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MPC with Nonlinear $\mathscr{H}_\infty$ Control for Path Tracking of a Quad-Rotor Helicopter

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Abstract: This paper presents a predictive and nonlinear robust control strategy to solve the path tracking problem for a quadrotor helicopter. The dynamic motion equations are obtained by the Lagrange-Euler formalism. The control structure is performed through a model-based predictive controller (MPC) to track the reference trajectory and a nonlinear $\mathscr{H}_\infty$ controller to stabilize the rotational movements. Simulations results in presence of aerodynamic disturbances and parametric uncertainty are presented to corroborate the effectiveness and the robustness of the proposed strategy. *Copyright © 2008 IFAC.*

1. INTRODUCTION

This paper deals with a quadrotor UAV, in which the VTOL (Vertical Take-Off and Landing) is one of the concepts usually used to develop control laws. This kind of helicopter tries to reach a stable hovering and flight using the forces equilibrium produced by the four rotors [Castillo et al., 2005b]. One of the advantages of the quadrotor configuration is its load capacity. Moreover, this helicopter is highly maneuverable, which allows take-off and landing as well as flight in hard environment. As a drawback, this type of UAV presents a weight and energy consumption augmentation due to the extra motors.

Many efforts have been made to control the quadrotor helicopter and many strategies have been developed to solve the path tracking problem for this type of system (see, for example, Mistler et al. [2001], Bouabdallah et al. [2004], Bouabdallah and Siegwart [2005], Castillo et al. [2005a]). Several control strategies have been tested on the quadrotor helicopter, but most of them do not consider external disturbances and parametric uncertainty of the model.

In some publications the quadrotor helicopter has been controlled using a linear $\mathscr{H}_\infty$ controller based on linearized models. In Chen and Huzmezan [2003], a simplified nonlinear model of the UAV movements was presented. The path tracking problem was divided into two parts, the first one to achieve the angular rates and vertical velocity stabilization by a 2DOF $\mathscr{H}_\infty$ controller using the loop shaping technique. The same technique was used to control the longitudinal and lateral velocities, the yaw angle and the height in the outer loop. A predictive control was designed to solve the path tracking problem.

In this paper a predictive and nonlinear robust control strategy to solve the path tracking problem of the quadrotor helicopter is proposed. A state space predictive controller based on the variant time error model is used to track the reference trajectory. A nonlinear $\mathscr{H}_\infty$ controller is synthesized to stabilize the helicopter rotational movements .

The objective of MPC is to compute a future control sequence in a defined horizon in such a way that the prediction of the plant output is driven close to the reference. This is accomplished by minimizing a multi-stage cost function in respect to the future control actions. An analytical solution can be obtained for a quadratic cost if the model is linear and there are no constraints, otherwise an iterative method of optimization should be used [Camacho and Bordons, 1998]. Because of its formulation MPC also allows the use of previously known references for the control law calculation [Normey-Rico et al., 1999].

The goal of the nonlinear $\mathscr{H}_\infty$ control theory, introduced by van der Schaft in his prominent article [van der Schaft, 1992], is to achieve a bounded ratio between the energy of the so-called error signals and the energy of the disturbance signals. In general, the nonlinear approach of this theory considers two Hamilton-Jacobi-Bellman-Isaacs partial derivative equations (HJBI PDEs), which replace the Riccati equations in the case of the linear $\mathscr{H}_\infty$ control formulation. The main problem in the nonlinear case is the absence of a general method to solve these HJBI PDEs.

In Ortega et al. [2005] a strategy to control mechanical systems considering the tracking error dynamic equation was proposed. In such strategies a nonlinear $\mathscr{H}_\infty$ control, formulated via game theory, was applied. This strategy provides, by an analytical solution, a constant gain similar to the results obtained with the feedback linearization procedures.

The remainder of the paper is organized as follows: in Section II, a description of the quadrotor helicopter modelling is given. The predictive controller for the translational movements is presented in Section III. In Section IV, the nonlinear $\mathscr{H}_\infty$ controller for the rotational subsystem is developed. Some simulation results are presented in Section V. Finally, the major conclusions of the work are drawn in Section VI.

2. SYSTEM MODELLING

2.1 Description

The autonomous aerial vehicle used in this paper is a miniature four rotor helicopter. The movement of the UAV results from changes in the velocities of the rotors. Longitudinal motions are achieved by means of front and rear rotors velocity, while lateral displacements are performed using the speed of the right and left propellers. Yaw movement is obtained from the difference in the counter-torque between each pair of propellers, i.e., accelerating the two clockwise turning rotors while decelerating the counter-clockwise turning rotors, and vice-versa.

The dynamic model of the system is obtained under the assumption that the vehicle is a rigid body in the space subject to one main force (thrust) and three torques. However, this type of vehicle is a flight system of lightweight structure and, therefore, gyroscopes effects resulting from the rotation of the rigid body and the four propellers must be included in the dynamic model [Bouabdallah et al., 2004].

Besides, a helicopter is an underactuated mechanical system with six degrees of freedom and only four control inputs. Due to the complexities presented, some assumptions are made for modelling purposes [Koo and Sastry, 1999]. The moment effects caused by the rigid body on the translational dynamic are neglected, as well as the ground effect. The center of mass and the body fixed frame origin are assumed coincident. Moreover, the helicopter structure is assumed to be symmetric, which results in a diagonal inertia matrix.

2.2 Helicopter Kinematics

The helicopter as a rigid body is characterized by a frame linked to it. Let $\mathscr{B} = \{B_1^b, B_2^b, B_3^b\}$ be the body fixed frame, where the B_1^b axis is the helicopter normal flight direction, B_2^b is orthogonal to B_1^b and positive to starboard in the horizontal plane, whereas B_3^b is oriented in ascendant sense and orthogonal to the plane $B_1^b O B_2^b$. The inertial frame $\mathscr{I} = \{E_x, E_y, E_z\}$ is considered fixed with respect to the earth (see Fig. 1).

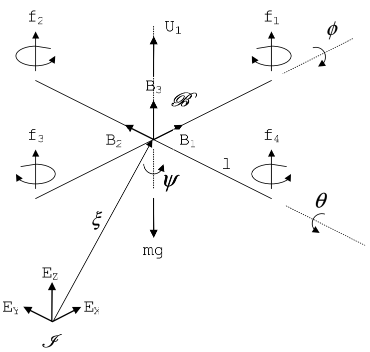

Fig. 1. Quadrotor helicopter scheme.

The vector $\xi = \{x, y, z\}$ represents the position of the helicopter mass center expressed in the inertial frame $\mathscr{I}$. The vehicle orientation is given by a rotation matrix $R_{\mathscr{I}} : \mathscr{B} \to \mathscr{I}$, where $R_{\mathscr{I}} \in SO(3)$ is an orthonormal rotation matrix [Fantoni and Lozano, 1995]. The rotation matrix is obtained through three successive rotations around the axes of the body fixed frame. The first one is given by a rotation around the E_x axis by roll angle, $(-\pi < \phi < \pi)$, followed by a rotation of pitch angle, $(-\pi/2 < \theta < \pi/2)$, around the E_y axis from the new axis B_2^b. Finally, a rotation of the yaw angle, $(-\pi < \psi < \pi)$, is carried out around the E_z axis from the new axis B_3^b to carry the helicopter to the final position.

From these three rotations, the following rotation matrix from $\mathscr{B}$ to $\mathscr{I}$ is obtained:

$$R_{\mathscr{I}} = \begin{bmatrix} C\psi C\theta & C\psi S\theta S\phi - S\psi C\phi & C\psi S\theta C\phi + S\psi S\phi \\ S\psi C\theta & S\psi S\theta S\phi + C\psi C\phi & S\psi S\theta C\phi - C\psi S\phi \\ -S\theta & C\theta S\phi & C\theta C\phi \end{bmatrix} \quad (1)$$

where $C\cdot = \cos(\cdot)$ and $S\cdot = \sin(\cdot)$.

The kinematic equations of the rotational and translational movements are obtained by means of the rotation matrix. The translational kinematic can be written as:

$$v = R_{\mathscr{I}} \cdot V \quad (2)$$

where $v = [u_0 \quad v_0 \quad w_0]^T$ and $V = [u_L \quad v_L \quad w_L]^T$ are linear velocities expressed in the inertial frame and body fixed frame, respectively.

The rotational kinematic can be obtained from the relationship between the rotation matrix and its derivative with an skew-symmetric matrix [Craig, 1989, Olfati-Saber, 2001] as follows:

$$\dot{R}_{\mathscr{I}} = R_{\mathscr{I}} \cdot S(\omega) \quad (3)$$

$$\dot{\eta} = W_\eta^{-1} \omega$$

$$\begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} = \begin{bmatrix} 1 & \sin\phi \tan\theta & \cos\phi \tan\theta \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi \sec\theta & \cos\phi \sec\theta \end{bmatrix} \begin{bmatrix} p \\ q \\ r \end{bmatrix} \quad (4)$$

where $\eta = (\phi, \theta, \psi)$, $\omega = (p, q, r)$ are the angular velocities in the body fixed frame.

2.3 Lagrange-Euler Equations

The helicopter motion equations can be expressed by the Lagrange-Euler formalism based on the kinetic and potential energy concept:

$$\Gamma_i = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} \quad (5)$$

$$L = E_c - E_p$$

where L is the Lagrangian, E_c is the total kinetic energy, E_p is the total potential energy, q_i is the generalized coordinate and Γ_i are the generalized forces/torques given by nonconservative forces/torques

The generalized coordinates for a rigid body rotating in the three-dimensional space can be written as [Castillo et al., 2005a]:

$$q = [x \quad y \quad z \quad \phi \quad \theta \quad \psi]^T \in \Re^6$$

The Lagrangian expression of the helicopter is given by:

$$L(q, \dot{q}) = E_{c_{Trans}} + E_{c_{Rot}} - E_p \quad (6)$$

where $E_{c_{Trans}}$ is the translational energy and $E_{c_{Rot}}$ is the rotational energy.

Firstly, the translational energy term is developed requiring the knowledge of each generalized coordinate velocity. The linear velocity is given by (2), where $\dot{\xi} = v$ and the quadratic velocity is $\dot{\xi}^2(x, y, z) = (\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$. Thus, the translational kinetic energy can be written as:

$$E_{c_{Trans}} = \frac{1}{2}\int \dot{\xi}^2(x,y,z)\, dm = \frac{m}{2}\dot{\xi}^2(x,y,z) = \frac{m}{2}\dot{\xi}^T\dot{\xi}$$

Let $E_{c_{Rot}}$ be the rotational kinetic energy in $\mathscr{B}$ expressed in $\mathscr{I}$, and let $dE_{c_{Rot}}$ be the kinetic energy of a particle with differential mass dm in $\mathscr{B}$. Then:

$$dE_{c_{Rot}} = \frac{1}{2}\left({}^{\mathscr{I}}v_{\mathscr{B}}^2\right) dm = \frac{1}{2}\left({}^{\mathscr{I}}v_{\mathscr{B}x}^2 + {}^{\mathscr{I}}v_{\mathscr{B}y}^2 + {}^{\mathscr{I}}v_{\mathscr{B}z}^2\right) dm \quad (7)$$

Therefore, the rotational kinetic energy can be obtained solving (7). Furthermore, from the hypothesis assumed on the inertia matrix, the cross products can be neglected and consequently the inertia matrix becomes diagonal. Like this the rotational kinetic energy is given by:

$$E_{c_{Rot}} = \frac{1}{2}\int {}^{\mathscr{I}}v_{\mathscr{B}}^2 dm = \frac{1}{2}I_{xx}\left(\dot{\phi} - \dot{\psi}\sin\theta\right)^2 + \frac{1}{2}I_{yy}\left(\dot{\theta}\cos\phi + \dot{\psi}\sin\phi\cos\theta\right)^2 + \frac{1}{2}I_{zz}\left(\dot{\theta}\sin\phi - \dot{\psi}\cos\phi\cos\theta\right)^2 \quad (8)$$

or in a compact form using (4):

$$E_{c_{Rot}} = \frac{1}{2}I_{xx}p^2 + \frac{1}{2}I_{yy}q^2 + \frac{1}{2}I_{zz}r^2 = \frac{1}{2}\omega^T J\omega \tag{9}$$

If the Jacobian from ω to $\dot{\eta}$ in (4) is named as W_η and the following matrix is defined:

$$\mathscr{J} = \mathscr{J}(\eta) = W_\eta^T J W_\eta \tag{10}$$

then the kinetic energy equation (9) can be rewritten as function of the generalized coordinate η as follows:

$$E_{c_{Rot}} = \frac{1}{2}\dot{\eta}^T \mathscr{J} \dot{\eta} \tag{11}$$

The potential energy E_p expressed in terms of the generalized coordinates is given by:

$$E_p = mgz \tag{12}$$

The complete movement equation is obtained from the Lagrangian expression (6), as follows:

$$\begin{bmatrix} F_\xi \\ \tau_\eta \end{bmatrix} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} \tag{13}$$

where $\tau_\eta \in \Re^3$ represents the roll, pitch and yaw moments and $F_\xi = R_{\mathscr{J}}\hat{F} + A_T$ is the translational force applied to the helicopter due to the main control input U_1 in z axis direction, with $R_{\mathscr{J}}\hat{F} = R_{\mathscr{J}_{E_3}}U_1$ and A_T the external disturbances.

Since the Lagrangian does not contain kinetic energy terms combining $\dot{\xi}$ and $\dot{\eta}$, the Lagrange-Euler equations can be divided into translational and rotational dynamics, being the Lagrange-Euler equations of the translational movement:

$$m\ddot{\xi} + mgE_3 = F_\xi \tag{14}$$

Then, (14) can be expressed by means of state vector ξ, yielding:

$$\begin{cases} \ddot{x} = \frac{1}{m}(\cos\psi\sin\theta\cos\phi + \sin\psi\sin\phi)U_1 + \frac{A_x}{m} \\ \ddot{y} = \frac{1}{m}(\sin\psi\sin\theta\cos\phi - \cos\psi\sin\phi)U_1 + \frac{A_y}{m} \\ \ddot{z} = -g + \frac{1}{m}(\cos\theta\cos\phi)U_1 + \frac{A_z}{m} \end{cases} \tag{15}$$

The Lagrange-Euler equations for the coordinate η, written in the general form, are [Castillo et al., 2005a]:

$$M(\eta)\ddot{\eta} + C(\eta,\dot{\eta})\dot{\eta} = \tau_\eta \tag{16}$$

where $M(\eta) = \mathscr{J}(\eta)$.

Thus, the mathematical model that describes the helicopter rotational movement obtained from the Lagrange-Euler formalism is given by:

$$\ddot{\eta} = M(\eta)^{-1}(\tau_\eta - C(\eta,\dot{\eta})\dot{\eta}) \tag{17}$$

3. ERROR BASED STATE SPACE CONTROLLER (E-SSPC) FOR PATH TRACKING

In this section a control law to solve the path tracking problem by translational movements is designed. A linear state space MPC strategy based on the error model is performed. From the error model, two predictive controllers are synthesized. The first one controls the height through of the input U_1, whereas the second one makes use of this signal as a time variant parameter in the linear x and y motions to compute the virtual inputs.

Thus, the system (15) is rewritten in state space form $\dot{\hat{x}}(t) = f(\hat{x}(t),\hat{u}(t))$ for the controller design, where $\hat{x}(t) = [z(t) \quad w_0(t) \quad x(t) \quad u_0(t) \quad y(t) \quad v_0(t)]^T$ stands for the state space vector of the system.

From (15) and the new state space vector, the system dynamic equation can be written in the following form:

$$\dot{\hat{x}}(t) = f(\hat{x}(t),\hat{u}(t)) = \begin{bmatrix} w_0(t) \\ -g + (\cos\theta(t)\cos\phi(t))\frac{U_1(t)}{m} \\ u_0(t) \\ u_x(t)\frac{U_1(t)}{m} \\ v_0(t) \\ u_y(t)\frac{U_1(t)}{m} \end{bmatrix} \tag{18}$$

with:

$$\begin{aligned} u_x(t) &= (\cos\psi(t)\sin\theta(t)\cos\phi(t) + \sin\psi(t)\sin\phi(t)) \\ u_y(t) &= (\sin\psi(t)\sin\theta(t)\cos\phi(t) - \cos\psi(t)\sin\phi(t)) \end{aligned} \tag{19}$$

Equations (15) show that the movement through the x and y axes depends on the control input U_1. In fact, U_1 is the designed total thrust vector to obtain the desired linear movement, while u_x and u_y can be considered as the orientations of U_1 that cause the movement through the x and y axes, respectively.

The objective of this approach is to guarantee that the UAV follows a previously defined reference trajectory without any displacement error. However, due to the fact that the destination coordinates varies in time, a reference virtual vehicle having the same quadrotor helicopter mathematical model is placed on the track:

$$\dot{\hat{x}}_{ref}(t) = f(\hat{x}_{ref}(t),\hat{u}_{ref}(t)) \tag{20}$$

where $\hat{x}_{ref}(t) = [z_{ref}(t) \quad w_{0_{ref}}(t) \quad x_{ref}(t) \quad u_{0_{ref}}(t) \quad y_{ref}(t) \quad v_{0_{ref}}(t)]^T$ and $\hat{u}_{ref}(t) = [U_{1_{ref}} \quad u_{x_{ref}} \, u_{y_{ref}}]^T$ are the reference states and control inputs, respectively.

The input control $U_1(t)$ is considered a time variant parameter to the reference x and y motions. Moreover, because the decentralized control structure, the roll, pitch and yaw angles are also considered as parameters that vary in time.

Thus, by subtracting the system (20) from the system (18) and using the Euler's method, the proposed translational error model, as a time variant discrete linear model, is given by:

$$\tilde{\boldsymbol{x}}(k+1) = \mathbf{A}\cdot\tilde{\boldsymbol{x}}(k) + \mathbf{B}(k)\cdot\tilde{\boldsymbol{u}}(k)\,. \tag{21}$$

where $\tilde{\boldsymbol{x}}(k) = \hat{\boldsymbol{x}}(k) - \hat{\boldsymbol{x}}_{\boldsymbol{ref}}(k)$ represents the vector error and $\tilde{\boldsymbol{u}}(k) = \hat{\boldsymbol{u}}(k) - \hat{\boldsymbol{u}}_{\boldsymbol{ref}}(k)$ the control input error.

Therefore, the error model (21) is split up in two subsystems: height error and x and y motions error as follows:

$$\tilde{x}_z(k+1) = \begin{bmatrix} 1 & 1 \\ 0 & T \end{bmatrix}\begin{bmatrix} \tilde{z}(k) \\ \tilde{w}_0(k) \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{T}{m}\cos(\theta(k))\cos(\phi(k)) \end{bmatrix}\tilde{U}_1(k) \tag{22}$$

$$\tilde{x}_{xy}(k+1) = \begin{bmatrix} 1 & T & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & T \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \tilde{x}(k) \\ \tilde{u}_0(k) \\ \tilde{y}(k) \\ \tilde{v}_0(k) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ \frac{T}{m}U_1(k) & 0 \\ 0 & 0 \\ 0 & \frac{T}{m}U_1(k) \end{bmatrix}\begin{bmatrix} \tilde{u}_x(k) \\ \tilde{u}_y(k) \end{bmatrix} \tag{23}$$

So that, from the height and longitudinal-lateral error models the control laws can be designed in such way that the system is forced to track the reference trajectory. The first one computes the control input U_1. The idea used in this controller consists in the computation of a control law in such a way that minimizes the cost defined by:

$$\mathbf{J}_z = \left[\hat{\mathbf{x}}_z - \tilde{\mathbf{x}}_{z_r}\right]'\mathbf{Q}_z\left[\hat{\mathbf{x}}_z - \tilde{\mathbf{x}}_{z_r}\right] + \left[\tilde{\mathbf{U}}_1 - \tilde{\mathbf{U}}_{1_r}\right]'\mathbf{R}_z\left[\tilde{\mathbf{U}}_1 - \tilde{\mathbf{U}}_{1_r}\right]\,, \tag{24}$$

where $\mathbf{Q}_z$ and $\mathbf{R}_z$ are diagonal definite positive weighting matrices and N_{2_z} and N_{u_z} are the horizons [Rossiter, 2003]. The predictions of the plant output $\hat{\tilde{\boldsymbol{x}}}_z(k+j|k)$ are computed using a linearized time variant state space model of the vehicle by the equation (22), obtaining:

$$\hat{\tilde{\mathbf{x}}}_z = \mathbf{P}_z(k|k)\cdot\tilde{\mathbf{x}}_z(k|k) + \mathbf{H}_z(k|k)\cdot\tilde{\mathbf{U}}_1\,, \tag{25}$$

where $\tilde{U}_1(k) = U_1(k) - U_{1_{ref}}(k)$ and $\tilde{x}_z(k)$ is the height state error, and the height reference vectors are:

$$\tilde{\mathbf{x}}_{z_r} \triangleq \begin{bmatrix} \hat{x}_{z_r}(k+1|k) - \hat{x}_{z_r}(k|k) \\ \vdots \\ \hat{x}_{z_r}(k+N_2|k) - \hat{x}_{z_r}(k|k) \end{bmatrix}, \; \tilde{\mathbf{U}}_{1_{ref}} \triangleq \begin{bmatrix} \hat{U}_{1_r}(k|k) - \hat{U}_{1_r}(k-1|k) \\ \vdots \\ \hat{U}_{1_r}(k+Nu-1|k) - \hat{U}_{1_r}(k-1|k) \end{bmatrix}$$

Minimizing the equation (24) when the constraints are not considered, the control law can be obtained as:

$$\tilde{\mathbf{U}}_1 = \left[\mathbf{H}_z' \cdot \mathbf{Q}_z \cdot \mathbf{H}_z + \mathbf{R}_z\right]^{-1} \cdot \left[\mathbf{H}_z' \cdot \mathbf{Q}_z \cdot (\tilde{\mathbf{x}}_{z_r} - \mathbf{P}_z \cdot \tilde{x}_z(k)) + \mathbf{R}_z \cdot \tilde{\mathbf{U}}_{1_{ref}}\right], \quad (26)$$

although only $\tilde{U}_1(k)$ is needed at each instant k [Camacho and Bordons, 1998]. In the constrained case, an optimization algorithm solves (24) at each sampling time. However, in this work constraints are not considered. So that, by (26), $U_1(k) = \tilde{U}_1(k) + U_{1_{ref}}(k)$ is obtained.

Then, the next step performs the x and y motion control inputs. The same previous procedure, using the error model (23), to compute the control law is carried out, and is given by:

$$\tilde{\mathbf{u}}_{xy} = \left[\mathbf{H}_{xy}' \cdot \mathbf{Q}_{xy} \cdot \mathbf{H}_{xy} + \mathbf{R}_{xy}\right]^{-1} \cdot \left[\mathbf{H}_{xy}' \cdot \mathbf{Q}_{xy} \cdot \left(\tilde{\mathbf{x}}_{xy_r} - \mathbf{P}_{xy} \cdot \tilde{x}_{xy}(k)\right) + \mathbf{R}_{xy} \cdot \tilde{\mathbf{u}}_{xy_r}\right], \quad (27)$$

where $\tilde{\mathbf{u}}_{xy} = [\tilde{u}_x(k) \quad \tilde{u}_y(k)]^T$, and

$$\begin{bmatrix} u_x(k) \\ u_y(k) \end{bmatrix} = \begin{bmatrix} \tilde{u}_x(k) \\ \tilde{u}_y(k) \end{bmatrix} + \begin{bmatrix} u_{x_{ref}}(k) \\ u_{y_{ref}}(k) \end{bmatrix}$$

The error reference states and control inputs are obtained from the same form that for height controller case.

From $u_x(k)$ and $u_y(k)$ the reference roll, ϕ_{ref}, and pitch, θ_{ref}, angles for the helicopter rotational loop using equation (19) are computed.

4. NONLINEAR $\mathscr{H}_\infty$ CONTROLLER FOR STABILIZATION

In this section the rotational subsystem control law to achieve robustness in presence of sustained disturbances and parametric uncertainty is developed. A nonlinear $\mathscr{H}_\infty$ controller is able to execute this task. The controller design for mechanical system models using Lagrange-Euler equations is carried out by a direct method.

4.1 Nonlinear $\mathscr{H}_\infty$ Control Theory

The dynamic equation of an nth order smooth nonlinear system which is affected by an unknown disturbance can be expressed as follows:

$$\dot{x} = f(x,t) + g(x,t)u + k(x,t)\omega, \quad (28)$$

where $u \in \Re^p$ is the vector of control inputs, $\omega \in \Re^q$ is the vector of external disturbances and $x \in \Re^n$ is the vector of states. Performance can be defined using the cost variable $z \in \Re^{(m+p)}$ given by the expression:

$$z = W \begin{bmatrix} h(x) \\ u \end{bmatrix}, \quad (29)$$

where $h(x) \in \Re^m$ represents the error vector to be controlled and $W \in \Re^{(m+p)\times(m+p)}$ is a weighting matrix. If states x are assumed to be available for measurement, then the optimal $\mathscr{H}_\infty$ problem can be posed as follows [van der Schaft, 1992]:

Find the smallest value $\gamma^* \geq 0$ such that for any $\gamma \geq \gamma^*$ exists a state feedback $u = u(x,t)$, such that the L_2 gain from ω to z is less than or equal to γ, that is:

$$\int_0^T \|z\|_2^2 dt \leq \gamma^2 \int_0^T \|\omega\|_2^2 dt. \quad (30)$$

The internal term of the integral expression on the left-hand side of inequality (30) can be written as:

$$\|z\|_2^2 = z^T z = \left[\, h^T(x) \;\; u^T \,\right] W^T W \begin{bmatrix} h(x) \\ u \end{bmatrix}.$$

and the symmetric positive definite matrix $W^T W$ can be partitioned as follows:

$$W^T W = \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \quad (31)$$

Matrices Q and R are symmetric positive definite and the fact that $W^T W > 0$ guarantees that $Q - SR^{-1}S^T > 0$.

Under these assumptions, an optimal control signal $u^*(x,t)$ may be computed for system (28) if there is a smooth solution $V(x,t)$, with $V(x_0,t) \equiv 0$ for $t \geq 0$, to the following HJBI equation [van der Schaft, 2000]:

$$\frac{\partial V}{\partial t} + \frac{\partial^T V}{\partial x} f(x,t) + \frac{1}{2}\frac{\partial^T V}{\partial x}\left[\frac{1}{\gamma^2}k(x,t)k^T(x,t) - g(x,t)R^{-1}g^T(x,t)\right]\frac{\partial V}{\partial x}$$
$$-\frac{\partial^T V}{\partial x} g(x,t)R^{-1}S^T h(x) + \frac{1}{2}h^T(x)\left(Q - SR^{-1}S^T\right)h(x) = 0 \quad (32)$$

for each $\gamma > \sqrt{\sigma_{\max}(R)} \geq 0$, where $\sigma_{\max}$ stands for the maximum singular value. In such a case, the optimal state feedback control law is derived as follows [W. Feng and I. Postlethwaite , 1994]:

$$u^* = -R^{-1}\left(S^T h(x) + g^T(x,t)\frac{\partial V(x,t)}{\partial x}\right). \quad (33)$$

4.2 Rotational Subsystem Nonlinear $\mathscr{H}_\infty$ Control

In order to develop the nonlinear $\mathscr{H}_\infty$ controller the rotational movements dynamic model (16), obtained from the Lagrange-Euler formalism, is used. τ_η adds the control torques and external disturbances, and is redefined as:

$$\tau_\eta = \tau_{\eta_a} + \tau_{\eta_d}$$

where τ_{η_a} is the applied torques vector and τ_{η_d} represents the total effect of system modelling errors and external disturbances.

As a first step to synthesize the control law, the tracking error vector is defined as follows:

$$\hat{x} = \begin{bmatrix} \dot{\tilde{\eta}} \\ \tilde{\eta} \\ \int \tilde{\eta}\, dt \end{bmatrix} = \begin{bmatrix} \dot{\eta} - \dot{\eta}^d \\ \eta - \eta^d \\ \int \left(\eta - \eta^d\right) dt \end{bmatrix} \quad (34)$$

where η^d and $\dot{\eta}^d \in \Re^n$ are the desired trajectory and the corresponding velocity, respectively. Note that an integral term has been included in the error vector. This term will allow to achieve a null steady-state error when persistent disturbances are acting on the system [Ortega et al., 2005].

The following control law is proposed for the rotational subsystem:

$$\tau_{\eta_a} = M(\eta)\ddot{\eta} + C(\eta,\dot{\eta})\dot{\eta} - T_1^{-1}\left(M(\eta)T\dot{\hat{x}} + C(\eta,\dot{\eta})T\hat{x}\right) + T_1^{-1}u \quad (35)$$

The proposed control law can be split up into three different parts: the first one consists of the first three terms of that equation, which are designed in order to compensate the system dynamics (16). The second part consists of terms including the error vector $\hat{x}$ and its derivative, $\dot{\hat{x}}$. Assuming $\tau_{\eta_d} \equiv 0$, these two terms of the control law enable perfect tracking, which means that they represent the *essential* control effort needed to perform the task. Finally, the third part includes a vector u, which represents the *additional* control effort needed for disturbance rejection.

It can also be pointed out that, despite the preceding control law might seem a not well posed system, it will be shown afterwards that the computed torque does not rely on joint accelerations, but on their references.

Matrix T in (35) can be partitioned as follows:

$$T = \begin{bmatrix} T_1 & T_2 & T_3 \end{bmatrix}$$

with $T_1 = \rho I$, where ρ is a positive scalar and I is the nth-order identity matrix.

Substituting the expression of the control law from (35) into the Lagrange-Euler equation of the system (16) and defining $\omega = M(\eta)T_1 M^{-1}(\eta)\tau_{\eta_d}$, one gets:

$$M(\eta)T\dot{\hat{x}} + C(\eta,\dot{\eta})T\hat{x} = u + \omega \tag{36}$$

This expression represents the *dynamic equation of the system error*. Taking into account this nonlinear equation, the nonlinear $\mathcal{H}_\infty$ control problem can be posed as follows:

"Find a control law $u(t)$ such that the ratio between the energy of the cost variable $z = W \left[h^T(\hat{x})\ u^T\right]^T$ and the energy of the disturbance signals ω is less than a given attenuation level γ".

Taking into account the definition of the vector error, $\hat{x}$, and the definition of the cost variable, z, the following structures are considered for matrices Q and S in (31):

$$Q = \begin{bmatrix} Q_1 & Q_{12} & Q_{13} \\ Q_{12} & Q_2 & Q_{23} \\ Q_{13} & Q_{23} & Q_3 \end{bmatrix}, \quad S = \begin{bmatrix} S_1 \\ S_2 \\ S_3 \end{bmatrix}.$$

To apply the theoretical results presented in Section 4.1, it is necessary to rewrite the nonlinear dynamic equation of the error (36) into the standard form of the nonlinear $\mathcal{H}_\infty$ problem (see (28)). This can be done by defining the following expressions:

$$\dot{\hat{x}} = f(\hat{x},t) + g(\hat{x},t)u + k(\hat{x},t)\omega\,, \tag{37}$$

$$f(\hat{x},t) = T_0^{-1}\begin{bmatrix} -M^{-1}C & O & O \\ T_1^{-1} & I - T_1^{-1}T_2 & -I + T_1^{-1}(T_2 - T_3) \\ O & I & -I \end{bmatrix} T_0,$$

$$g(\hat{x},t) = k(\hat{x},t) = T_0^{-1}\begin{bmatrix} M(\eta)^{-1} \\ O \\ O \end{bmatrix}$$

where I is the identity matrix, O the zero matrix, both of n-th order, and

$$T_0 = \begin{bmatrix} T_1 & T_2 & T_3 \\ O & I & I \\ O & O & I \end{bmatrix}. \tag{38}$$

As stated in Section 4.1, the solution of the HJBI equation depends on the choice of the cost variable, z, and particularly on the selection of function $h(\hat{x})$ (see (29)). In this paper, this function is taken to be equal to the error vector, that is, $h(\hat{x}) = \hat{x}$. Once this function has been selected, computing the control law, u, will require finding the Lyapunov function, $V(\hat{x},t)$, to the HJBI equation posed in the previous section (see (32)). The details to achieve this solution can be found in Ortega et al. [2005].

Matrix $T = \begin{bmatrix} T_1 & T_2 & T_3 \end{bmatrix}$ can be computed by solving some Riccati algebraic equations (see Ortega et al. [2005]).

Once matrix T is computed, substituting $V(\hat{x},t)$ in (33), control law u^* corresponding to the $\mathcal{H}_\infty$ optimal index γ is given by

$$u^* = -R^{-1}\left(S^T + T\right)\hat{x} \tag{39}$$

Finally, if the control law (39) is replaced into (35), and after some manipulations, the optimal control law can be written as:

$$\tau_{\eta_a}^* = M(\eta)\ddot{\eta}^d + C(\eta,\dot{\eta})\dot{\eta} - M(\eta)\left(K_D\dot{\tilde{\eta}} + K_P\tilde{\eta} - K_I\int\tilde{\eta}\,dt\right) \tag{40}$$

A particular case can be obtained when the components of weighting compound $W^T W$ verify:

$$Q_1 = \omega_1^2 I, \quad Q_2 = \omega_2^2 I, \quad Q_3 = \omega_3^2 I, \quad R = \omega_u^2 I, \tag{41}$$

$$Q_{12} = Q_{13} = Q_{23} = 0, \quad S_1 = S_2 = S_3 = 0.$$

In this case, the following analytical expressions for the gain matrices have been obtained:

$$K_D = \frac{\sqrt{\omega_2^2 + 2\omega_1\omega_3}}{\omega_1} I + M(\eta)^{-1}\left(C(\eta,\dot{\eta}) + \frac{1}{\omega_u^2} I\right),$$

$$K_P = \frac{\omega_3}{\omega_1} I + \frac{\sqrt{\omega_2^2 + 2\omega_1\omega_3}}{\omega_1} M(\eta)^{-1}\left(C(\eta,\dot{\eta}) + \frac{1}{\omega_u^2} I\right),$$

$$K_I = \frac{\omega_3}{\omega_1} M(\eta)^{-1}\left(C(\eta,\dot{\eta}) + \frac{1}{\omega_u^2} I\right).$$

These expressions have an important property: they do not depend on the parameter γ. So, we obtain an algebraic expression for computing the general optimal solution for this particular case.

5. SIMULATION RESULTS

The proposed control strategy has been tested by simulations in order to check the performance attained for the path tracking problem. Simulations has been performed considering external disturbances and parametric uncertainties.

The following vertical helix has been defined as the reference trajectory:

$$x_d = \frac{1}{2}\cos\left(\frac{t}{2}\right),\ y_d = \frac{1}{2}\sin\left(\frac{t}{2}\right),\ z_d = 1 + \frac{t}{10},\ \psi_d = \frac{\pi}{3}$$

The initial conditions of the helicopter are $(x,y,z) = (0{,}0{,}0.5)m$ and $(\phi,\theta,\psi) = (0{,}0{,}0.5)rad$. The values of the model parameters used for simulations are the following: $m = 0.74\ kg$, $l = 0.21\ m$, $g = 9.81 m/s^2$ and $I_{xx} = I_{yy} = 0.004\ Kg.m^2$, $I_{zz} = 0.0084\ Kg.m^2$. An amount of $\pm 20\%$ in the uncertainty of the elements of the inertia matrix has been considered in the simulations.

In the simulations external disturbances on the aerodynamic moments were considered. The following persistent steps were applied: $A_r = 0.5Nm$ at $t = 5s$; $A_p = 1Nm$ at $t = 15s$; and $A_q = 1Nm$ at $t = 25s$. The E-SSPC parameters were adjusted as follows:

$$N_{2_z} = N_{u_z} = 3I_{nz}\ ,\quad Q_z = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\ ,\quad R_z = 0.01$$

$$N_{2_{xy}} = N_{u_{xy}} = 3I_{nxy}\ ,\quad Q_{xy} = \begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\ ,\quad R_{xy} = \begin{bmatrix} 90 & 0 \\ 0 & 90 \end{bmatrix}$$

The nonlinear $\mathcal{H}_\infty$ controller gains were tuned with the following values: $\omega_1 = 0.05$, $\omega_2 = 0.5$, $\omega_3 = 5$ y $\omega_u = 0.7$.

Figs. 2 to 4 present a perfect tracking of the reference trajectory when external disturbance originated by aerodynamic moments are considered. The results illustrate the robust performance provided by the controller in the case of parametric uncertainty in the inertia terms. Using the E-SSPC a smooth reference tracking was performed, mainly, in the beginning of the track where the vehicle is far from the trajectory. This is due because the predictive controller considers the future reference in the computation of the control signal and thus, it tries to predict the path smoothing the displacement.

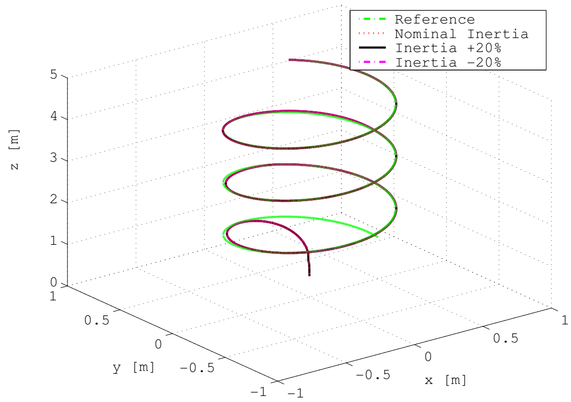

Fig. 2. Path following with external disturbances.

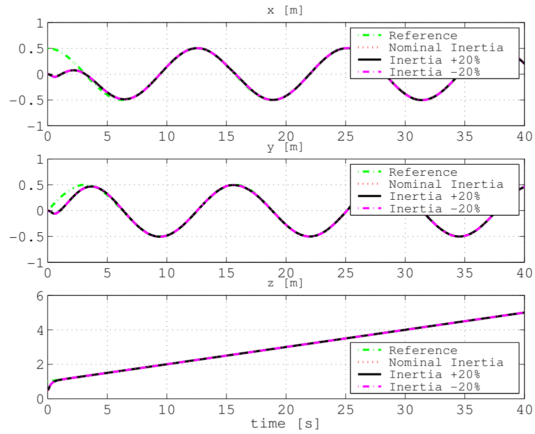

Fig. 3. Position (x, y, z) with external disturbances.

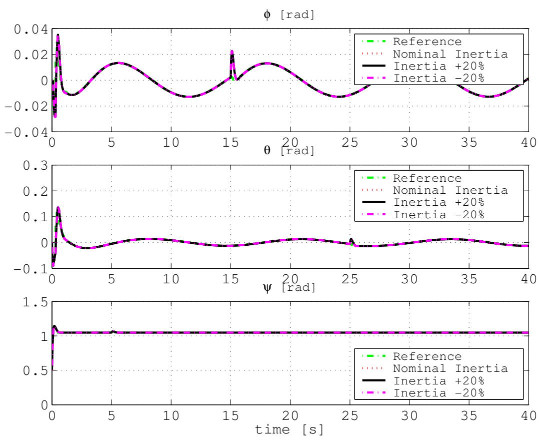

Fig. 4. Orientation (ϕ, θ, ψ) with external disturbances.

6. CONCLUSIONS

In this paper a predictive and robust control strategy to solve the path tracking problem for a quadrotor helicopter has been presented. The proposed strategy was designed in consideration of external disturbances like aerodynamic moments. Through the state space predictive controller for the linear movements a good and smooth performance in the reference tracking has been achieved. A robust control based on nonlinear $\mathscr{H}_\infty$ theory has been used for the helicopter stabilization, which is able to reject moment disturbances. Besides, the $\mathscr{H}_\infty$ controller robustness has been checked under uncertainty in the inertia terms. Finally, the robustness, the smoothness and the predictive feature of the presented control strategy has been also corroborated by simulations.

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