{i*} which satisfy the stackelberg equilibrium condition

$$\begin{gathered} J^1\left(\nu^{1*},\nu^{2*}\left(\nu^{1*}\right)\right)\leq J^1\left(\nu^1,\nu^{2^o}\left(\nu^1\right)\right) \\ \nu^{2^o}\left(\nu^1\right):=\arg\min_{\boldsymbol{\nu}_2\in S^N}J^2\left(\nu^1,\nu^2\right) \\ \text{for any } \nu^1,\nu^2\in S^M \end{gathered} \quad (42)$$

where

$$\begin{gathered} J^i\left(\nu^{1*},\nu^{2*}\right):=\max_{\alpha\in A}h^{i,\alpha}(u^1,u^2) \\ i=1,2 \end{gathered} \quad (43)$$

with u^i given by (40) parametrized by $\boldsymbol{\nu}^{1*}$ and $\boldsymbol{\nu}^{2*}$ ($\nu^{i*} \in S^{i,M}$) through (39).

Proof: Representing $\Gamma^2\boldsymbol{\psi}$, $\Gamma^1\boldsymbol{\lambda}_1$ and $\Gamma^1\boldsymbol{\lambda}_2$ as $\Gamma^2\boldsymbol{\psi} = -\mathbf{P}_{\nu^2}\mathbf{x} - \mathbf{p}_{\nu^2}$, $\Gamma^1\boldsymbol{\lambda}_1 = -\boldsymbol{\Lambda}_{1,\nu^1}\mathbf{x} - \mathbf{l}_{1,\nu^1}$, $\Gamma^1\boldsymbol{\lambda}_2 = -\boldsymbol{\Lambda}_{2,\nu^1}\mathbf{x} - \mathbf{l}_{2,\nu^1}$. and by (39) and (41) and by the commutation of the operators $\Gamma^i\mathbf{A}^\intercal = \mathbf{A}^\intercal\Gamma^i$ and $\Gamma^i\mathbf{Q}^i = \mathbf{Q}^i\Gamma^i$ the result follows.

4. NUMERICAL PROCEDURE

For the problem of finding the leader-follower equilibrium weights we propose the use of the next minimizing numerical procedure. Assuming that $J^i(\nu_1, \nu_2(\nu_1)) > 0$ for all $\nu^i \in S^{i,M}$ $(i = 1, 2)$, define the series of the vectors iterations $\{\nu^{i,k}\}$ (for any fixed n) as

$$\begin{gathered} \nu^{i,k+1}=\pi_{S^{i,M}}\left\{\nu^{i,k}+\frac{\gamma^{i,k}}{\tilde{J}^i(\nu^{1,k},\nu^{2,k})}\tilde{F}^i(\nu^{1,k},\nu^{2,k})\right\} \\ \nu^{i,0}\in S^{i,M},\ k=1,2,... \\ \tilde{F}^i(\nu^{1,k},\nu^{2,k})=\left[\tilde{h}^{1,i}(\nu^{1,k},\nu^{2,k}),\tilde{h}^{M,i}(\nu^{1,k},\nu^{2,k})\right] \\ \tilde{J}^i(\nu^{1,k},\nu^{2,k})=\max_{\alpha\in\overline{1,N}}\tilde{h}^{\alpha,i}(\nu^{1,k},\nu^{2,k}) \end{gathered} \quad (44)$$

where $\pi_{S^{i,M}}\{\cdot\}$ is the projector of an argument to the simplex $S^{i,M}$ and the new functional $\tilde{h}^{\alpha,i}$ is defined as:

$$\tilde{h}^{\alpha,i}(\nu^{1,k},\nu^{2,k}):=\frac{\delta}{2}\left\|\nu^{(i)}\right\|^2+h^{\alpha,i}\left(\nu^{1,k},\nu^{2,k}\right)$$

where δ is a small constant. Whether this algorithm converges to a unique point or not we let this discusion for a large version of this work.

4.1 Solving Coupled equations

For the solution of the set of coupled Riccati equations (39), we follow the work of Simman and Cruz [1973b], which is based on the solution of an auxiliary system with only terminal conditions. For the lack of space we omit the details of this method.

5. NUMERICAL EXAMPLE

Consider the next two-scenarios two-players LQ differential game given by

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}^1\mathbf{u}^1 + \mathbf{B}^2\mathbf{u}^2 + \mathbf{d}$$
$$\mathbf{A}= \begin{bmatrix} A^1 & 0 \\ 0 & A^2 \end{bmatrix}; \mathbf{B}^1 = \begin{bmatrix} B^{1,1} \\ B^{2,1} \end{bmatrix};$$
$$\mathbf{B}^2 = \begin{bmatrix} B^{1,2} \\ B^{2,2} \end{bmatrix}; A^1= \begin{pmatrix} 0.75 & 0 \\ 0.71 & -0.20 \end{pmatrix};$$
$$A^2= \begin{pmatrix} 0.80 & 0 \\ 0.70 & -0.10 \end{pmatrix}; x_0= \begin{pmatrix} 3 \\ -1 \end{pmatrix}$$
$$B^{1,1}= \begin{pmatrix} 2 \\ 0 \end{pmatrix}; B^{2,1}= \begin{pmatrix} 0 \\ 1 \end{pmatrix}; d^1=d^2= \begin{pmatrix} 0 \\ 0.3 \end{pmatrix};$$
$$B^{1,2} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}; B^{2,2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$Q^{1,1}=Q^{1,2}= \times I_{2\times 2}, Q^{2,1}=Q^{2,2} = 1.5 \times I_{2\times 2}$$
$$Q_f^{1,1}=3 \times I_{2\times 2}; Q_f^{1,2}=2 \times I_{2\times 2};$$
$$Q_f^{2,1} = 3 \times I_{2\times 2}; Q_f^{2,2} = 2 \times I_{2\times 2}$$
$$R^{ii} = R^{ij} = I_{n\times n} \quad i,j = 1,2$$

The tables below show the conevergence to dependence of the cost functionals on the weights $\nu^{(1)} := \left(\nu_1^{(1)}, 1-\nu_1^{(1)}\right)$, $\nu^{(2)} := \left(\nu_1^{(2)}, 1-\nu_1^{(2)}\right) \left(\nu_1^{(1)} \geq 0, \nu_1^{(2)} \geq 0\right)$.

k	$\nu^{1,1}$	$\nu^{1,2}$	$h^{1,1}$	$h^{1,2}$
1	0.5000	0.5000	9.5713	9.5615
2	0.5000	0.5000	9.5699	9.5623
3	0.5000	0.5000	9.5696	9.5630
4	0.5000	0.5000	9.5696	9.5635
5	0.5000	0.5000	9.5696	9.5639
⋮	⋮	⋮	⋮	⋮
24	0.5027	0.4973	9.5685	9.5685
25	0.5027	0.4973	9.5685	9.5685

Table 1 Cost Funtcion Leader

k	$\nu^{2,1}$	$\nu^{2,2}$	$h^{2,1}$	$h^{2,2}$
1	0.5000	0.5000	2.1545	2.1411
2	0.5931	0.4069	2.1540	2.1421
3	0.6345	0.3655	2.1537	2.1423
4	0.6607	0.3393	2.1534	2.1424
5	0.6799	0.3201	2.1532	2.1423
⋮	⋮	⋮	⋮	⋮
23	0.7430	0.2570	2.1488	2.1447
24	0.7430	0.2570	2.1488	2.1447
25	0.7430	0.2570	2.1488	2.1447

Table 2 Cost Funtcion Follower

As one can see in the tables 1 and 2 the numerical procedure works efficiently finding the robust strategies. At the end of the process the cost functionals arrives to the practically same values as we expected.

6. CONCLUTIONS

In this paper the formulation of a concept for a type of robust Leader-Follower equilibrium for a Multi-Plant differential game was presented. The dynamic of the game for this kind of problems is given by a set of N different possible differential equations. The problem is solved designing of min-max strategies for each player. As in the Nash equilibrium case the initial min-max differential game is converted into a standard static game given in a multidimensional simplex. The realization of the numerical procedure confirms the effectiveness of the suggested approach.

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