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Distributed Estimation for Spacecraft Formations Over Time-Varying Sensing Topologies

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Abstract: In this paper we present the analysis and design of distributed estimators for formation flying spacecraft with time-varying sensing topologies. We first develop a discrete-time, switched linear model of the formation translational dynamics in which the the measurement vector is characterized in terms of the edge matrix of a graph associated with the sensing topology. Then a switched, linear estimator is developed, called a λ-estimator, for a general class of discrete-time, switched linear systems. This estimator is replicated on each spacecraft to estimate the entire relative translational state of a formation, and estimator gain switching occurs as a function of the instantaneous sensing topology. These estimators guarantee that the mean of the estimation error decays to the origin with a prescribed decay rate and that the error covariance decays to an ultimate bound, also with a prescribed decay rate. In addition, linear matrix inequality-based design procedures are developed for λ-estimators. It is proven that a stable formation λ-estimator exists if all of the possible sensing topologies describe connected graphs. This observation leads to the design of opportunistic λ-estimators for formations switching among connected sensing topologies in which more sensing links are available than considered in estimator design.

1. INTRODUCTION

This paper presents the analysis and design of distributed estimators for formation flying spacecraft with time-varying sensing topologies. This research is motivated by NASA's formation flying missions, such as the Terrestrial Planet Finder Interferometer (TPF-I) [Lawson (2001)], in which several spacecraft operate in a coordinated manner to achieve a common objective. Each spacecraft in a formation is assumed to estimate a maximal, linearly independent set of inter-spacecraft (i.e., relative) translational states (see Smith and Hadaegh (2006) for a similar distributed estimator structure). The resulting state vector is referred to as the formation state. Each estimator uses all available inter-spacecraft measurements, which form a subset of the relative position vectors. This set of relative measurements defines a sensing topology and an associated sensing graph. It is also assumed that the overall measurement vector is instantaneously available to all spacecraft. Subsequent research will address extensions to account for communication delays. We consider systems whose dynamics are accurately modeled by linear, time-invariant ordinary differential equations, which includes formations of an arbitrary number of spacecraft both in deep space, such as TPF-I, and near-circular planetary orbits.

Formation maneuvers, such as reconfigurations, will change the sensing topology. Further, with multiple sensing levels, specific sensors can go in and out of lock. Previous work related to TPF-I developed a steady-state-Kalman-based estimator for the three levels of sensing available in the baseline TPF-I design [Scharf et al. (2004)]. Mode changes and assumptions on timing were used to ensure TPF-I estimator performance as sensors were added or removed from the measurement vector. Both more operational flexibility and more rigorous performance guarantees are desired. To this end, we assume the sensing topology can vary arbitrarily in time within a specified set of topologies. However, no a priori knowledge of the time sequence of topologies from the set is assumed. An estimator determines the instantaneous sensing topology in real-time upon receiving the overall measurement vector. While the Kalman filter addresses this scenario, flight computers on-board formation flying spacecraft will perform a variety of autonomous operations that restrict the complexity of formation estimation algorithms. Computationally efficient algorithms are required. In this regard, simply matching the steady-state Kalman filter gain to the instantaneous sensing topology, as done previously, provides no guarantee of stability. Further, the transient performance of Kalman-based estimators can be significantly degraded by errors in the initial covariance due to, for example, delays or errors in inter-spacecraft communication of measurements.

Our objective is to develop formation state estimators that are: (i) stable, (ii) exponentially convergent, (iii) precise, and (iv) computationally inexpensive. Here, stability simply means that the dynamics of the expected estimation error (mean error) are asymptotically stable. Exponential convergence of the estimator requires that the mean error converges to the origin at least as fast as a prescribed decay rate. Precision is determined by the error variance, and the estimator must minimize the error variance in a sense described subsequently.

In the following sections, the dynamics of the formation state are first formulated in discrete time. The measurements are then expressed in terms of edge matrices and Laplacians of the sensing graph. This system is shown to be observable when the sensing graph is connected. Next, we describe a class of fast estimators, termed λ-estimators, with desirable properties of stability, fast decay, precision, and simplicity. The scalar $\lambda \in [0,1]$ specifies the decay rate.

For formation estimation, the λ-estimator on-board each spacecraft contains a copy of the relative state dynamics and a feedback term that utilizes the measurement error (i.e., the difference between the measurement vector and the current estimate of the measurement vector). Hence, the λ-estimator has the same structure as a Luenberger observer [Luenberger (1964)] or a Kalman filter [Kalman (1960)]. However, the the λ-estimator gain is constant for each sensing topology, changing only as the sensing topology changes, whereas the Kalman gain is always varying. Also, the Luenberger observer does not consider stochastic optimality of the estimation error. For λ-estimator design, a linear matrix inequality (LMI)-based [Boyd et al. (1994)] synthesis method minimizes the ultimate variance of the estimation error vector while guaranteeing a decay rate in the mean error that is specified by λ. The estimation error covariance matrix also converges to an ultimate bound with a decay rate determined by λ.

Related work in LMI-based estimator synthesis for switched, discrete-time linear systems includes Luenberger-type observer synthesis for linear [Alessandri and Coletta (2003); Alessandri et al. (2005)] and nonlinear systems [Açıkmeşe and Corless (2005)]. These LMI-synthesized observers establish globally stable error dynamics but do not have stochastic performance measures. The work presented here extends the LMI-design methods to optimize such measures and adds a guaranteed, prescribed decay rate. Such fast estimators can be useful in practice when the estimator dynamics drive performance limits, such as on the Spitzer Space Telescope [Bayard (1998)]. Another contribution is to augment λ-estimator to utilize measurements in addition to those specified in the design sensor topologies. This opportunistic use of additional measurements preserves stability and the exponential decay properties as well as improves the error covariance beyond the designed level.

A partial list of notation is as follows: $P = P^T > (\geq) 0$ implies P is a positive (semi-) definite matrix; $\mathrm{diag}(A_1, \ldots, A_n)$ is a block-diagonal matrix with matrix entries $A_1, \ldots, A_n$; $\mathrm{tr}A$ is the trace of square matrix A; $A \succ 0$ indicates each entry of matrix A is strictly positive; $\lambda_{max}(P)$ and $\lambda_{min}(P)$ are the largest and smallest eigenvalues of P; $\otimes$ is the Kronecker product; $\sigma(A)$ is the spectral radius of matrix A; I is the identity matrix of appropriate dimension and I_n is $n \times n$ identity matrix; 0_n is $n \times n$ zero matrix and $0_{n\times m}$ is the $n \times m$ zero matrix; $\mathcal{Z}_+$ is the set of positive integers; $\mathrm{E}\{\cdot\}$ is the expectation operator; for random vector $x \in \mathbb{R}^n$, $\bar{x} = \mathrm{E}\{x\}$ is its mean, $P = \mathrm{E}\left\{(x-\bar{x})(x-\bar{x})^T\right\}$ is its covariance matrix, and $\mathrm{tr}P$ is its variance; two random vectors x and y are called independent when $\mathrm{E}\left\{(x-\bar{x})(y-\bar{y})^T\right\} = 0$ and $\mathrm{E}\left\{(y-\bar{y})(x-\bar{x})^T\right\} = 0$; imA denotes the range space of A; $\ker A$ denotes the null space of A; $|A|$ is the matrix with the absolute values of the entries in matrix A; $\|\cdot\|$ is a vector norm, and $|||\cdot|||$ is the matrix norm induced by it.

Let $G(V,E)$ represent an undirected graph with set of vertices V and edges E. The elements of V and E are distinct. A sequence of vertices and distinct edges define a *path*. $G(V,E)$ is *connected* if there exists a path between any two vertices. A *cycle* is a path of length greater than one that starts and ends at the same vertex. An *acyclic* graph has no cycles. A *tree* is a connected acyclic graph, that is, every two vertices are connected by a unique path [Deo (1974)]. For any sensing topology, the corresponding sensing graph is constructed by considering each spacecraft as a vertex, and by putting an edge between any two vertices where the corresponding relative position vector is one of the measurements.

2. PROBLEM FORMULATION

The inertial dynamics of spacecraft in deep space or in a circular planetary orbit can be expressed as

$$\dot{\xi}_l = A_0 \xi_l + B_0(\eta_l + \theta_l) \qquad l = 1, .., n_s, \tag{1}$$

where $\xi_l \in \mathbb{R}^6$ is the translational state vector of lth spacecraft with the first three entries describing the position vector and the last three describing the velocity vector, $\eta_l \in \mathbb{R}^3$ is the control input, $\theta_l \in \mathbb{R}^3$ is a zero-mean, random disturbance vector, n_s is the total number of spacecraft,

$$A_0 = \begin{bmatrix} 0_3 & I_3 \\ \omega^2 D_0 & \omega S_0 \end{bmatrix}, \qquad B_0 = \begin{bmatrix} 0_3 \\ I_3 \end{bmatrix}, \tag{2}$$

$$D_0 = \mathrm{diag}(3, 0, -1), \qquad S_0 = \begin{bmatrix} 0 & 2 & 0 \\ -2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

and ω is a scalar determined by the orbit: $\omega = 0$ for deep space and $\omega = \sqrt{\mu/R^3}$, where μ is the gravitational parameter for the planetary body and R is the orbital radius. The inertial dynamics of the entire formation can be expressed compactly as

$$\dot{\xi} = (A_0 \otimes I_{n_s})\xi + (B_0 \otimes I_{n_s})(\eta + \theta) \tag{3}$$

where, noting $\xi_l = [\xi_{l,1}, \ldots, \xi_{l,6}]^T$, the "augmented" *inertial formation state vector* $\xi \in \mathbb{R}^{6n_s}$ is given by

$$\xi = [\xi_{1,1}, \ \ldots, \ \xi_{n_s,1}, \ \xi_{1,2}, \ \ldots, \ \xi_{n_s,2}, \ \ldots, \ \xi_{1,6}, \ \ldots, \ \xi_{n_s,6}]^T$$

and similarly, $\eta \in \mathbb{R}^{3n_s}$ and $\theta \in \mathbb{R}^{3n_s}$ are defined as

$$\eta = [\eta_{1,1}, \ \ldots, \ \eta_{n_s,1}, \ \ldots, \ \eta_{1,3}, \ \ldots, \ \eta_{n_s,3}]^T$$
$$\theta = [\theta_{1,1}, \ \ldots, \ \theta_{n_s,1}, \ \ldots, \ \theta_{1,3}, \ \ldots, \ \theta_{n_s,3}]^T.$$

The control of formations is typically partitioned into control of the overall formation location, in which a formation is treated as a single object, and control of the relative positions within a formation. Further, in deep space often only on-board, relative measurements are available to the necessary precision. Hence, we focus on estimating relative spacecraft positions. As there is no unique, linearly-independent set of relative position vectors, the designer must select the relative states that will be estimated. For each maximal, linearly independent set, there is an onto matrix $T \in \mathcal{Z}_+{}^{n_s-1\times n_s}$ ($TT^T > 0$) that relates the inertial positions to the relative ones, that is, $r = (I_3 \otimes T)p$ where $r \in \mathbb{R}^{3(n_s-1)}$ is the "augmented" vector of all the relative position vectors, and p is the "augmented" vector of all the inertial position vectors. Note that $Te = 0$ where e is a vector of ones. The *formation state vector* $x \in \mathbb{R}^{6(n_s-1)}$, consisting of the relative positions and velocities selected by T, is related to the inertial formation state vector by

$$x = (I_2 \otimes (I_3 \otimes T))\xi = (I_6 \otimes T)\xi. \tag{4}$$

With this relationship, the formation dynamics are given by

$$\dot{x} = A_c x + B_c(u + w) \qquad \text{where} \tag{5}$$

$$\begin{aligned} u &= (I_3 \otimes T)\eta, \qquad & w &= (I_3 \otimes T)\theta \\ A_c &= A_0 \otimes I_{n_s-1}, \qquad & B_c &= B_0 \otimes I_{n_s-1}. \end{aligned} \tag{6}$$

Discretizing with time step Δt and a zero-order hold for the control input, we obtain

$$x_{k+1} = Ax_k + B(u_k + w_k) \qquad \text{where}$$

$$A = \underbrace{e^{A_0 \Delta t}}_{:= A_d} \otimes I_{n_s-1}, \quad B = \underbrace{\int_0^{\Delta t} e^{A_0(t-\tau)} B_0 d\tau}_{:= B_d} \otimes I_{n_s-1}. \tag{7}$$

Recall the sensing topology can vary arbitrarily over a finite number of specified topologies. Each sensing topology determines a distinct set of relative position measurements described by the *edge matrix*, $E \in \mathcal{Z}_+{}^{q\times n_s}$, where q is the number of relative position vectors measured. A *sensing link* exists between the ith and jth spacecraft if their relative position vector is measured. For each sensing link, a row is added to the edge matrix with l^{th} entry +1, the m^{th} entry -1 (assuming $m > l$), and zero otherwise. The measurement vector y is then given in terms of the inertial position vector as

$$y = (I_3 \otimes E)p.$$

Since all relative measurements can be expressed by means of the relative position vector r, we have $\mathrm{imE^T} \subset \mathrm{imT^T}$. This inclusion implies that there exists some matrix H such that $E = HT$. One such matrix is $H = ET^T(TT^T)^{-1}$, which gives

$$y = (I_3 \otimes HT)p = (I_3 \otimes H)(I_3 \otimes T)p = (I_3 \otimes ET^T(TT^T)^{-1})r.$$

Hence,

$$y = [I_3 \otimes ET^T(TT^T)^{-1} \quad 0_{3(n_s-1)}]x. \tag{8}$$

As a result, the discrete-time relative dynamics of the formation with switched sensing topology are

$$x_{k+1} = Ax_k + B(u_k + w_k) \tag{9}$$
$$y_k = C_{\mathcal{T}(k)}x_k + v_k, \qquad \mathcal{T}(k) \in \mathcal{S} \tag{10}$$

where $\mathcal{S} = \{1, 2, ..., q_s\}$ is the index set of sensing topologies, q_s is the number of sensing topologies, $\mathcal{T} : \mathcal{Z}_+ \to \mathcal{S}$ maps the time index k into the sensing topology,

$$C_i = [I_3 \otimes E_iT^T(TT^T)^{-1} \quad 0_{3(n_s-1)}], \quad i \in \mathcal{S}, \tag{11}$$

and the process and measurement noise vectors are zero mean independent random vectors with

$$\mathrm{E}\left\{v_kv_k^T\right\} = R_{\mathcal{T}(k)} > 0 \quad \text{and} \quad \mathrm{E}\left\{w_kw_k^T\right\} = Q \geq 0.$$

In addition to the edge matrix E_i, a sensing topology can be uniquely specified by the graph Laplacian $\mathcal{L}_i$, where

$$\mathcal{L}_i = E_i^TE_i, \qquad i = 1, ..., q_s. \tag{12}$$

Intuitively, a sensing topology must be connected for the formation dynamics (C_i, A) to be observable. From graph theory, a sensing topology is connected if and only if $\mathrm{sgn}(|\mathcal{L}_{\mathrm{i}}^{\mathrm{n_s}-1}|) \succ 0$, which leads to the following result.

Lemma 1. The pair (C_i, A) is observable, where C_i and A are given by (11), (7), and (2), if the sensing graph corresponding to the matrix C_i is connected and $\omega\Delta t \in [0, 2\pi)$. □

Proof: First we show that $\ker H_i = \{0\}$ where $H_i = E_iT^T(TT^T)^{-1}$. Suppose that $H_iv = 0$ for some v. Since T^T is one-to-one, $w \neq 0$ when $v \neq 0$ where $w := T^T(TT^T)^{-1}v$. Now suppose that $E_iw = 0$ that is $w^TE_i^TE_iw = w^T\mathcal{L}_iw = 0$. Since $\mathcal{L}_i$ corresponds to a connected graph, it has 0 as a non-repeating eigenvalue with $e = [1, ..., 1]^T$ as the corresponding eigenvector, and all the other eigenvalues are positive [Deo (1974)]. This implies that $w = \alpha e$ for some scalar α. If $\alpha \neq 0$, this implies that, since $w = T^T(TT^T)^{-1}v$, there must be some vector z such that $e = T^Tz$, which implies that $TT^Tz = Te$. Note that $Te = 0$, which can easily be obtained by noting that the relative positions of point which are all the same location is zero vectors. Hence $TT^Tz = 0$. Since TT^T is invertible, this implies that $z = 0$, which leads to a contradiction proving that $\alpha = 0$. Hence $w = 0$ and then $v = 0$. Hence $\ker H_i = \{0\}$. This implies that $\ker I_3 \otimes H_i = \{0\}$. Consequently $C_ix = 0$ implies that $x_1 = 0$ where $x = [x_1^T, \ x_2^T]^T$. Now consider C_iAx for $x = [0, \ x_2^T]^T$. Partitioning matrix A_d in (7) into square blocks as follows

$$A_d = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}$$

$C_iAx = (I_3 \otimes H_i)(A_2 \otimes I_{n_s-1})x_2$. Since $\ker H_i = \{0\}$, this implies that $C_iAx = 0$ for some $x_2 \neq 0$ if and only if A_2 is singular. Note that $A_2 = I_3$ when $\omega = 0$. For $\omega\Delta t \in (0, 2\pi)$ $\det A_2 = 0$ if and only if $g(\omega\Delta t) = 0$ (see p.112 in Kaplan (1976) for an expression of A_2 that leads to this observation) where

$$g(\theta) := \theta\sin(\theta)(4\sin(\theta)/(\theta) - 3) + 4(1 - \cos(\theta))^2.$$

Since $g(\theta) > 0$ for all $\theta \in (0, 2\pi)$ (can be shown simply by evaluating it), A_2 is nonsingular. Hence

$$\ker \begin{bmatrix} C_i \\ C_iA \end{bmatrix} = \{0\},$$

which implies the observability of the pair (C_i, A). ■

3. ESTIMATOR ANALYSIS AND SYNTHESIS

In this section we introduce an algorithm to estimate the formation state vector x_k of (9). The estimation algorithm is developed for a more general class of systems of the form

$$\begin{aligned} x_{k+1} &= A_\tau x_k + B_\tau u_k + G_\tau w_k \\ y_k &= C_\tau x_k + v_k \end{aligned} \qquad \tau = \mathcal{T}(k), \tag{13}$$

where $\mathcal{T} : \mathcal{Z}_+ \to \mathcal{S}$ maps the time index k into the index set $\mathcal{S} = \{1, 2, \ldots, M\}$, x_k is the state vector having random initial condition x_0 with a mean $\bar{x}_0$ and variance P_0, y_k is the measured output vector, u_k is the vector of known inputs, and w_k and v_k are zero mean and independent random vectors with

$$\mathrm{E}\{w_kw_l\} = \delta_{kl}Q_{\mathcal{T}(k)}, \qquad \mathrm{E}\{v_kv_l\} = \delta_{kl}R_{\mathcal{T}(k)}$$

where δ_{kl} is the Kronecker delta

$$\delta_{kl} = \begin{cases} 1 & \text{when } k = l \\ 0 & \text{otherwise} \end{cases}.$$

For the formation dynamics (9), $M = q_s$, $A_\tau = A$, $B_\tau = G_\tau = B$, and $Q_\tau = Q$ for $\tau = 1, \ldots, M$. Our objective is to design a linear estimator for the state x_k, which is a random vector for each k, of the following form:

$$\hat{x}_{k+1} = A_\tau\hat{x}_k + L_\tau(C_\tau\hat{x}_k - y_k) + B_\tau u_k \quad \text{where} \quad \tau = \mathcal{T}(k) \tag{14}$$

where L_τ is the *estimator gain matrix*. Let $e_k := \hat{x}_k - x_k$ be the estimation error. Its propagation is given by

$$e_{k+1} = (A_\tau + L_\tau C_\tau)e_k - L_\tau v_k - G_\tau w_k. \tag{15}$$

Letting $P_k := \mathrm{E}\left\{(e_k - \bar{e}_k)(e_k - \bar{e}_k)^T\right\}$ be the estimation error covariance matrix and noting that

$$\bar{e}_{k+1} = (A_\tau + L_\tau C_\tau)\bar{e}_k, \tag{16}$$

we have

$$e_{k+1} - \bar{e}_{k+1} = (A_\tau + L_\tau C_\tau)(e_k - \bar{e}_k) - L_\tau v_k - G_\tau w_k.$$

Since e_k depends on the process and measurement noise vectors for only time steps $0, ..., k-1$,

$$\mathrm{E}\left\{(e_k - \bar{e}_k)v_k^T\right\} = 0 \quad \text{and} \quad \mathrm{E}\left\{(e_k - \bar{e}_k)w_k^T\right\} = 0.$$

The previous two relations imply

$$P_{k+1} = (A_\tau + L_\tau C_\tau)P_k(A_\tau + L_\tau C_\tau)^T + L_\tau R_\tau L_\tau^T + G_\tau Q_\tau G_\tau^T. \tag{17}$$

The following definition describes a class of estimators that have the properties stated in the Introduction.

Definition 1. For $\lambda \in [0, 1]$, $P = P^T > 0$, and any switching function $\mathcal{T}$, a filter of the form (14) is a λ-*estimator with ultimate covariance* P for the system (13) if

(1) For any e_0

$$\lim_{k \to \infty} \bar{e}_k = 0 \quad \text{and} \quad \exists c > 0 \text{ s.t. } \|\bar{e}_k\| \leq c\lambda^k\|\bar{e}_0\| \tag{18}$$

(2) For any e_0 and P_0 the covariance sequence $\{P_k\}_{k=0}^\infty$ is bounded and

$$\forall \varepsilon > 0, \ \exists n \geq 1 \text{ s.t. } P_k \leq P + \varepsilon I \quad \forall k \geq n, \tag{19}$$
$$P_k \leq P \quad \text{for} \quad k > 0 \text{ when } P_0 \leq P. \tag{20}$$

□

For any ultimate covariance P, the covariance $Y = Y^T \geq P$ is also an ultimate bound. Therefore, since $P > 0$, there exists an infimal ultimate covariance for any λ-estimator. This observation leads to the definition of an optimal λ-estimator.

Definition 2. A λ-estimator with ultimate covariance P is *optimal* if, for any other λ-estimator with ultimate covariance Q, $\mathrm{tr}P \leq \mathrm{tr}Q$. □

Remark 1. If a system (13) has singleton $\mathcal{S}$ (no switching), (C,A) detectable, and $(A, GQ^{1/2})$ reachable, then the optimal 1-estimator is the steady-state Kalman filter. □

The next two theorems establish sufficient conditions for the existence of a λ-estimator for the system (13). The first theorem considers $\lambda \in [0,1)$ and the second, $\lambda = 1$. The majority of proofs are omitted for brevity.

Theorem 1. Given $\lambda \in [0,1)$ and $P = P^T > 0$, a filter of the form (14) with gain matrices $L_1,...,L_M$ is a *λ-estimator with ultimate covariance P for the system (13)* if the following matrix inequalities are satisfied with some $F = F^T > 0$ for $i = 1,...,M$,

$$P - (A_i + L_iC_i)P(A_i + L_iC_i)^T - G_iQ_iG_i^T - L_iR_iL_i^T > 0 \quad (21)$$

$$\lambda^2 F - (A_i + L_iC_i)F(A_i + L_iC_i)^T \geq 0. \quad (22)$$

Further, for any initial covariance P_0,

$$P_k - P \leq c^2\lambda^{2k}|||P_0 - P|||I \qquad \text{for } k \geq 1 \quad (23)$$

where $c = \sqrt{\frac{\lambda_{max}(F)}{\lambda_{min}(F)}}$ is the c of condition (18). □

Proof: A candidate Lyapunov function for the mean error dynamics (16) is

$$V_k = \bar{e}_k^T F^{-1}\bar{e}_k.$$

Then,

$$\begin{aligned}\lambda^2 V_k - V_{k+1} = \bar{e}_k^T[\lambda^2 F^{-1} - \\ \left(A_{\mathcal{T}(k)} + L_{\mathcal{T}(k)}C_{\mathcal{T}(k)}\right)^T F^{-1}\left(A_{\mathcal{T}(k)} + L_{\mathcal{T}(k)}C_{\mathcal{T}(k)}\right)]\bar{e}_k\end{aligned} \quad (24)$$

where $\mathcal{T}(k) \in \mathcal{S}$. By using Schur complements twice, the inequality (22) is equivalent to

$$\lambda^2 F^{-1} - (A_i + L_iC_i)^T F^{-1}(A_i + L_iC_i) \geq 0 \qquad \forall i \in \mathcal{S}.$$

Since $\mathcal{T}(k) \in \mathcal{S}$, the previous two inequalities imply all solutions of the mean error dynamics satisfy

$$V_{k+1} \leq \lambda^2 V_k, \qquad \forall k \geq 0.$$

Consequently,

$$\begin{aligned} V_k \leq \lambda^{2k}V_0 \;\Rightarrow\; \tfrac{\|\bar{e}_k\|^2}{\lambda_{max}(F)} \leq \lambda^{2k}\tfrac{\|\bar{e}_0\|^2}{\lambda_{min}(F)} \\ \Rightarrow\; \|\bar{e}_k\| \leq c\lambda^k\|\bar{e}_0\| \;\text{ where }\; c = \sqrt{\tfrac{\lambda_{max}(F)}{\lambda_{min}(F)}}.\end{aligned} \quad (25)$$

This proves the satisfaction of the condition (18).

Suppose that, for $n \geq 0$, $P_n \leq P$ that is $P = P_n + H$ for some $H = H^T \geq 0$. Then, by using the inequality (21), there is some $i \in \mathcal{S}$ such that

$$P_{n+1} = A_{ci}(P - H)A_{ci}^T + S_i \leq P - A_{ci}HA_{ci}^T \leq P$$

where

$$A_{ci} = A_i + L_iC_i \qquad \text{and} \qquad S_i = L_iR_iL_i^T + G_iQ_iG_i^T.$$

When $P_0 \leq P$, the above implies (by induction) that $P_k \leq P$ for all $k \geq 0$. Hence, if $P_n \leq P$ for some $n \geq 0$ then $P_k \leq P$ for all $k \geq n$.

Now consider the case when $P_0 \geq P$ and let $\Delta_k := P_k - P$. Since $P = P^T > 0$, there exists some $r > 1$ such that $rP \geq P_0$. Since P satisfies the inequality (21) and $r > 1$

$$rP \geq A_{ci}(rP)A_{ci}^T + rS_i \geq A_{ci}(rP)A_{ci}^T + S_i.$$

Hence rP satisfies the inequality (21) for all $r \geq 1$. Furthermore, by using the earlier arguments

$$P_k \leq rP \qquad \forall k \geq 0.$$

This proves the boundedness of P_k, $k = 0,1,...$ (also note that $P_k \geq 0$). Now we claim that, for any solution of the error dynamics (15) there exists some $n \geq 1$ such that $P_n \leq P$. This will be proved by contradiction. Suppose this is not the case, that is, for any integer $n \geq 1$ there exists some vector $x \neq 0$ such that $x^T\Delta_n x > 0$. Observe that the inequality (21) implies that there exists some $\alpha > 0$ such that

$$P - A_{ci}PA_{ci}^T - S_i \geq \alpha I \qquad \forall i \in \mathcal{S}. \quad (26)$$

Let $\tilde{A}_k := A_{\mathcal{T}(k)} + L_{\mathcal{T}(k)}C_{\mathcal{T}(k)}$,

$$\Delta_{k+1} = \tilde{A}_kP_k\tilde{A}_k^T + G_{\mathcal{T}(k)}Q_{\mathcal{T}(k)}G_{\mathcal{T}(k)}^T + L_{\mathcal{T}(k)}R_{\mathcal{T}(k)}L_{\mathcal{T}(k)}^T - P.$$

Since $P_k = \Delta_k + P$ and $\tilde{A}_k = A_{ci}$ for some i and $\mathcal{T}(k) \in \mathcal{S}$, the above equality with the inequalities (26) imply that, for any $\mathcal{T}$,

$$\Delta_{k+1} \leq \tilde{A}_k\Delta_k\tilde{A}_k^T - \alpha I, \qquad \forall k \geq 0.$$

Then, for any function $\mathcal{T} : \mathcal{Z}_+ \to \mathcal{S}$,

$$\begin{aligned}
\Delta_1 &\leq \tilde{A}_0\Delta_0\tilde{A}_0^T - \alpha I \\
\Delta_2 &\leq \tilde{A}_1\Delta_1\tilde{A}_1^T - \alpha I \\
&\leq \tilde{A}_1(\tilde{A}_0\Delta_0\tilde{A}_0^T - \alpha I)\tilde{A}_1^T - \alpha I \leq \tilde{A}_1\tilde{A}_0\Delta_0\tilde{A}_0^T\tilde{A}_1^T - \alpha I \\
&\vdots \quad \vdots \\
\Delta_k &\leq \tilde{A}_{k-1}...\tilde{A}_0\Delta_0\underbrace{\tilde{A}_0^T...\tilde{A}_{k-1}^T}_{:=\Gamma_k} - \alpha I.
\end{aligned}$$

Here for any vector $x \neq 0$, define the following dynamics

$$x_{k+1} = \tilde{A}_k^T x_k \qquad \text{with} \quad x_0 = x. \quad (27)$$

Since all the solutions of the mean error dynamics exponentially converge to the origin as shown earlier, where the mean error dynamics can be expressed as $\bar{e}_{k+1} = \tilde{A}_k\bar{e}_k$, all the solutions of the system (27) also converge exponentially to the origin, which can easily be proven with Lyapunov function $V_k = x_k^T F x_k$. In particular, for any $x \neq 0$

$$\|x_k\| \leq c\lambda^k\|x\|.$$

Note that $x_k = \Gamma_k x$. This implies that

$$|||\Gamma_k||| \leq c\lambda^k.$$

Hence

$$\Delta_k \leq \Gamma_k^T\Delta_0\Gamma_k - \alpha I \leq (c^2\lambda^{2k}|||\Delta_0||| - \alpha)I. \quad (28)$$

Since there exists some $n_* \geq 1$ such that $c^2\lambda^{2k}|||\Delta_0||| < \alpha$ for $k \geq n_*$, we have

$$x^T\Delta_k x < 0 \qquad \forall x \neq 0 \qquad \text{when } k \geq n_*.$$

Consequently, this establishes the contradiction, hence proving that there exists some $n \geq 1$ such that $P_k \leq P$ for all $k \geq n$. The inequality (28) also proves the condition (23). ■

Theorem 2. Given $P = P^T > 0$, a filter of the form (14) with gain matrices $L_1,...,L_M$ is a *1-estimator with ultimate covariance P for system (13)* if matrix inequalities (21) are satisfied. □

Proof: The inequality (21) implies that there exists a small enough positive scalar $1 > \varepsilon > 0$ such that, for $i = 1,...,M$,

$$P - (A_i + L_iC_i)P(A_i + L_iC_i)^T - B_iQ_iB_i^T - L_iR_iL_i^T \geq \varepsilon P.$$

Since $L_iR_iL_i^T \geq 0$ and $B_iQ_iB_i^T \geq 0$, this implies that

$$\underbrace{(1-\varepsilon)}_{\lambda}P - (A_i + L_iC_i)P(A_i + L_iC_i)^T \geq 0, \qquad (29)$$

where $\lambda \in (0,1)$. Now the inequalities (29) and (21) are same as the inequalities (21) and (22) with F replaced by P, hence the conclusions of Theorem 3 apply with some $\lambda \in (0,1)$. ∎

An LMI approach to design λ-estimator is given as follows.

Theorem 3. Given $\lambda \in [0,1)$, suppose there exist matrices $S = S^T > 0, X = X^T > 0$, and Y_i, $i = 1,...M$, such that the following LMIs hold:

$$\begin{bmatrix} S & SA_i+Y_iC_i & Y_iR_i^{1/2} & SG_iQ_i^{1/2} \\ A_i^TS+C_i^TY_i^T & S & 0 & 0 \\ R_i^{1/2}Y_i^T & 0 & I & 0 \\ Q_i^{1/2}G_i^TS & 0 & 0 & I \end{bmatrix} > 0, \quad i=1,...M \qquad (30)$$

$$\begin{bmatrix} \lambda^2X & A_i^TS+C_i^TY_i^T \\ SA_i+Y_iC_i & 2S-X \end{bmatrix} \geq 0, \quad i=1,...M. \qquad (31)$$

Then the filter of the form (14) is a λ-estimator for system (13) with ultimate covariance P and gains L_i, $i = 1,...m$, given by

$$L_i = S^{-1}Y_i \qquad \text{and} \qquad P = S^{-1}. \qquad (32)$$

Condition (23) is also satisfied with $F = X^{-1}$. □

Proof: First we prove that the satisfaction of LMIs (30) implies the satisfaction of the inequalities (21). To do that pre and post-multiply (30) by $\mathrm{diag}(P,I,I,I)$ where $P = S^{-1}$ and let $L_i = PY_i$ to obtain

$$\begin{bmatrix} P & (A_i+L_iC_i) & L_iR_i^{1/2} & G_iQ_i^{1/2} \\ (A_i+L_iC_i)^T & P^{-1} & 0 & 0 \\ R_i^{1/2}L_i^T & 0 & I & 0 \\ Q_i^{1/2}G_i & 0 & 0 & I \end{bmatrix} > 0, \quad i = 1,..,M.$$

Using Schur complements twice, the matrix inequality above can be reduced to

$$\begin{bmatrix} P - L_iR_iL_i^T - G_iQ_iG_i^T & (A_i+L_iC_i) \\ (A_i+L_iC_i)^T & P^{-1} \end{bmatrix} > 0 \;\Rightarrow$$

$$P - (A_i+L_iC_i)P(A_i+L_iC_i)^T - L_iR_iL_i^T - G_iQG_i^T > 0, \; i = 1,...,M.$$

To prove that the LMIs (31) imply the inequalities (22), pre and post-multiply each of the LMIs (31) by $[I \;\; (A_i+L_iC_i)]$, $i = 1,...,M$, to obtain

$$\lambda^2X - (A_i+L_iC_i)^TX(A_i+L_iC_i), \quad i = 1,...,M.$$

By using Schur complements several times, the above inequalities imply

$$\begin{bmatrix} \lambda^2X & (A_i+L_iC_i)^T \\ (A_i+L_iC_i) & X^{-1} \end{bmatrix} \geq 0, \; i = 1,..,M$$

$$\Rightarrow \; \lambda^2F - (A_i+L_iC_i)F(A_i+L_iC_i)^T \geq 0, \; i = 1,..,M,$$

where $F = X^{-1}$. Now we can conclude the proof by using Theorem 1. ∎

The following is a specialized LMI result for $1-$estimators and its proof follows the same steps as in the proof of Theorem 3.

Theorem 4. Suppose there exist matrices $S = S^T > 0, X = X^T > 0$, and Y_i, $i = 1,...M$, such that the LMIs (30) hold for $\lambda = 1$. Then the filter of the form (14) is a *1-estimator* for the system (13) with ultimate covariance P and gains L_i, $i = 1,...M$, given by (32). □

These results suggest the following optimization problem to design λ-estimators.

$$\max_{S,X,Y_i} \mathrm{tr}S \quad \text{subject to}$$
$$S = S^T > 0, \;\; X = X^T > 0, \;\; \text{and} \qquad (33)$$
$$\text{LMIs (30) and, when } \lambda \in [0,1), \text{(31).}$$

Since the LMI conditions are only sufficient, the resulting ultimate covariance is suboptimal in general. However, the following corollary establishes that the LMI optimization problem (33) produces the optimal 1-estimator [1] .

Corollary 1. Suppose that there exists a feasible solution of the inequalities (21) for $P = P^T > 0$ and $L_1,...,L_M$. Then the solution of the optimization problem (33) gives the *optimal 1-estimator* for system (13). □

4. ESTIMATION OVER CONNECTED SENSING TOPOLOGIES WITH ARBITRARY ADDITIONAL LINKS

In this section the previous design techniques are extended to the case in which sensing topologies are not known a priori. However, it is assumed that there are *persistent, connected sensing topologies* with edge matrices E_i, $i = 1,...,q_s$, such that the actual sensing topology at any given time contains one of these persistent topologies as a sub-graph. Equivalently, at any time step k, we have $\mathrm{imE_i^T} \subset \mathrm{imE_k^T}$ for some $1 \leq i \leq q_s$. The following theorem shows a λ-estimator exists for a spacecraft formation with connected sensing graphs.

Theorem 5. Consider the system (9) with $\omega\Delta t \in [0,2\pi)$ and measurements (10) in which the C_i given by (11) result from connected sensing graphs. Then there exists a λ-estimator with some ultimate covariance $P = P^T > 0$ for any $\lambda \in [0,1]$.

Proof: Since each C_i defines a connected graph, for any C_i and C_j, it is straight forward to show that there exists some V_{ij} such that $C_i = V_{ij}C_j$. Pick any of the sensing tree and consider the corresponding C_i with R_i. Since (C_i,A) is observable, we can place the poles of $A+L_iC_i$ in any circle in the complex plane. So pick L_i such that $\sigma(A+L_iC_i) < \lambda$, which implies that $\sigma((A+L_iC_i)/\lambda) < 1$, which is equivalent to the existence of $F = F^T > 0$ satisfying

$$F - \frac{(A+L_iC_i)}{\lambda}F\frac{(A+L_iC_i)^T}{\lambda} > 0$$
$$\Rightarrow \; \lambda^2F - (A+L_iC_i)F(A+L_iC_i)^T > 0.$$

Since $C_i = V_{ij}C_j$, $L_iC_i = L_iV_{ij}C_j$, which implies that $A+L_iC_i = A+L_jC_j$ with $L_j = L_iV_{ij}$. Then choosing all L_j by using L_i in this manner, the matrix inequalities (22) are satisfied with F and L_j, $j = 1,..,q_s$.

Now since $\tilde{A} := A+L_iC_i = A+L_jC_j$ for any i and j, and $\tilde{A}$ has its eigenvalues strictly in the unit circle,

$$P - \tilde{A}P\tilde{A}^T > W$$

has a positive definite solution for $P = P^T$ when $W = W^T \geq 0$. Let W be such that

$$W \geq GQG^T + L_iR_iL_i^T.$$

[1] The proof of Corollary 1 is involved and it is omitted for brevity.

Then, P is a feasible solution of the inequality (21). Now the proof is completed by using Theorem 1 for $\lambda \in [0,1)$ or Theorem 2 for $\lambda = 1$. ■

Each measurement can now be decomposed into two parts:

- $y_k = C_k x + v_k$ with C_k corresponding to one of the persistent connected sub-graphs, and
- Additional measurements beyond those available in the current persistent sub-graph given by
$$z_k = H_k x_k + n_k \quad \text{where} \quad \mathrm{E}\left\{n_k n_k^T\right\} = N_k \qquad (34)$$
such that $[C_k^T \;\; H_k^T]^T$ gives the full measurement vector.

The random vectors v_k and n_k are independent. The persistent sub-graphs characterized by C_k are used to design a λ-estimator. Existence of a λ-estimator is guaranteed by Theorem 5. The H_k are unknown a priori and are determined in real-time as measurements become available.

To incorporate the "opportunistic" information z_k, the filter form (14) is augmented to
$$\hat{x}_{k+1} = A_\tau \hat{x}_k + L_\tau(C_\tau \hat{x}_k - y_k) + K_k(H_k \hat{x}_k - z_k) + B_\tau u_k, \qquad (35)$$
where $\tau = \mathcal{T}(k)$. The following corollary (see the Appendix for a proof) of Theorems 1 and 2 establishes the theoretical basis for this filter form and specifies the *opportunistic gain matrix* K_k.

Corollary 2. Given $\lambda \in [0,1)$, $P = P^T > 0$, and possible, additional measurements (34), a filter of the form (35) with gain matrices $L_1,\ldots,L_M$ is a *λ-estimator with ultimate covariance P and opportunistic measurements for the system (13)* if there exist γ_i such that following matrix inequalities are satisfied for $i = 1,..,M$,
$$\begin{aligned} &\lambda^2 P - (A_i + L_i C_i) P (A_i + L_i C_i)^T - G_i Q_i G_i^T - L_i R_i L_i^T \geq \gamma_i I, \\ &\gamma_i \geq 0 \text{ for } \lambda \in [0,1) \text{ and } \gamma_i > 0 \text{ for } \lambda = 1, \end{aligned} \qquad (36)$$
where the opportunistic gain matrix is given by
$$K_k = -\tilde{A}_k P H_k^T (H_k P H_k^T + N_k)^{-1}, \qquad (37)$$
where $\tilde{A}_k = A_{\mathcal{T}(k)} + L_{\mathcal{T}(k)} C_{\mathcal{T}(k)}$. Further, if the estimator exists, then the error covariance satisfies the inequality (23) when $\lambda \in [0,1)$. Finally, let $\{\hat{P}_k\}_{k=0}^{\infty}$ be the sequence of error covariance matrices when $H_k \equiv 0 \Rightarrow K_k \equiv 0$. Then,
$$P_k \;\leq\; \hat{P}_k, \qquad k \geq 0. \qquad (38)$$
□

Remark 2. Corollary 2 is used to set up the following optimization problem to obtain estimator gains as in Theorem 3.

$$\begin{aligned} &\max_{S,Y_i} \operatorname{tr} S \quad \text{subject to} \\ &S = S^T > 0, \;\text{ and} \\ &\text{LMIs (30) with } S \text{ in } 1\times 1 \text{ block diagonal} \\ &\text{entry replaced by } \lambda^2 S. \end{aligned} \qquad (39)$$

Clearly the ultimate covariance obtained from the design procedure above, which is S^{-1}, is at least as large as the ultimate covariance obtained from the optimization problem (33). They are guaranteed to be identical only for $\lambda = 1$. Hence, the estimator performance can suffer by using the measurements in an opportunistic fashion when $\lambda < 1$. ◇

The inequality (38) shows that including opportunistic measurements will not reduce performance. A key step in the proof of Corollary 2, namely, that conditions (23) and (38) are satisfied, is showing that the inequalities (36) imply
$$\lambda^2 P - (A_k + L_k C_k) P (A_k + L_k C_k)^T - G_k Q_k G_k^T - L_k R_k L_k^T \geq 0 \quad (40)$$
for all $k \geq 0$ and $\lambda \in [0,1)$ (similar result for $\lambda = 1$). Then, the inequalities (40) combined with the choice of K_k in (37) implies
$$\begin{aligned} \lambda^2 P - (A_k + L_k C_k + K_k H_k) P (A_k + L_k C_k + K_k H_k)^T \\ - G_k Q_k G_k^T - L_k R_k L_k^T - K_k N_k K_k^T \geq 0, \end{aligned}$$
for all $k \geq 0$, which leads to the corollary.

Corollary 2 can be repeatedly applied within a single time step. In particular, if z_k has a large dimension, then inverting the matrix $N_k + H_k P H_k^T$ is computationally expensive. To reduce computation, z_k can be partitioned and incorporated in smaller pieces within the same time step. More precisely, suppose that we have the following description of the additional measurements z_k
$$z_k = \left[\, z_{1,k}^T \;\ldots\; z_{p,k}^T \,\right]^T \quad \text{where} \quad z_{k,j} = H_{j,k} x_k + n_{j,k} \qquad (41)$$
where $n_{j,k}$, $j = 1,..,p$, are independent, zero mean random vectors with covariances $N_{j,k}$. Then, extending (35), the *opportunistic λ-estimator with partitioned update* is
$$\begin{aligned} &\hat{x}_{k+1} = A_\tau \hat{x}_k + L_\tau(C_\tau \hat{x}_k - y_k) + \sum_{i=1}^{p} K_{i,k}(H_{i,k}\hat{x}_k - z_{i,k}), \quad \tau = \mathcal{T}(k), \\ &K_{1,k} = -\tilde{A}_k P H_{1,k}^T (H_{1,k} P H_{1,k}^T + N_{1,k})^{-1}, \quad \tilde{A}_k = A_{\mathcal{T}(k)} + L_{\mathcal{T}(k)} C_{\mathcal{T}(k)} \\ &K_{i,k} = -\left(\tilde{A}_k + \sum_{l=1}^{i-1} K_{l,k} H_{l,k}\right) P H_{i,k}^T (H_{i,k} P H_{i,k}^T + N_{i,k})^{-1}, \quad i = 2,\ldots,p. \end{aligned} \qquad (42)$$

The requirement that partitions $z_{j,k}$ have independent measurement noise $n_{j,k}$ is not limiting for spacecraft formations. Relative position measurements can be partitioned based on the originating, physically-independent sensors.

4.1 Sensor Topology-Independent Formation Estimation

Corollary 2 suggests a design methodology for a universal formation estimator. Assume a formation's sensing topology is always connected as is required for observability. Then the sensing graph always contains a tree [Deo (1974)]. In practice then, a tree sub-graph is selected at every time step for y_k, and the remainder of measurements are collected into z_k. Now also assume that the relative position measurements between any two spacecraft have the same noise properties. This implies that for any tree in the sensing graph with the corresponding measurement $y_k = C_k x + n_k$, we have $\mathrm{E}\left\{n_k n_k^T\right\} = R = I_{n_s-1} \otimes R_0$ where $R_0 \in \mathbb{R}^{3\times 3}$ is the measurement error covariance matrix per relative position measurement. Since y_k has the same dimension for any tree, $\mathrm{E}\left\{n_k n_k^T\right\}$ is identical among different tree sub-graphs (but C_k can be different). Consider any two different trees with corresponding matrices C_i and C_j, then there exists an invertible matrix Λ_{ij} such that $C_i = \Lambda_{ij} C_j$ and $\Lambda_{ji} := \Lambda_{ij}^{-1}$. Note that $\Lambda_{ij} = \tilde{\Lambda}_{ij} \otimes I_3$ where $\tilde{\Lambda}_{ij} \in \mathbb{R}^{n_s-1 \times n_s-1}$. When these trees correspond to the same unlabeled graph, it can be shown that $\Lambda_{ji} = \Lambda_{ij}^T$. In this case, suppose an estimator gain L_i defines a $\lambda-$estimator for the measurement matrix C_i with an ultimate covariance P satisfying the inequality (36). Then $A + L_i C_i = A + L_j C_j$ where $L_j = L_i \Lambda_{ij}$. Additionally
$$\begin{aligned} L_j R L_j^T &= L_i \Lambda_{ij} R \Lambda_{ij}^T L_i^T \\ &= L_i(\tilde{\Lambda}_{ij} \otimes I_3)(I_{n_s-1} \otimes R_0)(\tilde{\Lambda}_{ij} \otimes I_3)^T L_i^T \\ &= L_i(\tilde{\Lambda}_{ij} \otimes R_0)(\tilde{\Lambda}_{ij} \otimes I_3) L_i = L_i(\tilde{\Lambda}_{ij}\tilde{\Lambda}_{ij}^T \otimes R_0) L_i^T = L_i R L_i^T. \end{aligned}$$

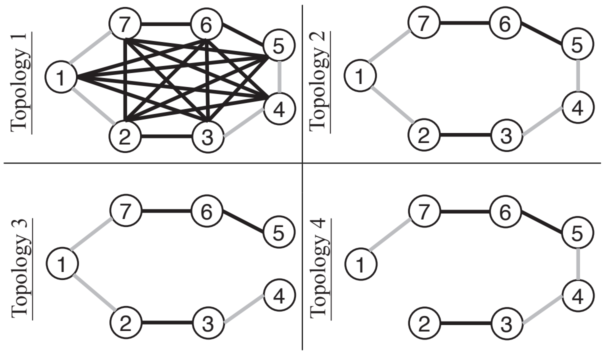

Fig. 1. Sensing Topologies for Seven-Spacecraft Formation. Links are solid lines, and gray links are noisier.

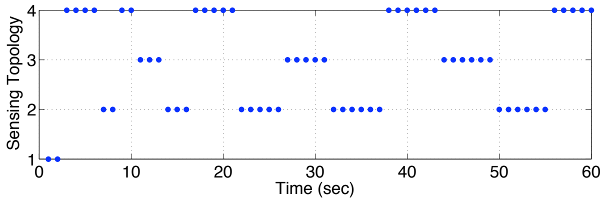

Fig. 2. Sensing-Topology Switching Sequence for Simulations.

This together with $A+L_iC_i = A+L_jC_j$ show that L_j with C_j also satisfies the inequality (36), that is, L_j defines a λ-estimator with ultimate covariance P as well. Note that this is not the case if C_i and C_j correspond to different unlabeled trees where $\Lambda_{ij}\Lambda_{ij}^T \neq I$. Hence, a λ-estimator can address *every connected sensing topology of a formation* by designing for all unlabeled trees only. Performance will necessarily be sub-optimal. However, considering ten spacecraft, there are 11,716,571 connected, unlabeled graphs, but only 106 unlabeled trees [Sloane (2007)], which is a considerable design simplification.

5. SIMULATION RESULTS

The preceding theory is demonstrated with simulations of a seven-spacecraft formation switching between four connected sensing topologies. The topologies are depicted in Figure 1. Sensing links are solid lines; the lighter-shaded lines had higher noise. Figure 2 provides the switching sequence used in the provided simulations. Comparisons are made between (i) LMI-designed λ-estimators, (ii) the Kalman filter, and (iii) the Switched Steady-State Kalman Filter (SSKF). The SSKF has the same form (14) as a λ-estimator, but with gains L_τ given by the steady-state Kalman gain corresponding to the instantaneous sensing topology. SSKF has no performance guarantees.

An uncertainty is assumed in the initial error covariance, which noticeably affects the Kalman filter. In this case, Figure 3 shows that the λ-estimator converges to its ultimate variance faster than the Kalman filter. The ultimate variance is slightly larger than Kalman due to the decay rate constraint of $\lambda = 0.9$. Figure 4 shows a case for $\lambda = 0.4$, which requires faster decay. The λ-estimator ultimate variance is significantly worse than for the Kalman filter. However, the mean error goes to zero much faster, as shown in Figure 5. These two figures illustrate the trade-off between convergence of the mean error and steady-state variance.

6. SUMMARY AND FUTURE WORK

A new class of computationally-efficient estimators, called λ-estimators, has been developed for switched, discrete-time linear systems. An explicit constraint on the convergence rate of

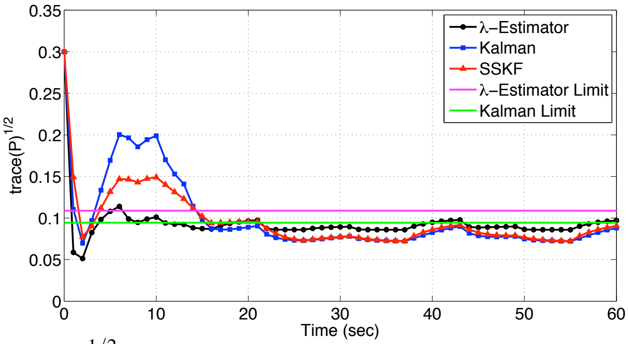

Fig. 3. $\mathrm{tr}P_k^{1/2}$, $\lambda = 0.9$.

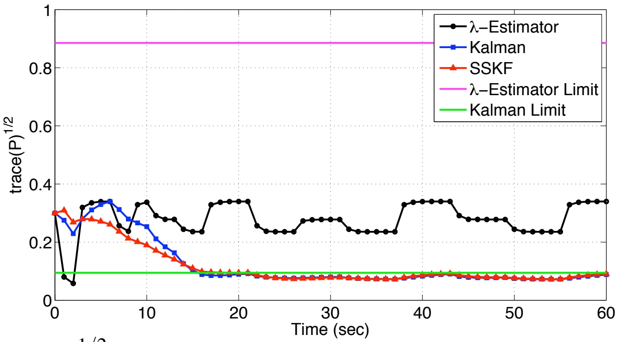

Fig. 4. $\mathrm{tr}P_k^{1/2}$, $\lambda = 0.4$.

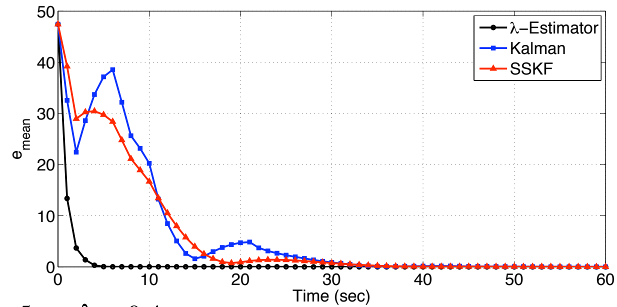

Fig. 5. $\bar{e}_k$, $\lambda = 0.4$.

the mean estimation error allows a trade-off between speed and an ultimate bound on variance. These estimators were then applied to spacecraft formations in deep space or near-circular planetary orbits that have time-varying sensor topologies. An LMI-based synthesis technique was described. Further, by dividing sensor measurements into persistent and opportunistic categories, sensor topology-independent λ-estimators can be designed efficiently for large formations. These formation estimators provide guaranteed performance under arbitrary sensor topology variations with significantly less computation than a traditional Kalman filter.

Future work includes: (i) incorporating dwell time constraints, which limit how fast topologies can vary, thereby reducing the ultimate variance, (ii) extending to time-varying state dynamics for formations in elliptical orbits, and (iii) developing time-based sequences of λ-estimators with increasing λ that allow faster convergence to smaller ultimate variances. Finally, delays due to communicated measurements are being included.

APPENDIX

Proof of Corollary 2

First note that the inequalities (36) for $\lambda = 1$ are the same as the inequalities (21). For $\lambda < 1$, the inequalities (36) are also same as the inequalities (21) and (22). To see this, first note that the inequalities (36) imply that, for $i = 1, .., M$,

$$P-(A_i+L_iC_i)P(A_i+L_iC_i)^T-B_iQ_iB_i^T-L_iR_iL_i^T \geq (1-\lambda)P > 0,$$

which are the inequalities (21). Also, letting $F=P$ the inequalities (36) clearly imply the satisfaction of the inequalities (22).

Now we follow similar steps as in the proof of Theorem 1 to prove the corollary for $\lambda \in [0,1)$. Let $V_k = \bar{e}_k^T P^{-1} \bar{e}_k$. Then,

$$\lambda^2 V_k - V_{k+1} = \bar{e}_k^T \left[\lambda^2 P^{-1} - \left(\tilde{A}_k + K_k H_k\right)^T P^{-1}(\tilde{A}_k + K_k H_k\right] \bar{e}_k,$$

where $\tilde{A}_k := A_{\mathcal{T}(k)} + L_{\mathcal{T}(k)} C_{\mathcal{T}(k)}$. By using Schur complements twice, the following inequality holds

$$\lambda^2 P^{-1} - \left(\tilde{A}_k + K_k H_k\right)^T P^{-1}(\tilde{A}_k + K_k H_k) \geq 0$$

if and only if

$$W_k := \lambda^2 P - \left(\tilde{A}_k + K_k H_k\right) P(\tilde{A}_k + K_k H_k)^T \geq 0.$$

By noting that $\tilde{A}_k = A_i + L_i C_i$ for some $i \in \{1,...,M\}$, the inequalities (36) imply that

$$\lambda^2 P - \tilde{A}_k P \tilde{A}_k^T \geq 0.$$

Here

$$W_k = \underbrace{\lambda^2 P - \tilde{A}_k P \tilde{A}_k^T}_{\geq 0} - Z_k$$

where

$$Z_k = \tilde{A}_k P H_k^T K_k^T + K_k H_k P \tilde{A}_k^T + K_k H_k P H_k^T K_k^T.$$

Note that

$$Z_k \leq Z_k + K_k N_k K_k^T. \tag{43}$$

By using the equality (37),

$$\begin{aligned} Z_k &\leq Z_k + K_k N_k K_k^T \\ &= -\tilde{A}_k P H_k^T (H_k P H_k^T + N_k)^{-1} H_k P \tilde{A}_k^T \leq 0 \;\Rightarrow\; W_k \geq 0. \end{aligned} \tag{44}$$

This inequality implies that

$$\lambda^2 V_k - V_{k+1} \geq 0.$$

Then, as done in the equation (25), we prove the decay properties of the vector $\bar{e}_k$.

Suppose that for $n \geq 1$, $P_n \leq P$ and $P = P_n + X$ where $X + X^T \geq 0$. Then,

$$\begin{aligned} P_{n+1} &= (\tilde{A}_n + K_k H_k)(P-X)(\tilde{A}_n + K_k H_k)^T + S_n + K_n N_n K_n^T \\ &= \tilde{A}_n (P-X) \tilde{A}_n^T + S_n + Z_n + K_n N_n K_n^T \\ &\leq \tilde{A}_n (P-X) \tilde{A}_n^T + S_n \\ &\leq \tilde{A}_n P \tilde{A}_n^T - \tilde{A}_n X \tilde{A}^T \\ &\leq \lambda^2 P \leq P \end{aligned}$$

where $S_n = L_{\mathcal{T}(n)} R_{\mathcal{T}(n)} L_{\mathcal{T}(n)}^T + B_{\mathcal{T}(n)} Q_{\mathcal{T}(n)} B_{\mathcal{T}(n)}^T$. By induction, this implies that $P_k \leq P$ for all $k \geq n$. Similarly we can prove that the sequence of error covariance matrices $\{P_k\}_{k=0}^{\infty}$ is bounded.

By using the inequalities (36), since $\mathcal{T}(k) \in \mathcal{S}$, there exists some $\alpha > 0$

$$P - \tilde{A}_k P \tilde{A}_k^T - S_k \geq \alpha I \qquad \forall k \geq 0.$$

Since

$$\begin{aligned} &P - (\tilde{A}_k + K_k H_k) P (\tilde{A}_k + K_k H_k)^T - S_k - K_k N_k K_k^T \\ &= P - \tilde{A}_k P \tilde{A}_k^T - S_k - \underbrace{(Z_k + K_k N_k K_k^T)}_{\leq 0} \geq \alpha I \\ &\Rightarrow P \geq \tilde{A}_k P \tilde{A}_k^T + S_k + \alpha I. \end{aligned}$$

Letting $\Delta_k := P_k - P$,

$$\begin{aligned} \Delta_{k+1} &= \tilde{A}_k P_k \tilde{A}_k^T + S_k + (Z_k + K_k N_k K_k^T) - P \\ &\leq \tilde{A}_k \Delta_k \tilde{A}_k^T - \alpha I. \end{aligned}$$

Once we have the above inequality, we can use the same arguments in the proof of Theorem 1 after the inequality (26) to conclude the decay properties of the estimator for $\lambda \in [0,1)$.

The recursive relationship for P_k can be written as

$$P_{k+1} = \tilde{A}_k P_k \tilde{A}_k^T + S_k + \Phi(K_k)$$

where

$$\Phi(K_k) = \Phi(K_k)^T := Z_k + K_k N_k K_k^T \leq 0.$$

Note that if $K_k = 0$ then $\Phi(K_k) = 0$. Now suppose that $\hat{P}_k \geq P_k$ (where P_k is the error covariance when $K_k \neq 0$). Then

$$\begin{aligned} P_{k+1} &= \tilde{A}_k P_k \tilde{A}_k^T + S_k + \Phi(K_k) \leq \tilde{A}_k \hat{P}_k \tilde{A}_k^T + S_k + \Phi(K_k) \\ &\leq \hat{P}_{k+1} + \Phi(K_k) \leq \hat{P}_{k+1}. \end{aligned}$$

Since $\hat{P}_0 = P_0$, by induction, this implies that

$$P_k \leq \hat{P}_k, \qquad \forall k \geq 0.$$

The proof of the case with $\lambda = 1$ uses the results for $\lambda \in [0,1)$ exactly as it is done in the proof of Theorem 2 which uses the results of Theorem 1.

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