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Stable Swarming by Mutual Interactions of Attraction/Alignment/Repulsion: Fixed Topology ⋆

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Abstract: In this paper we present a general decentralized controller for a swarm of mobile agents with fixed topology to move in a given environment. The controller utilizes the widely accepted hypothesis of Attraction/Alignment/Repulsion (A/A/R) interactions for fish schools in mathematical biology community. We assume that during the swarm's motion, each agent can sense and interact with its neighbors via A/A/R interactions, while follow the path clue of the environment. The environment is assumed to have identical effect on all agents. Under the assumptions of connected graph, the controller is proved to make the velocities of all swarm members asymptotically converge to a common value. The advantage of this controller is that all the information it needs can be locally sensed, therefore, communication link and associated issues (such as communication noise and time delay) are avoided. Simulations of a swarm with a fixed topology are presented to verify the proposed controller.

1. INTRODUCTION

The natural phenomena of swarming, such as schooling fish, have invoked intensive research interests in diverse areas for decades. The collective group behaviors are believed to have certain advantages over individual ones, for example, increasing the survival chances for the whole group under the danger from predators [1][2]. The inspiring point in these phenomena is that although the intelligence of the individual member is limited, the sophisticated group behavior can be achieved without a global coordinator.

Biologists have observed and analyzed the swarming behaviors of different species for decades [1]-[12]. Some interesting phenomena were first observed and recorded by them. For example, tuna shoals are observed to school together with a separation of 0.16-0.25 body length in shapes of 1D "soldier", 2D "surface", and 3D "ball" [3].

Two main approaches are used in the literature to model and analyze how the collective behaviors are self-organized in different environments. In reference to the Lagrangian and Eulerian descriptions of fluid motion, they are referred to as Eulerian and Lagrangian approaches. The Eulerian approach applies partial differential equations to describe the evolving swarm density [2][6]; while the Lagrangian approach uses certain individual-based rules or the classical Newtonian mechanics law to study the motion of swarm members [2][5][12][13].

The individual-based interaction rules used in most models of fish schools in the literature include short-distance repulsion, long-distance attraction and middle-range alignment (also called "parallel orientation") [10]-[13]. It is commonly believed that individual fish senses and adjusts its motion according to certain neighbors through the Attraction/Alighment/Repulsion (A/A/R) interactions [10]-[13].

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In [15] Jadbabaie *et al.* presented a discrete kinematic model and a decentralized averaging controller to prove the convergence of agents' velocities. In their later work [16]-[18], the authors used a continuous dynamic model and proposed a decentralized controller for fixed and dynamic topologies. The controller includes heading and velocity adjustment components, both of which are based on nearest neighbors' states. Further theoretical extensions of this work are presented in [25]. It is shown that consensus is achieved asymptotically if the union of the "information exchange graph" is connected. However, the controllers in [15]-[18] do not explicitly consider the environment.

In [20] Gazi used a continuous kinematic model for individual agents and proposed a decentralized controller for swarm aggregation in n-dimensional space. An explicit bound of the swarm size is also derived. The results in [20] are extended to a class of virtual force functions in [21]. Their later work [22]-[24] demonstrate the collective behavior of swarms moving in different environments. In [19], Liu *et al.* used a second-order dynamic model to study the stable foraging of swarms in certain noisy environments. However, all the controllers proposed in [19]-[24] require each agent to know the global states of all other members.

In this paper, we present a general decentralized controller for a swarm of mobile agents to achieve the collective group behavior in given environments. The controller utilizes the widely accepted concept of A/A/R mutual interactions [2][10]-[13]. We assume that each agent can sense and interact with its neighbors via A/A/R forces. Moreover, according to the biological facts that many natural species swarm in dynamic environments, such as reef fish school along ocean currents [8][9] and migrate birds flock to the south by the guidance of the earth's magnetic field [7], we assume that during the swarm's motion, each agent can perceive and follow the path clue of the environment. We explicitly consider the environment and assume that it has identical effects on all agents. The swarm's topological graph is assumed to be always connected. This paper discusses the case when the swarm's topology is fixed. By some tools of nonsmooth analysis theory [26]-[30], the controller is proved to enable all agents' velocities to asymptotically converge to a common value.

The advantage of the controller is that all the information it needs can be locally sensed, therefore, communication modules

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are not needed for individual agents. Subsequently, all issues related to communication links (such as time delay and communication noise) are avoided.

This paper is organized as follows. In section 2 we present a simplified dynamic model for individual agents and a graph representation for swarm's fixed topology. The general decentralized controller and its asymptotic stability analysis are illustrated in section 3. In section 4 a set of simulation result is presented. This paper ends with some conclusions in section 5.

2. MODELLING OF SWARMS WITH FIXED TOPOLOGY

Consider a swarm of N agents moving in a 2D or 3D Euclidean space. For simplicity, we do not consider each agent's dimension. We assume no disturbance upon each agent. For the $i^{th}(i = 1, 2, ..., N)$ agent in the swarm, its dynamics can be modelled as

$$\begin{aligned} \dot{r_i} &= v_i \\ \dot{v_i} &= u_i \end{aligned} \tag{1}$$

where $r_i \in \mathbb{R}^2$ or $\mathbb{R}^3$ is its position vector relative to ground coordinates, v_i is its velocity vector, and u_i is the control input.

Define

$$\bar{v} = \frac{1}{N}\sum_{i=1}^{N} v_i \tag{2}$$

to represent the average velocity of all swarm members. We will show that all agents' velocity vectors converge to $\bar{v}$ by the proposed controller.

Let

$$r_{ij} = r_i - r_j, \tag{3}$$

and $\|r_{ij}\| = \|r_{ij}\|_2$ is the distance between two agents i and j.

Let $r = [r_1^T, r_2^T, ..., r_N^T]^T$ and $v = [v_1^T, v_2^T, ..., v_N^T]^T$ represent the position and velocity vector of the whole swarm, respectively.

The swarm's topology can be represented by algebraic graph. According to how the information is exchanged among the agents, the graph embodies either communication or sensing relations of the swarm members. As shown in this paper, only local and relative sensing information are needed by the proposed controller, thus, we rather consider the swarm's topological graph as a sensing graph.

Definition 2.1 (Swarm's Topological Graph) The topological graph of a swarm with a fixed topology is an undirected graph, denoted $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$, consisting of:

(1) a set of vertices, $\mathcal{V} = \{1, ..., N\}$, indexed by the agents in the swarm;
(2) a set of edges, $\mathcal{E} = \{(i,j) \in \mathcal{V} \times \mathcal{V} \mid i \sim j\}$ ($\sim$ denotes adjacency), in which each undirected edge is given by initial condition and represents the sensing relation between two vertices.

Define

$$\mathbb{N}_i \triangleq \{j \mid (i,j) \in \mathcal{E}\} \subseteq \mathcal{V} \setminus \{i\} \tag{4}$$

to represent the set of agent i's neighbors.

We assume that during the swarm's motion, $\mathcal{G}$ is always connected. For swarms with fixed topology, the graph $\mathcal{G}$ and $\mathbb{N}_i$ are given by initial condition and keep unchanged.

3. CONTROLLER AND STABILITY ANALYSIS

In this section, we illustrate the general decentralized controller for swarms with fixed topology and prove that it can make all agents' velocities to asymptotically converge to a common value ($\bar{v}$).

The hypothesis of mutual interactions of A/A/R for fish schools has been widely accepted in mathematical biology community for decades [2][5][10][11][12]. A fish is generally assumed to adopt different interactions (attraction, repulsion, or alignment) according to the range in which the perceived neighbor fish is positioned. Fig.1 shows the diagram of two neighbored agents and the mutual interaction between them. The force vector $\overrightarrow{g}_{ij}$ is along the direction of r_{ij}, in which $\overrightarrow{g}_{ij} \triangleq g(\|r_{ij}\|)\frac{r_{ij}}{\|r_{ij}\|}$. The amplitude $g(\|r_{ij}\|)$ is a scalar function that only depends on the relative distance $\|r_{ij}\|$.

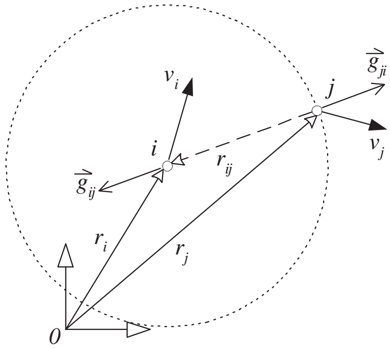

Fig. 1. Two neighbors (agent i and j) and their mutual interaction.

Depending on the relative distance between two neighbored agents, the interaction has different dominated effects. Fig.2 shows the three non-overlapping interaction zones associated with each agent, in which d_0, d_1 and d_2 are the respective radius. Since it is not expensive to implement a sensing or communication module that has an omnidirectional field of view by current technology, we assume that each agent has no any blind angle as in some models of fish schools in the literature [11]-[13].

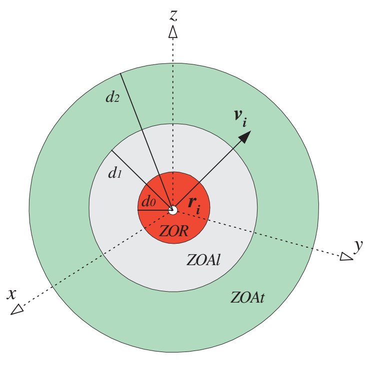

Fig. 2. Interaction zones associated with agent i: zone of repulsion (ZOR), zone of alignment (ZOAl), zone of attraction (ZOAt).

On the other hand, although each agent hardly has the full knowledge about the environment, it is still reasonable to assume that it knows about the local environment around its current position. This assumption can be justified by some phenomena in biological systems. For example, some tropical reef fish can perceive and ride along the ocean current [8][9]. Assume the swarm moves in an environment with a global potential function $J(r)$, and its gradient at r_i is denoted by $\nabla_{r_i} J(r)$. Although each agent hardly knows $J(r)$, but local information $\nabla_{r_i} J(r)$ is assumed to be known. We assume that the environment has identical effects on all agents, i.e., $\nabla_{r_i} J(r)$ is the same for $i = 1, ..., N$.

Based on the above discussion, we propose a general decentralized controller for each agent as

$$u_i = -k_p[v_i - \nabla_{r_i} J(r)] + \sum_{j\in\mathbb{N}_i} g(\|r_{ij}\|)\frac{r_{ij}}{\|r_{ij}\|}, \quad (5)$$

where $k_p > 0$ is a design constant, and $g(\|r_{ij}\|)$ is the amplitude of interactions between two neighbored agents. The implication of this controller is that each agent perceives and follows the environmental "path clue" ($\nabla_{r_i} J(r)$), and at the same time affects its neighbors via A/A/R interactions.

According to the A/A/R zones, for a swarm with fixed topology, in order to keep the whole group cohesive and avoid collision among swarm members, the mutual interactions should satisfy

$$g(\|r_{ij}\|) = \begin{cases} > 0 & 0 \le \|r_{ij}\| < d_0, \\ = 0 & d_0 \le \|r_{ij}\| \le d_1, \\ < 0 & \|r_{ij}\| > d_1. \end{cases} \quad (6)$$

For simplicity, we assume that $g(\|r_{ij}\|)$ is continuous inside each interaction zone, but may not be continuous along the boundaries. Moreover, let $g(\|r_{ij}\|) \neq \infty$.

Note that for swarms with fixed topology, each agent's neighborhood is given by initial condition and not changing. Hence, we do not need to consider the limit of the upper boundary (d_2) of attraction zone.

We will show that for a swarm with fixed topology, no matter which specific functions the mutual interactions are, the general controller (5) can make all agents' velocities asymptotically converge to a common value ($\bar{v}$), as long as the condition in (6) is satisfied.

Define error state

$$e_{v_i} = v_i - \bar{v}. \quad (7)$$

It is straightforward to have

$$\dot{v}_i - \dot{\bar{v}} = -k_p(v_i - \bar{v}) + \sum_{j\in\mathbb{N}_i} g(\|r_{ij}\|)\frac{r_{ij}}{\|r_{ij}\|} + k_p[\nabla_{r_i} J(r) - \frac{1}{N}\sum_{i=1}^{N}\nabla_{r_i} J(r)] - \frac{1}{N}\sum_{i=1}^{N}\sum_{j\in\mathbb{N}_i} g(\|r_{ij}\|)\frac{r_{ij}}{\|r_{ij}\|}.$$

Since $g(\|r_{ij}\|)\frac{r_{ij}}{\|r_{ij}\|} = -g(\|r_{ji}\|)\frac{r_{ji}}{\|r_{ji}\|}$, also because the graph $\mathcal{G}$ is assumed to be always connected and $\mathbb{N}_i$ is symmetric, so we have

$$\sum_{i=1}^{N}\sum_{j\in\mathbb{N}_i} g(\|r_{ij}\|)\frac{r_{ij}}{\|r_{ij}\|} = 0. \quad (8)$$

Thus,

$$\dot{v}_i - \dot{\bar{v}} = -k_p e_{v_i} + \sum_{j\in\mathbb{N}_i} g(\|r_{ij}\|)\frac{r_{ij}}{\|r_{ij}\|} + k_p[\nabla_{r_i} J - \frac{1}{N}\sum_{i=1}^{N}\nabla_{r_i} J(r)].$$

Since the swarm is assumed to move in an environment that has identical effect on each agent, i.e., $\nabla_{r_i} J(r) = \nabla_{r_j} J(r), \forall i \neq j$. For example [24], $J(r) = \sum_{i=1}^{N} J(r_i) = \sum_{i=1}^{N} a^T \cdot r_i + b$, where $a \in \mathbb{R}^n$ and $b \in \mathbb{R}$. Then $\nabla_{r_i} J(r) - \frac{1}{N}\sum_{i=1}^{N}\nabla_{r_i} J(r) = 0$. So we have

$$\dot{v}_i - \dot{\bar{v}} = -k_p e_{v_i} + \sum_{j\in\mathbb{N}_i} g(\|r_{ij}\|)\frac{r_{ij}}{\|r_{ij}\|}. \quad (9)$$

Define

$$E_{ij}(\|r_{ij}\|) = \int_{\|r_{ij}\|}^{d_0} g(\tau)d\tau. \quad (10)$$

Clearly,

$$E_{ij} = \begin{cases} \|r_{ij}\| < d_0 : & = \int_{\|r_{ij}\|}^{d_0} g(\tau)d\tau > 0, \\ d_0 \le \|r_{ij}\| \le d_1 : & = 0, \\ \|r_{ij}\| > d_1 : & = -\int_{d_1}^{\|r_{ij}\|} g(\tau)d\tau > 0. \end{cases} \quad (11)$$

We consider both cases when $g(\|r_{ij}\|)$ is continuous and discontinuous along the boundaries of interaction zones (i.e., d_0 and d_1).

Theorem 3.1 Consider a swarm of N mobile agents moving in an environment that has identical effects on all agents. The topological graph $\mathcal{G}$ of the swarm is given by initial condition and keeps fixed. Assume $\mathcal{G}$ is always connected. Then with any set of continuous interactions $g(\|r_{ij}\|)$ that satisfy the condition (6), the general decentralized controller (5) makes all agents' velocity asymptotically converge to a common value ($\bar{v}$).

Proof: For continuous $g(\|r_{ij}\|)$, the error dynamics in uniform environments is also continuous and

$$\dot{e}_{v_i} = -k_p e_{v_i} + \sum_{j\in\mathbb{N}_i} g(\|r_{ij}\|)\frac{r_{ij}}{\|r_{ij}\|}. \quad (12)$$

Since $g(\|r_{ij}\|)$ is continuous, E_{ij} is continuously differentiable and

$$\dot{E}_{ij} = -g(\|r_{ij}\|)[v_i^T \cdot \nabla_{r_{ij}}\|r_{ij}\| + v_j^T \cdot \nabla_{r_{ji}}\|r_{ji}\|]. \quad (13)$$

Use the candidate Lyapunov function

$$V_1 = \frac{1}{2}\sum_{i=1}^{N} e_{v_i}^T e_{v_i} + \frac{1}{2}\sum_{i=1}^{N}\sum_{j\in\mathbb{N}_i} E_{ij}(\|r_{ij}\|). \quad (14)$$

From (11) we know $V \geq 0$. And

$$\dot{V}_1 = -k_p\sum_{i=1}^{N} e_{v_i}^T e_{v_i} + \sum_{i=1}^{N}(v_i - \bar{v})^T \sum_{j\in\mathbb{N}_i} g(\|r_{ij}\|)\frac{r_{ij}}{\|r_{ij}\|} - \sum_{i=1}^{N}\sum_{j\in\mathbb{N}_i} g(\|r_{ij}\|)(v_i^T \cdot \nabla_{r_{ij}}\|r_{ij}\|) \quad (15)$$

Similarly to (8), we have $\sum_{i=1}^{N}\sum_{j\in\mathbb{N}_i} g(\|r_{ij}\|)\bar{v}^T \cdot \frac{r_{ij}}{\|r_{ij}\|} = 0$. So,

$$\dot{V}_1 = -k_p\sum_{i=1}^{N} e_{v_i}^T e_{v_i} + \sum_{i=1}^{N}\sum_{j\in\mathbb{N}_i} g(\|r_{ij}\|)(v_i^T \cdot \frac{r_{ij}}{\|r_{ij}\|}) - \sum_{i=1}^{N}\sum_{j\in\mathbb{N}_i} g(\|r_{ij}\|)(v_i^T \cdot \nabla_{r_{ij}}\|r_{ij}\|). \quad (16)$$

For a time variant vector $\boldsymbol{P} \in \mathbb{R}^n$, we know

$$\nabla_{\boldsymbol{P}}\|\boldsymbol{P}\| = \frac{\boldsymbol{P}}{\|\boldsymbol{P}\|}, \quad (17)$$

Therefore,

$$\dot{V}_1 = -k_p\sum_{i=1}^{N} e_{v_i}^T e_{v_i}. \quad (18)$$

Clearly $\dot{V}_1 \le 0$, then e_{v_i} is stable for any agent. Moreover, from LaSalle's invariance principle, we know that the error states will converge to the largest invariant set in which $e_{v_i} = 0$. This means that the agents' velocities will asymptotically converge to $\bar{v}$. □

For more general case, if $g(\|r_{ij}\|)$ is not continuous at d_0 and d_1, but still assumed to be continuous inside the interaction zone. We have the following theorem.

Theorem 3.2 Consider a swarm of N mobile agents with a fixed topology given by initial condition. Assume the environment has identical effects on all agents. Assume the topological graph $\mathcal{G}$ of the swarm is always connected. Then with any set of mutual interactions that satisfy the condition (6) and discontinuous at d_0 and d_1 but continuous inside the interaction zones, the decentralized controller (5) enables all agents' velocities to asymptotically converge to a common value ($\bar{v}$).

Proof: Due to the discontinuous mutual interactions, the error dynamics is non-smooth. Under the assumption that the environment has identical effects on all agents, we have the following differential inclusion [26][30] for the error dynamics :

$$\dot{e}_{v_i} \in^{a.e.} K[e_{v_i}] = -k_p e_{v_i} + \sum_{j\in\mathbb{N}_i} g(\|r_{ij}\|)\frac{r_{ij}}{\|r_{ij}\|}. \quad (19)$$

Because $g(\|r_{ij}\|)$ is not continuous at d_0 and d_1, then $E_{ij}(\|r_{ij}\|)$ is discontinuous and non-differentiable at d_0 and d_1. We have its general gradient [30] as

$$\partial E_{ij} = \begin{cases} \|r_{ij}\| < d_0 : & = -g(\|r_{ij}\|), \\ \|r_{ij}\| = d_0 : & = \overline{co}[-g(d_0^-),\ 0], \\ d_0 < \|r_{ij}\| < d_1 : & = 0 \\ \|r_{ij}\| = d_1 : & = \overline{co}[-g(d_1^+),\ 0], \\ \|r_{ij}\| > d_1 : & = -g(\|r_{ij}\|). \end{cases} \quad (20)$$

in which $\overline{co}[\ \cdot\ ]$ is the closed convex hull.

Use the candidate Lyapunov function

$$V_2 = \frac{1}{2}\sum_{i=1}^{N} e_{v_i}^T e_{v_i} + \frac{1}{2}\sum_{i=1}^{N}\sum_{j\in\mathbb{N}_i} E_{ij}(\|r_{ij}\|). \quad (21)$$

From (11) we know $V_2 \geq 0$. Clearly V_2 is smooth about e_{v_i}; but because of nonsmooth E_{ij}, it is not smooth about $\|r_{ij}\|$.

Because $g(\|r_{ij}\|) \neq \infty$, E_{ij} is locally Lipschitz, then V_2 is also locally Lipschitz. From Rademacher's Theorem [30], we know that it is differentiable almost everywhere. In order to derive the set-valued Lie derivative of V_2 [28][29], we need to show it is regular everywhere [30].

Lemma 3.1 The function V_2 in (21) is regular everywhere in its domain.

Proof: Because $e_{v_i}^T e_{v_i}$ is convex, it is regular [28]; and since for fixed topology, $\mathbb{N}_i$ is time-invariant for every i, thus, we just need to prove E_{ij} is regular in order to show V_2 is regular. Because E_{ij} is smooth everywhere except at d_0 and d_1, we only need to prove it is regular at d_0 and d_1. To show its regularity at d_0 and d_1, we need to prove $E_{ij}^{\circ}(d_1, w) = E_{ij}'(d_1, w)$ [30], where $E_{ij}'(d_1, w) = \lim_{h\downarrow 0}\frac{E_{ij}(d_1+hw)-E_{ij}(d_1)}{h}$ and $E_{ij}^{\circ}(d_1, w) = \lim_{y\to d_1}\sup_{h\downarrow 0}\frac{E_{ij}(y+hw)-E_{ij}(y)}{h}$.

For the sake of brevity, the rest of this proof is omitted. One can refer to [18] for similar details.

Since V_2 is locally Lipschitz, we have its generalized gradient : $\partial V_2 = \overline{co}\{\lim \nabla V_2(e_{v_i}, \|r_{ij}\|),\ \|r_{ij}\| \notin \Omega_V,\ i, j = 1, ..., N\}$, in which Ω_V is the set of measure zero where the gradient of V_2 is not defined. Specifically,

$$\partial V_2 = [e_{v_1}^T, ..., e_{v_N}^T, \frac{1}{2}\partial E_{11}, ..., \frac{1}{2}\partial E_{ij}, ..., \frac{1}{2}\partial E_{NN}]^T. \quad (22)$$

For simplicity, denote $\zeta_{ij} = \frac{1}{2}\partial E_{ij}$, then

$$\partial V_2 = [e_{v_1}^T, ..., e_{v_N}^T, \zeta_{11}, ..., \zeta_{ij}, ..., \zeta_{NN}]^T. \quad (23)$$

From the chain rule of set-valued Lie derivative of V_2 [27], we know

$$\frac{dV_2}{dt} \in^{a.e.} \dot{\widetilde{V}}_2, \quad (24)$$

where

$$\dot{\widetilde{V}}_2 = \bigcap_{\xi\in\partial V_2} \xi^T \cdot \{K[e_{v_1}], .., K[e_{v_N}], \frac{d\|r_{11}\|}{dt}, .., \frac{d\|r_{ij}\|}{dt}, .., \frac{d\|r_{NN}\|}{dt}\}^T.$$

Using (23) we have

$$\dot{\widetilde{V}}_2 = \bigcap_{\xi\in\partial V_2}\{\sum_{i=1}^{N} e_{v_i}^T \cdot K[e_{v_i}] + \sum_{i=1}^{N}\sum_{j\in\mathbb{N}_i}\zeta_{ij}\frac{d\|r_{ij}\|}{dt}\}. \quad (25)$$

To find out $\dot{\widetilde{V}}_2$ on the whole domain of $\|r_{ij}\|$, we discuss it piece-wisely. For simplicity, let

$$\Gamma = \sum_{i=1}^{N} e_{v_i}^T \cdot K[e_{v_i}] + \sum_{i=1}^{N}\sum_{j\in\mathbb{N}_i}\zeta_{ij}\frac{d\|r_{ij}\|}{dt}. \quad (26)$$

If for $\forall i$, $\|r_{ij}\| < d_0$ or $\|r_{ij}\| > d_1$ where $j \in \mathbb{N}_i$, i.e., in the domain of attraction and repulsion zones, we have $K[e_{v_i}] = -k_p e_{v_i} + \sum_{j\in\mathbb{N}_i} g(\|r_{ij}\|)\frac{r_{ij}}{\|r_{ij}\|}$ and $\zeta_{ij} = -\frac{1}{2}g(\|r_{ij}\|)$. Then,

$$\Gamma = -k_p\sum_{i=1}^{N} e_{v_i}^T e_{v_i} - \sum_{i=1}^{N}\sum_{j\in\mathbb{N}_i} g(\|r_{ij}\|)\bar{v}^T\frac{r_{ij}}{\|r_{ij}\|} + \sum_{i=1}^{N}\sum_{j\in\mathbb{N}_i} g(\|r_{ij}\|)(v_i^T \cdot \frac{r_{ij}}{\|r_{ij}\|}) - \frac{1}{2}\sum_{i=1}^{N}\sum_{j\in\mathbb{N}_i} g(\|r_{ij}\|)\frac{d\|r_{ij}\|}{dt}. \quad (27)$$

Because

$$\sum_{i=1}^{N}\sum_{j\in\mathbb{N}_i} g(\|r_{ij}\|)\frac{d\|r_{ij}\|}{dt} = \sum_{i=1}^{N}\sum_{j\in\mathbb{N}_i} g(\|r_{ij}\|)\{v_i^T \cdot \nabla_{r_{ij}}\|r_{ij}\| + v_j^T \cdot \nabla_{r_{ji}}\|r_{ji}\|\} = 2\sum_{i=1}^{N}\sum_{j\in\mathbb{N}_i} g(\|r_{ij}\|)v_i^T \cdot \nabla_{r_{ij}}\|r_{ij}\|,$$

and since $\sum_{i=1}^{N}\sum_{j\in\mathbb{N}_i} g(\|r_{ij}\|)\bar{v}^T\frac{r_{ij}}{\|r_{ij}\|} = 0$, then by using (17), (27) becomes

$$\Gamma = -k_p\sum_{i=1}^{N} e_{v_i}^T e_{v_i}. \quad (28)$$

If for $\forall i$, $d_0 < \|r_{ij}\| < d_1$ where $j \in \mathbb{N}_i$, i.e., in the domain of alignment zone, we have $K[e_{v_i}] = -k_p e_{v_i}$ and $\zeta_{ij} = 0$, so

$$\Gamma = -k_p\sum_{i=1}^{N} e_{v_i}^T e_{v_i} + \sum_{i=1}^{N}\sum_{j\in\mathbb{N}_i} 0 \cdot \frac{d\|r_{ij}\|}{dt} = -k_p\sum_{i=1}^{N} e_{v_i}^T e_{v_i}. \quad (29)$$

If for $\forall i$, $\|r_{ij}\| = d_0$ where $j \in \mathbb{N}_i$, then $\zeta_{ij} \in \mathcal{Q} \triangleq \overline{co}[-\frac{1}{2}g(d_0^-), 0]$, and $K[e_{v_i}] = -k_p e_{v_i} + \overline{co}[g(d_0^-), 0]\sum_{j\in\mathbb{N}_i}\frac{r_{ij}}{\|r_{ij}\|}$. Then we have:

$$\dot{\widetilde{V}}_2\,|_{\|r_{ij}\|=d_0} = \bigcap_{\zeta_{ij}\in\mathcal{Q}}\{\sum_{i=1}^{N} e_{v_i}^T \cdot K[e_{v_i}] + \sum_{i=1}^{N}\sum_{j\in\mathbb{N}_i}\zeta_{ij}\frac{d\|r_{ij}\|}{dt}\}$$

$$= \bigcap_{\zeta_{ij}\in\mathcal{Q}}\{\sum_{i=1}^{N} -k_p e_{v_i}^T e_{v_i} + \sum_{i=1}^{N}\sum_{j\in\mathbb{N}_i}\overline{co}[g(d_0^-),\ 0] (v_i - \bar{v})^T \cdot \frac{r_{ij}}{\|r_{ij}\|} + \sum_{i=1}^{N}\sum_{j\in\mathbb{N}_i}\zeta_{ij}[\frac{dr_{ij}}{dt}]^T \cdot \nabla_{r_{ij}}\|r_{ij}\|\}$$

$$= \bigcap_{\zeta_{ij}\in\mathcal{Q}} \{\sum_{i=1}^{N} -k_p e_{v_i}^T e_{v_i} + (\overline{co}[g(d_0^-),\ 0] + 2\zeta_{ij})$$
$$\sum_{i=1}^{N}\sum_{j\in\mathbb{N}_i} v_i^T \cdot \frac{r_{ij}}{\|r_{ij}\|}\} \subseteq \{\sum_{i=1}^{N} -k_p e_{v_i}^T e_{v_i}\} + \bigcap_{\zeta_{ij}\in\mathcal{Q}}$$
$$\{(\overline{co}[g(d_0^-),\ 0] + 2\zeta_{ij})\sum_{i=1}^{N}\sum_{j\in\mathbb{N}_i} v_i^T \cdot \frac{r_{ij}}{\|r_{ij}\|}\}.$$

Since

$$\bigcap_{\zeta_{ij}\in\overline{co}[-\frac{1}{2}g(d_0^-),\ 0]} \{\overline{co}[g(d_0^-),\ 0] + 2\zeta_{ij}\} = \{0\}, \qquad (30)$$

then

$$\dot{\widetilde{V_2}}\ |_{\|r_{ij}\|=d_0} \subseteq \{\sum_{i=1}^{N} -k_p e_{v_i}^T e_{v_i}\}. \qquad (31)$$

For $\|r_{ij}\| = d_1$ where $j \in \mathbb{N}_i$, $\zeta_{ij} \in \overline{co}[-\frac{1}{2}g(d_1^+),\ 0]$, and $K[e_{v_i}] = -k_p e_{v_i} + \overline{co}[g(d_1^+),\ 0]\sum_{j\in\mathbb{N}_i}\frac{r_{ij}}{\|r_{ij}\|}$. Similarly to the above, we have:

$$\dot{\widetilde{V_2}}\ |_{\|r_{ij}\|=d_1} \subseteq \{\sum_{i=1}^{N} -k_p e_{v_i}^T e_{v_i}\}. \qquad (32)$$

Therefore, on the whole domain, we have

$$\dot{\widetilde{V_2}} \subseteq \{\alpha \mid \alpha = \sum_{i=1}^{N} -k_p e_{v_i}^T e_{v_i} \leq 0\}. \qquad (33)$$

From (24) we know all $\frac{d}{dt}V_2 \leq 0$, which means that e_{v_i} is stable for any agent. Furthermore, since the swarm's topology is fixed, so the system is autonomous. Thus, from the nonsmooth version of LaSalle's invariance principle [27][28], we know that $(e_{v_i}, \|r_{ij}\|)$ approaches the largest invariant set in

$$\bar{S} = cl(\{(e_{v_i}, \|r_{ij}\|) \mid 0 \in \dot{\widetilde{V_2}},\ i,j = 1,...,N\})$$
$$= cl(\{(0, \|r_{ij}\|) \mid i,j = 1,...,N\}). \qquad (34)$$

where $cl(\cdot)$ is the closure of a set. This means all agents' velocities asymptotically converge to a common value $(\bar{v})$. □

Remark 1. For Theorem 3.2, if $g(\cdot)$ is continuous at d_0 and d_1, the differential inclusion $K[e_{v_i}]$ and general gradient ∂E_{ij} will be singletons, and the calculus of set inclusion associated with the general gradient will become equalities. So one can consider Theorem 3.1 is a special case of Theorem 3.2.

Remark 2. Note that all the information the controller (5) needs can be locally sensed, thus, the topological graph in this paper refers to sensing graph rather than communication graph [14][25]. The advantage of this configuration is that by the proposed controller, the agents do not communicate their states with each other, i.e., communication modules are not needed. Subsequently all the issues caused by communication setup (such as time delay and communication noise) are avoided.

4. SIMULATIONS

In this section, simulation results of a swarm moving with a fixed topology are presented to verify the proposed controller.

We select the mutual interaction to be piece-wise linear function as: for $\|r_{ij}\| < d0$: $g(\|r_{ij}\|) = -30\|r_{ij}\| + 320$; and for $\|r_{ij}\| > d1$: $g(\|r_{ij}\|) = -30\|r_{ij}\| + 400$, in which $d_0 = 10$ and $d_1 = 14$. The alignment zone lies in $[d_0,\ d_1]$. Clearly $g(\|r_{ij}\|)$ is not discontinuous at d_0 and d_1. The design constant is $k_p = 5$. We assume that the environment has identical effect on all agents.

Fig. 3–6 show a swarm ($N = 12$) with a fixed topology to move in a 2D linear environment. The potential profile of the environment is $\nabla_{r_i} J(r) = [-1.4,\ -1.4]^T$. Agents' initial positions and velocities are random. The swarm's topological graph is determined by agents' initial conditions. Fig. 3 shows agents' trajectories on $x - y$ plane. The stars and circles represent agents' initial and final positions, respectively. The convergence of agents' velocities is shown in Fig. 4. It is clear to see that all agents' velocities are asymptotically converged.

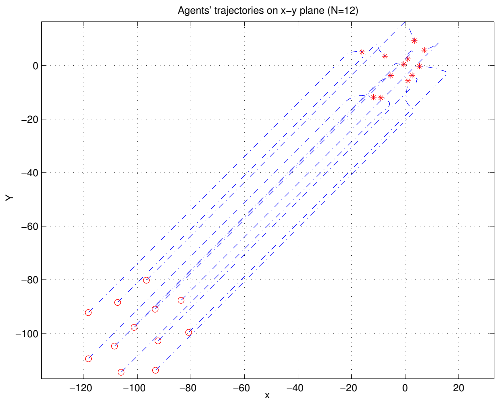

Fig. 3. Agents' trajectories on x-y plane when the swarm moves in a 2D linear environment ($N = 12$).

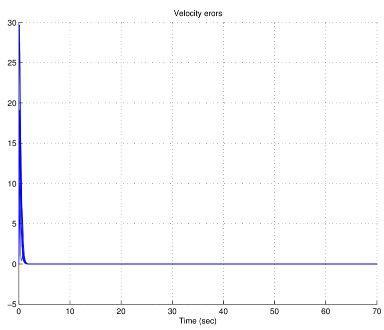

Fig. 4. Agents' velocity errors when the swarm moves in a 2D linear environment ($N = 12$).

The swarm's topologies at initial and final stages are shown in Fig. 5 and 6, respectively. It is not hard to see that the adjacent matrices of the graphs in Fig. 5 and 6 are the same, i.e., the swarm's topology keeps unchanged. Note that due to the mutual interactions among agents, the spacing between two adjacent agents may change. In the final stage, agents are more evenly dispersed in the group.

5. CONCLUSIONS AND FUTURE WORKS

In this paper, we study a swarm of mobile agents with fixed topology moving in a uniform environment. We propose a general decentralized controller that utilizes Attraction/Alignment/Replusion (A/A/R) interactions among neighbors to achieve the collective group behavior. Under the assumption of connected graph, We show that the proposed decentralized controller leads to all agents' velocities to asymptotically converge to a common value.

The case of swarms with dynamic topologies will be discussed in continued work. Future research will focus on issues arising from practical applications, such as sensing noise, disturbance and fluctuation of the environment.

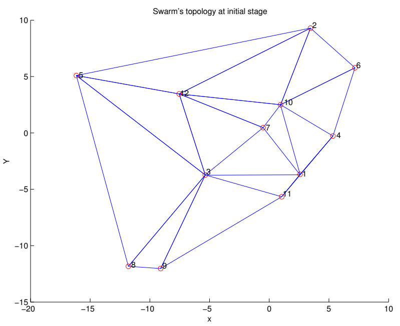

Fig. 5. Swarm's initial topology on x-y plane ($N = 12$).

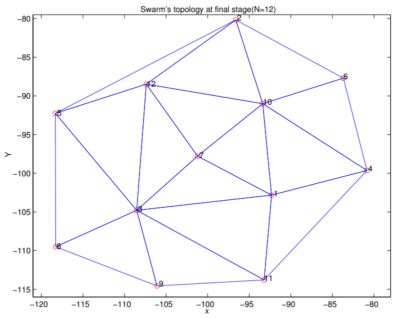

Fig. 6. Swarm's topology at final stage ($N = 12$).

REFERENCES

[1] E. Shaw, "Schooling in fishes: critique and review", in: L. Arson (editor), *Development and evolution of behaviour*, pp.452-480, 1970.
[2] A. Mogilner, L. Edelstein-keshet, "A non-local model for a swarm", *J. Math. Biol.*, vol. 38, pp. 534-570, 1999.
[3] N. Newlands, PhD Dissertation, "Shoaling dynamics and abundance estimation: Atlantic Bluefin Tuna (Thunnus thynnus)", Fisheries Center, UBC, Vancouver BC, Canada, 2002.
[4] H. S. Niwa, "Self-organizing dynamic model of fish schooling", *J. Theor. Biol.*, vol. 171, pp. 123-136, 1994.
[5] H. S. Niwa, "Newtonian dynamical approach to fish schooling", *J. Theor. Biol.*, vol. 181, pp. 47-63, 1996.
[6] E. Holmes, M. Lewis, and J. Banks, "PDE in ecology: spatial interactions and populationh dynamics", *Ecology*, 75(1), pp.17-29, 1994.
[7] W. Wiltschko and R. Wiltschko, "Magnetic orientation in birds", *J. Experim. Biol.*, 199, pp. 29-38, 1996.
[8] A. Muss, D. R. Robertson, A. Stepien, P. Wirtz and B. W. Bowen, "Phylogeography of Ophioblennius: The role of ocean currents and geography in reef fish evolution", *Evolution*, 55(3), pp. 561-571, 2001.
[9] P. R. Armsworth, "Directed motion in the sea: Efficient swimming by reef fish larvae", *J. Theor. Biol.* , (2001)210, pp. 81-91, 2001.
[10] J. K. Parrish, S. V. Viscido, and D. Grunbaum, "Self-organized fish schools: An examination of emergent properties", *Biol. Bull.* 202: 296-305, June 2002.
[11] S. V. Viscido, J. K. Parrish, and D. Grunbaum, "Individual behavior and emergent properties of fish schools: a comparison of observation and theory", *Mar. Ecol. Prog. Ser.*, Vol.273: 239-249, 2004.
[12] I. D. Couzin, J. Krause, R. James, G. D. Ruxton and N. R. Franks, "Collective memory and spatial sorting in animal groups", *J. Theor. Biol.* 218: 1-11, 2002.
[13] H. Kunz and C. K. Hemelrijk, "Artificial fish schools: Collective effects of school size, body size, and body form", *Artificial Life*, 9:237-253, 2003.
[14] R. Olfati-Saber, "Flocking for Multi-Agent Dynamic Systems: Algorithms and Theory". *IEEE Trans. on Autom. Contr.*, 51(3), Mar. 2006.
[15] A. Jadbabai, J. Lin and A. S. Morse, "Coordiantion of groups of mobile autonomous agents using nearest neghborhood rules", *IEEE Trans. on Autom. Contr.*, 48(6), pp. 988-1001, 2003
[16] H. G. Tanner, A. Jababaie, and G. J. Pappas, "Flocking in fixed and switching networks", *submitted to IEEE Trans. on Autom. Contr.*, 2006.
[17] H. G. Tanner, A. Jadbabaie and G. J. Pappas, "Stable flocking of mobile agent, Part I: Fixed toplogy", *Proc. of Conf. Decision & Contr.* Maui, Hawaii, pp. 2010-2015, 2003.
[18] H. G. Tanner, A. Jadbabaie and G. J. Pappas, "Stable flocking of mobile agent, Part II: Dyanmic toplogy", *Proc. Conf. Decision & Contr.* Maui, Hawaii, pp. 2016-2021, 2003.
[19] Y. Liu and Kevin M. Passino, "Stable social foraging swarms in a noisy environment", *IEEE Trans. on Autom. Contr.*, vol. 49, no. 1, pp. 30-44, Jan 2004.
[20] V. Gazi and K. Passino, "Stability analysis of swarms", *IEEE Trans. on Autom. Contr.*, vol. 48, pp. 692-697, 2003.
[21] V. Gazi and K. Passino, "A class of attraction/repulsion functions for stable swarm aggregations", *Proc. of Conf. Decision & Contr.*, Los Vegas, NV, Dec 2002, pp.2842-2847.
[22] V. Gazi and K. Passino, "Stability analysis of swarms in an environment with an attractant/repellent profile", *Proc. of Amer. Contr. Conf.*, Anchorage, AK, 2002, pp. 1819-1824.
[23] V. Gazi and K. Passino, "Stability analysis of social foraging swarms: combined effects of attactrant/replellent profiles", *Proc. of Conf. Decision & Contr.*, Las Vegas, NV, Dec 2002, pp. 114-123.
[24] V. Gazi and K. Passino, "Stability analysis of social foraging swarm", *IEEE Trans. on Syst. Man, and Cyber. - Part B: Cyber.*, vol. 34, no. 1, pp. 539-557, Feb 2004.
[25] L. Moreau, "Stability of multiagent systems with time-dependent communication links". *IEEE Trans. on Autom. Contr.*, 50(2):169-182, 2005.
[26] A. F. Filippov, *Differential Equations with Discontinuous Righthand Sides*, vol. 18 of Mathematics and Its Applications (Soviet Series), Kluwer Academic Publishers, Dordrecht, The Netherlands, 1988.
[27] D. Shevitz and B. Paden, "Lyapunov Stability Theory of Nonsmooth Systems", *IEEE Trans. on Autom. Contr.*, vol. 39, No. 9, pp. 1910-1914, 1994.
[28] A. Bacciotti and F. Ceragioli,"Stability and Stabilization of Discontinuous Systems and Nonsmooth Lyapunov Functions", *ESAIM J. of Contr. Optimis. and Cal. of Var.*, vol.4, pp.361-376, 1999.
[29] J. Cortes, "Achieving coordination tasks in finite time via nonsmooth gradient flows", *Proc. of Conf. on Decision & Contr., and the Euro. Contr. Conf.*, Dec. 2005 Seville, Spain, pp. 6376-6381.
[30] F. H. Clarke, *Optimization and Nonsmooth Analysis*, Classics in Applied Mathematics: 5, SIAM, Philadelphia, 1990.
[31] Xiaohai Li, Zhijun Cai and Jizhong Xiao, "Biologically inspired flocking of swarms with dynamic topology in uniform environments", *Proc. of Conf. Decision & Contr. (CDC'07)*, New Orleans, LA, pp. 2522-2527, 2007.
[32] H. K. Khalil, *Nonlinear Systems*, 3rd edition, Prentice Hall, 2002.