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Nonlinear Adaptive H_∞ Output Feedback Tracking Control for Robotic Systems

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Abstract: This paper presents a solution to the tracking control problem of robotic systems in the presence of exogenous disturbances and model uncertainty with partial state information. The solution yields a Linear Matrix Inequalities (LMIs) based tracking output feedback controller. The main contribution of this paper lies in its particular approach which facilitates an application of the linear H_∞ control theory without linearizing the underlying system. This yields a relatively simple and elegant design procedure. In addition, a relatively low gain controller is achieved. Simulation results of application this control algorithm in a two-degree of freedom robot demonstrates the design procedure feasibility.

1. INTRODUCTION

This paper introduces a solution to the trajectory tracking control of robotic manipulators which is based on the H_∞ control and LMI methods. It is assumed that only a noisy partial state information is available, and that a model uncertainty and exogenous disturbances are present.

There are numerous papers which present studies of this subject, see, e.g. [1]-[7] for a state feedback utilization, and [8], [9], [10] for output feedback applications. Studies of this subject which deal with model uncertainty and assume partial information while using adaptive and robust control may be found in [11], [12], [13].

To the best of our knowledge all the studies (excluding those that take the H_∞ approach) do not assume the presence of exogenous disturbances, neither a plant noise, nor a measurement noise. The works of Acho et. al ([26]) and Zasadzinski et. al ([10]) which take the H_∞ approach, although they assume a presence of exogenous disturbances they do not consider model uncertainty as the theories they develop do not account for it.

The novelty of this paper is in its particular approach and in its extent of generality. In particular: 1. the results achieved in this work apply to robotic systems with model uncertainty and with exogenous disturbances that include both, noise associated with the plant and noisy measurements. 2. A particular choice of a storage function which facilitates an application of the linear H_∞ control theory and the LMI methods without linearizing the underlying system.

In view of the theory of nonlinear H_∞ control (see, e.g. [17]-[21]), we formulate the tracking problem as an H_∞ control problem, and use the interrelations among the the l_2-gain property, dissipativity and the Hamilton-Jacobi Inequality (HJI) to derive an output feedback controller, first for the case of absence of uncertainty, and then utilizing these results, we develop a controller that accounts for model uncertainty, achieves L_2-gain$< \gamma$ for a prescribed γ, and a semi-global asymptotic stability. As mentioned above, all this is facilitated by the particular choice of a storage function that takes an advantage of some certain structural properties the underlying system enjoys. This yields sufficient conditions, in terms of certain LMIs for the semi-global asymptotic stability and for the L_2-gain property to hold. The advantage of these sufficient conditions is that they turn to be exactly as the usual ones for an appropriate linear system (see, e.g [14]-[16]). We also introduce an example which demonstrates the algorithm performances by an application in a two-degree of freedom robot where gravity and the model-parameters are only approximately known, while relatively large uncertainties are assumed.

2. PROBLEM FORMULATION: NO MODEL UNCERTAINTY

In this section we consider the tracking problem of an n-link robot manipulator with no model uncertainty.

In section 2.1 below we introduce a convenient state space representation of the underlying system. The nonlinear H_∞ control problem is formulated in section 2.2, while the solution to the nonlinear HJI is introduced in section 2.3.

2.1 The System dynamics

The dynamical equations of an n-link robot manipulator with exogenous disturbances is commonly described by the following (see, e.g. Spong and Vidyasagar [1])

$$M(q)\ddot{q} + (C(q,\dot{q}) + H)\dot{q} + G(q) = \tau + \omega \tag{1}$$

where $q \in \Re^n$ is the robot's joint angular position, $M(q) \in \Re^{n\times n}$ is the symmetric positive definite inertia matrix, $C(q,\dot{q})\dot{q}$ is the centripetal and coriolis forces, $H\dot{q}$ represents the linear frictional forces, $G(q)$ consists of the gravitational forces, τ is the torque applied to the various links at the corresponding joints by means of electrical motors and ω represents exogenous disturbances, which are assumed to be in L_2, that is $\int_0^\infty ||\omega(t)||^2 dt < \infty$.

The objective is a design of an output feedback which drives the system's states along a desired trajectory $q_r(t)$ starting at a given initial position. For this we define the following error vector:

$$e = \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} q - q_r \\ \dot{q} - \dot{q}_r \end{bmatrix} \quad (2)$$

We take:

$$\tau = M(q_r)\ddot{q}_r + (C(q_r, \dot{q}_r) + H)\dot{q}_r + G(q_r) + u \quad (3)$$

where u is defined in section 2.2 below. Define

$$\Delta W = [M(q) - M(q_r)]\ddot{q}_r + [C(q, \dot{q}_r) - C(q_r, \dot{q}_r)]\dot{q}_r + [G(q) - G(q_r)]. \quad (4)$$

Using these in (1), yields the following tracking problem.

$$\dot{e} = A(q, \dot{q}, \dot{q}_r)e + B(q)(u - \Delta W + \omega) \quad (5)$$

where

$$A(q, \dot{q}, \dot{q}_r) = \begin{bmatrix} 0_{n\times n} & I_{n\times n} \\ 0_{n\times n} & -M^{-1}(q)(C(q, \dot{q}) + C(q, \dot{q}_r) + H) \end{bmatrix}$$
$$B(q) = \begin{bmatrix} 0_{n\times n} \\ M^{-1}(q) \end{bmatrix} \quad (6)$$

2.2 Nonlinear H_∞ control problem

Consider the nonlinear system:

$$\begin{cases} \dot{e} = A(q, \dot{q}, \dot{q}_r)e + B(q)(u - \Delta W) + B(q)\omega \\ y = C_2 e + D_{21}\omega \\ z = C_1 e + D_{12} u \end{cases}$$

Where $e \in \Re^{2n}, u \in \Re^m, y \in \Re^s$ and $\omega \in \Re^d$ are the state, the control input, the measurement output and disturbances, respectively, while $z \in \Re^h$ is an objective variable (controlled output).

The H_∞ output-feedback control objective is a synthesis of an output-feedback that renders the underlying system L_2-gain$< \gamma$. In order to achieve this goal the following controller structure is assumed

$$\begin{cases} \dot{\xi} = T(q)^{-1}[A_k\xi + B_k y] \\ u = C_k\xi + D_k y \end{cases} \quad (7)$$

where $\xi \in \Re^{2n}$, $T(q)$ is a $2n \times 2n$ matrix, and A_k, B_k, C_k, D_k are constant matrices. Let

$$x = \begin{bmatrix} e \\ \xi \end{bmatrix} \quad (8)$$

Thus the closed-loop system admits

$$\begin{cases} \dot{x} = A_{cl}(q, \dot{q}, \dot{q}_r)x - B_{cl}(q)\Delta W + B_{1cl}(q)\omega \\ z = C_{cl}x + D_{cl}\omega \end{cases} \quad (9)$$

where

$$\left[\begin{array}{c|c} A_{cl}(q, \dot{q}, \dot{q}_r) & B_{1cl}(q) \\ \hline C_{cl} & D_{cl} \end{array}\right] = \left[\begin{array}{cc|c} A(q, \dot{q}, \dot{q}_r) + B(q)D_kC_2 & B(q)C_k & B(q)(D_kD_{12} + I) \\ T(q)^{-1}B_kC_2 & T(q)^{-1}A_k & T(q)^{-1}B_kD_{21} \\ \hline C_1 + D_{12}D_kC_2 & D_{12}C_k & D_{12}D_kD_{21} \end{array}\right] \quad (10)$$

and

$$B_{cl}(q) = \begin{bmatrix} B(q) \\ 0_{\ 2n\times n} \end{bmatrix}. \quad (11)$$

2.3 Solution To The Nonlinear Hamilton-Jacobi Inequality (HJI)

Consider the nonlinear system (9) with the following storage function

$$S_o(x, q) = \frac{1}{2}x^T P_o(q)x \quad (12)$$

where $P_o(q) \in \Re^{4n\times 4n}$ is a positive C^1 matrix, (note that $P_o(q)$ is not necessarily a symmetric matrix). Define,

$$M_s(q) = blockdiag\{I_{n\times n}, M(q)\} \quad (13)$$

where $M(q)$ is the inertia matrix and $blockdiag\{\cdot\}$ denotes a diagonal block matrix. The notation $*$ will be used frequently in the sequel and will denote a symmetric entry of a matrix.

We have now the following theorem, the proof of which is omitted for the lack of space.

Theorem 1. Given $\delta > 0$. Assume $P_o(q)$ has the following structure:

$$P_o(q) = P_{o.c}M_o(q) \quad (14)$$

where

$$M_o(q) = blockdiag\{M_s(q), T(q)\}, \quad (15)$$

$M_s(q)$ given in (13) and $P_{o.c} \in \Re^{4n\times 4n}$ is a positive symmetric matrix that is to be determined. Then the closed-loop system (9) is L_2-gain $< \gamma$, and the controller (7) renders the closed-loop system semi-global exponentially stable if the following LMI's

$LMI\ (1):$

$$\begin{bmatrix} P_{o.c}A_{cl} + A_{cl}^T P_{o.c} & P_{o.c}B_{1cl} & C_{cl}^T & P_{o.c}B_{cl} & \Delta\tilde{W} \\ * & -\gamma^2 I & D_{cl}^T & 0 & 0 \\ * & * & -I & 0 & 0 \\ * & * & * & -\frac{1}{2}\delta I & 0 \\ * & * & * & * & -\delta^{-1}I \end{bmatrix} < 0 \quad (16)$$

$LMI\ (2):$

$$P_{o.c}M_o(q) + (P_{o.c}M_o(q))^T > 0 \quad (17)$$

hold for $e_2 \in B_r$ with an arbitrarily fixed $r > 0$ and for all q, where

$$\left[\begin{array}{c|c} A_{cl} & B_{1cl} \\ \hline C_{cl} & D_{cl} \end{array}\right] = \left[\begin{array}{cc|c} A + BD_kC_2 & BC_k & B(D_kD_{12} + I) \\ B_kC_2 & A_k & B_kD_{21} \\ \hline C_1 + D_{12}D_kC_2 & D_{12}C_k & D_{12}D_kD_{21} \end{array}\right],$$
$$\Delta\tilde{W} = blockdiag\{\Delta\tilde{W}_1, 0_{2n\times 2n}\},$$
$$\Delta\tilde{W}_1 = blockdiag\{b_{mcg}I_{n\times n}, b_{\tilde{c}}I_{n\times n}\}, \quad b_{mcg}, b_{\tilde{c}} > 0 \quad (18)$$

and

$$B_{cl} = \begin{bmatrix} B \\ 0_{\ 2n\times n} \end{bmatrix}, \quad B = \begin{bmatrix} 0_{n\times n} \\ I_{n\times n} \end{bmatrix}$$
$$A = \begin{bmatrix} 0_{n\times n} & I_{n\times n} \\ 0_{n\times n} & -H \end{bmatrix}. \quad (19)$$

Remark 1. The L_2-gain$< \gamma$ means in this theorem that for any initial state x_0 there is a neighborhood $B_{x_0} \subset L_2$ of 0 such that

$$\int_0^\infty ||z(t)||^2 dt < \gamma^2\{||x_0||^2 + \int_0^\infty ||\omega(t)||^2 dt\}, \quad \forall t \geq 0.$$

Theorem 2 below provides sufficient conditions for Theorem 1 to hold.

Theorem 2. Fix $\delta > 0, \gamma > 0, r > 0$. Assume the following LMI's

$$LMI\ 1: \begin{bmatrix} \Phi_{11} & \Phi_{12} & \Phi_{13} & \Phi_{14} & XB & \Delta\tilde{W}_1 \\ * & \Phi_{22} & \Phi_{23} & \Phi_{24} & B & Y\Delta\tilde{W}_1 \\ * & * & -\gamma^2 I & 0 & 0 & 0 \\ * & * & * & -I & 0 & 0 \\ * & * & * & * & -\frac{1}{2}\delta I & 0 \\ * & * & * & * & * & -\delta^{-1}I \end{bmatrix} < 0 \quad (20)$$

$$\Phi_{11}=A^T X+XA+\hat{B}_k C_2+C_2^T \hat{B}_k^T, \quad \Phi_{12}=\hat{A}_k+A^T+C_2^T \hat{D}_k^T B^T$$
$$\Phi_{13}=XB+\hat{B}_k D_{21}, \quad \Phi_{14}=C_1^T+C_2^T \hat{D}_k^T D_{12}^T$$
$$\Phi_{22}=YA^T+AY+B\hat{C}_k+\hat{C}_k^T B^T, \quad \Phi_{23}=B+B\hat{D}_k D_{21}$$
$$\Phi_{24}=YC_1^T+\hat{C}_k^T D_{12}^T$$

$$LMI\ 2: \begin{bmatrix} XM_s^{(j)} + M_s^{(j)}X & M_s^{(j)} \\ * & M_s^{(j)}Y + YM_s^{(j)} \end{bmatrix} > 0 \quad (21)$$

hold for some symmetric positive definite matrices $X, Y \in \Re^{2n\times 2n}$ and for some $\hat{A}_k \in \Re^{2n\times 2n}, \hat{B}_k \in \Re^{2n\times n}, \hat{C}_k \in \Re^{n\times 2n}, \hat{D}_k \in \Re^{n\times n}$, where each $M_s^{(j)}$ is the $M_s(q)$ evaluated at the vertex j of the polytope generated by $M_s(q)$. Then the closed-loop system (9), with the controller (3), (7), is L_2-gain$<\gamma$, and semi-global exponentially stable, where $T(q)$ is given by

$$T(q) = -M^{-1}XM_s(q)YN^{-T} \quad (22)$$

If a solution to these *LMIs* exist, the output feedback gains are given by

$$\hat{A}_k = XAY+MA_kN^T+XBC_kN^T+MB_kC_2Y+XBD_kC_2Y$$
$$\hat{B}_k = MB_k+XB_2D_k$$
$$\hat{C}_k = C_kN^T+D_kC_2Y$$
$$\hat{D}_k = D_k \quad (23)$$

3. PROBLEM FORMULATION WITH MODEL UNCERTAINTIES

This section deals with the tracking problem of an n-link robot manipulator with model uncertainties.

In section 3.1 below we introduce the state space equations. The nonlinear H_∞ control problem is formulated in section 3.2, while the solution to the nonlinear *HJI* is introduced in section 3.3.

3.1 The System Dynamics

Generally, in addition to external disturbances, there are uncertainties present in the system's model which must be accounted for. Let $\hat{M}(q)$ denote an estimate of $M(q)$, $\hat{C}(q,\dot{q})$ an estimate of $C(q,\dot{q})$, $\hat{H}$ an estimate of H and $\hat{G}(q)$ an estimate of the gravitational forces $G(q)$.

The inverse dynamics control law for the nominal system (1) is given by

$$\tau = \hat{M}(q_r)\ddot{q}_r + (\hat{C}(q_r,\dot{q}_r) + \hat{H})\dot{q}_r + \hat{G}(q_r) + u. \quad (24)$$

Substituting (24) into (1) and subtract $M(q)\ddot{q}_r + (C(q,\dot{q}) + H)\dot{q}_r$ from both sides of the equation we obtain

$$M(q)\dot{e}_2+(C(q,\dot{q}) + H)e_2 = u + \omega + [\hat{M}(q_r) - M(q)]\ddot{q}_r +$$
$$[\hat{C}(q_r,\dot{q}_r)-C(q,\dot{q})+\hat{H}-H]\dot{q}_r + [\hat{G}(q_r) - G(q)]. \quad (25)$$

It is easy to show now that (25) may now be expressed as

$$M(q)\dot{e}_2 + (C(q,\dot{q}) + C(q,\dot{q}_r) + H)e_2 = u + \omega +$$
$$[\hat{M}(q_r) - M(q)]\ddot{q}_r + [\hat{C}(q_r,\dot{q}_r) - C(q,\dot{q}_r)]\dot{q}_r +$$
$$[\hat{H}-H]\dot{q}_r + [\hat{G}(q_r) - G(q)]. \quad (26)$$

from which one obtains

$$M(q)\dot{e}_2 + (C(q,\dot{q}) + C(q,\dot{q}_r) + H)e_2 = u + \omega +$$
$$Y_1(q_r,\dot{q}_r,\ddot{q}_r)P_{o.c} - \Delta W. \quad (27)$$

where $\tilde{p}=\hat{p}-p$ is the parameter error vector, $Y_1(q_r,\dot{q}_r,\ddot{q}_r)$ is the regressor matrix and ΔW given by (4). Thus, the state space can be written as

$$\dot{e} = A(q,\dot{q},\dot{q}_r)e + B(q)(u + \omega + Y_1(q_r,\dot{q}_r,\ddot{q}_r)\tilde{p} - \Delta W) \quad (28)$$

where $A(q,\dot{q},\dot{q}_r), B(q)$ are given in (6).

3.2 The Nonlinear H_∞ control problem

Consider the nonlinear system:

$$\begin{cases} \dot{e} = A(q,\dot{q},\dot{q}_r)e+B(q)(u+Y_1(q_r,\dot{q}_r,\ddot{q}_r)\tilde{p}-\Delta W)+B(q)\omega \\ y = C_2e + D_{21}\omega \\ z = C_1e + D_{12}u \end{cases} \quad (29)$$

In order to obtain an adaptive H_∞ output-feedback control objective our goal is to compute a dynamical output-feedback controller in the same form as given in (7),i.e

$$\begin{cases} \dot{\xi} = \bar{T}(q)^{-1}[A_k\xi + B_ky] \\ u = C_k\xi + D_ky \end{cases} \quad (30)$$

where $\xi \in \Re^{2n}$, $\bar{T}(q)$ is a $2n \times 2n$ matrix (to be determined below) and A_k, B_k, C_k, D_k are constant matrices.

In addition, we choose the parameter estimator of the form

$$\dot{\hat{p}} = A_x(q_r,\dot{q}_r,\ddot{q}_r)x \quad (31)$$

where x given by (8) and $A_x(q_r,\dot{q}_r,\ddot{q}_r)$ will be determined later. Let $\tilde{x}$ be defined by

$$\tilde{x} = \begin{bmatrix} x \\ \tilde{p} \end{bmatrix}. \quad (32)$$

Then the closed-loop system admits

$$\begin{cases} \dot{\tilde{x}}=\tilde{A}_{cl}(q,\dot{q},\dot{q}_r)\tilde{x}+\tilde{B}_{cl}(q)(Y_1(q_r,\dot{q}_r,\ddot{q}_r)\tilde{p}-\Delta W)+\tilde{B}_{1cl}(q)\omega \\ z=\tilde{C}_{cl}\tilde{x} + \tilde{D}_{cl}\omega \end{cases} \quad (33)$$

where

$$\left[\begin{array}{c|c} \tilde{A}_{cl}(q,\dot{q},\dot{q}_r) & \tilde{B}_{1cl}(q) \\ \hline \tilde{C}_{cl} & \tilde{D}_{cl} \end{array}\right] = \left[\begin{array}{cc|c} A_{cl}(q,\dot{q},\dot{q}_r) & 0_{r\times r} & B_{1cl}(q) \\ A_x(q_r,\dot{q}_r,\ddot{q}_r) & 0_{n\times r} & 0_{r\times d} \\ \hline C_{cl} & 0_{h\times r} & D_{cl} \end{array}\right]$$

$$\tilde{B}_{cl}(q) = \begin{bmatrix} B_{cl}(q) \\ 0_{r\times n} \end{bmatrix}. \quad (34)$$

and $A_{cl}(q,\dot{q},\dot{q}_r), B_{cl}(q), C_{cl}, D_{cl}, B_{1cl}(q)$ are given in (10,11) with $\bar{T}(q)$ instead of $T(q)$.

3.3 Solution To The Nonlinear HJI

Consider the nonlinear system (33) with the following storage function

$$\bar{S}_o(x,\tilde{p},q) = \frac{1}{2}\{x^T\bar{P}_o(q)x + \tilde{p}^T\Lambda\tilde{p}\} \quad (35)$$

where Λ is a positive definite weighting matrix and $\bar{P}_o(q)$ is a positive C^1 matrix, (note that $\bar{P}_o(q)$ is not necessary a symmetric matrix).

Theorem 3. *Given* $\delta > 0$. Assume the following structure for $\bar{P}_o(q)$:

$$\bar{P}_o(q) = P_{o.c}\bar{M}_o(q) \tag{36}$$

where

$$\bar{M}_o(q) = blockdiag\{M_s(q), \bar{T}(q)\}, \tag{37}$$

$M_s(q)$ given in (13) and $P_{o.c} \in \Re^{4n\times 4n}$ is a positive symmetric matrix to be determined. Then the closed-loop system (33) is $L_2 - gain \leq \gamma$, and the controller (30) renders the closed-loop system semi-global asymptotically stable if the following LMI's

LMI (1) :

$$\begin{bmatrix} P_{o.c}A_{cl} + A_{cl}^T P_{o.c} & P_{o.c}B_{1cl} & C_{cl}^T & P_{o.c}B_{cl} & \Delta\tilde{W} \\ * & -\gamma^2 I & D_{cl}^T & 0 & 0 \\ * & * & -I & 0 & 0 \\ * & * & * & -\frac{1}{2}\delta I & 0 \\ * & * & * & * & -\delta^{-1}I \end{bmatrix} < 0 \tag{38}$$

LMI (2) :

$$P_{o.c}\bar{M}_o(q) + (P_{o.c}\bar{M}_o(q))^T > 0 \tag{39}$$

hold for $e_2 \in B_r$ with an arbitrarily fixed $r > 0$ and for all q, where

$$\left[\begin{array}{c|c} A_{cl} & B_{1cl} \\ \hline C_{cl} & D_{cl} \end{array}\right] = \left[\begin{array}{cc|c} A+BD_kC_2 & BC_k & B(D_kD_{12}+I) \\ B_kC_2 & A_k & B_kD_{21} \\ \hline C_1+D_{12}D_kC_2 & D_{12}C_k & D_{12}D_kD_{21} \end{array}\right],$$

$$\Delta\tilde{W} = blockdiag\{\Delta\tilde{W}_1, 0_{2n\times 2n}\},$$
$$\Delta\tilde{W}_1 = blockdiag\{b_{mcg}I_{n\times n}, b_{\tilde{c}0}I_{n\times n}\}, \quad b_{mcg}, b_{\tilde{c}0} > 0 \tag{40}$$

and

$$B_{cl} = \begin{bmatrix} B \\ 0_{\,2n\times n} \end{bmatrix}, \quad B = \begin{bmatrix} 0_{n\times n} \\ I_{n\times n} \end{bmatrix}$$
$$A_0 = \begin{bmatrix} 0_{n\times n} & I_{n\times n} \\ 0_{n\times n} & 0_{n\times n} \end{bmatrix}. \tag{41}$$

In what follows we utilize the algorithm introduced in [16] in order to solve the LMI's of Theorem 3 via LMI's optimization toolbox in MATLAB.

The following notations will be used in the sequel

$$M(q) = \begin{bmatrix} m_{11}(p,q) & \cdots & m_{1n}(p,q) \\ * & \ddots & \vdots \\ * & * & m_{nn}(p,q) \end{bmatrix}$$

where $m_{ik}(p,q)$ are bounded, with known bounds. It is well known that the parameters vector p is a function of the physical system's parameters like: masses, lengths etc.. We take f_i to be the i-th physical parameter of the system, therefore if we assume that the system has $-l-$ physical parameters then the vector p may be written as $p = \mathcal{F}(f_1, f_2, ..., f_l)$. We denote the upper bound of f_i by f_i^+ (i.e. $f_i \in [f_i^-, f_i^+]$) and the lower bound of f_i by f_i^- and define the following: Let f_i^{av} be the *average physical parameter* of f_i, and p^{av} be the *average vector parameter* of p which are given by:

$$f_i^{av} = \frac{1}{2}(f_i^- + f_i^+) \quad \forall i = 1, ..., l.$$
$$p^{av} = \mathcal{F}(f_1^{av}, f_2^{av}, ..., f_l^{av}). \tag{42}$$

Remark 2. Note that if the uncertainty range of f_i shrinks to zero then $p^{av} \to p$.

Define,

$$M^{av}(q) = \begin{bmatrix} m_{11}(p^{av},q) & \cdots & m_{1n}(p^{av},q) \\ * & \ddots & \vdots \\ * & * & m_{nn}(p^{av},q) \end{bmatrix} \tag{43}$$

where $M^{av}(q)$ is the average matrix of the inertia matrix $M(q)$. Obviously, by the above definition of $M^{av}(q)$, this matrix agrees with assumption **A1**. Finally we define the matrix

$$M_s^{av}(q) = blockdiag\{I_{2n\times 2n}, M^{av}(q)\} \tag{44}$$

We have now the following result.

Theorem 4. Consider the closed-loop system (33) with the storage function (35) where

$$\bar{T}(q) = -M^{-1}XM_s^{av}(q)YN^{-1} \tag{45}$$

Given the scalars $\delta > 0, \gamma > 0, r > 0, \varepsilon > 0$, there is an output-feedback controller given by (30) with the parameter update process given by (31), where $Y_1^T(q_r, \dot{q}_r, \ddot{r}_r)B_{cl}^T\tilde{P}_o^T(q) = A_x(q_r, \dot{q}_r, \ddot{q}_r)$. If the following LMI's

$$LMI\ 1: \begin{bmatrix} \Phi_{11} & \Phi_{12} & \Phi_{13} & \Phi_{14} & XB & \Delta\tilde{W}_1 \\ * & \Phi_{22} & \Phi_{23} & \Phi_{24} & B & Y\Delta\tilde{W}_1 \\ * & * & -\gamma^2 I & 0 & 0 & 0 \\ * & * & * & -I & 0 & 0 \\ * & * & * & * & -\frac{1}{2}\delta I & 0 \\ * & * & * & * & * & -\delta^{-1}I \end{bmatrix} \leq 0 \tag{46}$$

$\Phi_{11}=A^TX+XA+\hat{B}_kC_2+C_2^T\hat{B}_k^T$, $\Phi_{12}=\hat{A}_k+A^T+C_2^T\hat{D}_k^TB^T$
$\Phi_{13}=XB+\hat{B}_kD_{21}$, $\Phi_{14}=C_1^T+C_2^T\hat{D}_k^TD_{12}^T$
$\Phi_{22}=YA^T+AY+B\hat{C}_k+\hat{C}_k^TB^T$, $\Phi_{23}=B+B\hat{D}_kD_{21}$
$\Phi_{24}=YC_1^T+\hat{C}_k^TD_{12}^T$

LMI (2) :

$$\begin{bmatrix} \Psi_{11} & M_s^{(j)}(q) & X(M_s^{(j)}-M_s^{av(j)}) & 0 \\ * & \Psi_{22} & 0 & Y \\ * & * & \varepsilon I & 0 \\ * & * & * & \varepsilon^{-1}I \end{bmatrix} > 0 \tag{47}$$

$$\Psi_{11} = XM_s^{(j)}(q)+M_s^{(j)}(q),$$
$$\Psi_{22} = M_s^{(j)}(q)Y+YM_s^{(j)}(q)$$

hold for some symmetric positive definite matrices $X, Y \in \Re^{2n\times 2n}$ and for some $\hat{A}_k \in \Re^{2n\times 2n}, \hat{B}_k \in \Re^{2n\times n}, \hat{C}_k \in \Re^{n\times 2n}, \hat{D}_k \in \Re^{n\times n}$, where each $M_s^{(j)}, M_s^{av(j)}$ are the $M_s(q), M_s^{av}(q)$ (given by (13),(44) respectively) evaluated at the vertex j of the polytope generated by $M_s(q), M_s^{av}(q)$, respectively. If a solution to these *LMIs* exists, then the closed-loop system (33) has L_2-gain$\leq \gamma$ (from ω to z) and the tracking error $e \to 0$ as $t \to \infty$ semi-globally.

In this case, the output-feedback is given by (24,30,23) with the parameter update process

$$\hat{p}(t) = \hat{p}(0) + A_x(q_r(t), \dot{q}_r(t), \ddot{q}_r(t))x_I(t) - \int_0^t \dot{A}_x(q_r(\sigma), \dot{q}_r(\sigma), \ddot{q}_r(\sigma))x_I(\sigma)d\sigma \tag{48}$$

where

$$x_I(t) = \int_0^t x(\tau)d\tau = \begin{bmatrix} \begin{bmatrix} \int_0^t e_1(\tau)d\tau \\ e_1(t) \end{bmatrix} \\ \int_0^t \xi(\tau)d\tau \end{bmatrix} \quad (49)$$

$$(50)$$

4. EXAMPLE

The feasibility of the design of the foregoing sections is demonstrated via simulations of a two-link manipulator. The system is assumed to have known parameters and external disturbances. The H_∞ tracking control is then designed according to the proposed procedure. The system's parameters are: the links' masses: $m_1, m_2(kg)$, the links' lengths: $l_1, l_2(m)$, masses' centers: l_{c1}, l_{c2}, the angular positions: $q_1, q_2(rad)$, $q_1, q_2(rad)$, the viscosity coefficient : $h(kgm^2)$ and the applied torques: $\tau_1, \tau_2(Nm)$. By (1) we have:

$$M(q)=\begin{bmatrix} m_1l_{c1}^2+m_2l_1^2+I_{zz1}+I_{zz2} & m_2l_1l_{c2}cos(q_1-q_2)+I_{zz2} \\ m_2l_1l_{c2}cos(q_1-q_2)+I_{zz2} & m_2l_{c2}^2+I_{zz2} \end{bmatrix}$$

$$C(q,\dot{q}) = m_2l_1l_{c2}sin(q_1-q_2)\begin{bmatrix} 0 & \dot{q}_2 \\ -\dot{q}_1 & 0 \end{bmatrix}$$

$$G(q) = \begin{bmatrix} -(m_1l_{c1}+m_2l_1)gsin(q_1) \\ -m_2l_{c2}gsin(q_2) \end{bmatrix}$$

$$H = \begin{bmatrix} h & 0 \\ 0 & h \end{bmatrix} \quad (51)$$

where $q \in \Re^2$ and $\tau \in \Re^2$. The nominal parameters of the manipulator are taken to be: $m_1 = 1(kg), m_2 = 5(kg), l_1 = l_2 = 0.2(m), l_{c1} = l_{c2} = 0.1(m), g = 9.8(ms^{-2}), h = 5((kgm^2))$ where $I_{zz1} = I_{zz2} = \frac{1}{3}m_il^2$ (i=1,2) and the initial conditions are $q_1(0) = 30°, q_2(0) = 100°, \dot{q}_1(0) = 0, \dot{q}_2(0) = 0$. The desired position is: $q_{r1} = 30° sin(2\pi t)$, $q_{r2} = 60° sin(2\pi t)$. The exogenous disturbances $\omega = \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix}$ are chosen to be square wave with period 2π, that is

$$\omega_1 = \begin{cases} 1 & ,0 \le t < \pi \\ 0 & ,\pi \le t < 2\pi \end{cases}, \omega_2 = \begin{cases} 0 & ,0 \le t < \pi \\ -1 & ,\pi \le t < 2\pi \end{cases} \quad (52)$$

For the purpose of simulations, C_1 and D_{12} were chosen as:

$$C_1 = 2\begin{bmatrix} I_{2\times2} & 0_{2\times2} & 0_{2\times1} & 0_{2\times1} \\ 0_{2\times2} & I_{2\times2} & 0_{2\times1} & 0_{2\times1} \end{bmatrix}^T, D_{12} = 0.001\begin{bmatrix} 0_{1\times2} & 0_{1\times2} & 1 & 0 \\ 0_{1\times2} & 0_{1\times2} & 0 & 1 \end{bmatrix}^T$$

and

$$\Delta\tilde{W}_1 = \begin{bmatrix} 25I_{2\times2} & 0_{2\times2} \\ 0_{2\times2} & I_{2\times2} \end{bmatrix}, \delta = 10^{-6}$$

In this case:

$$M^{(j)} = \begin{bmatrix} m_1l_{c1}^2+m_2l_1^2+I_{zz1}+I_{zz2} & m_2l_1l_{c2}\cdot\delta_j+I_{zz2} \\ m_2l_1l_{c2}\cdot\delta_j+I_{zz2} & m_2l_{c2}^2+I_{zz2} \end{bmatrix}$$

$$j=1,2, \quad \delta_1=1, \ \delta_2=-1$$

By applying *Theorem* 2 we obtain $\gamma_{min} = 4.0467$. However, $\gamma = 4.05$ was selected to avoid an undesirable high-gain controller design corresponding to γ which is close to the optimum. (see Fig.1).

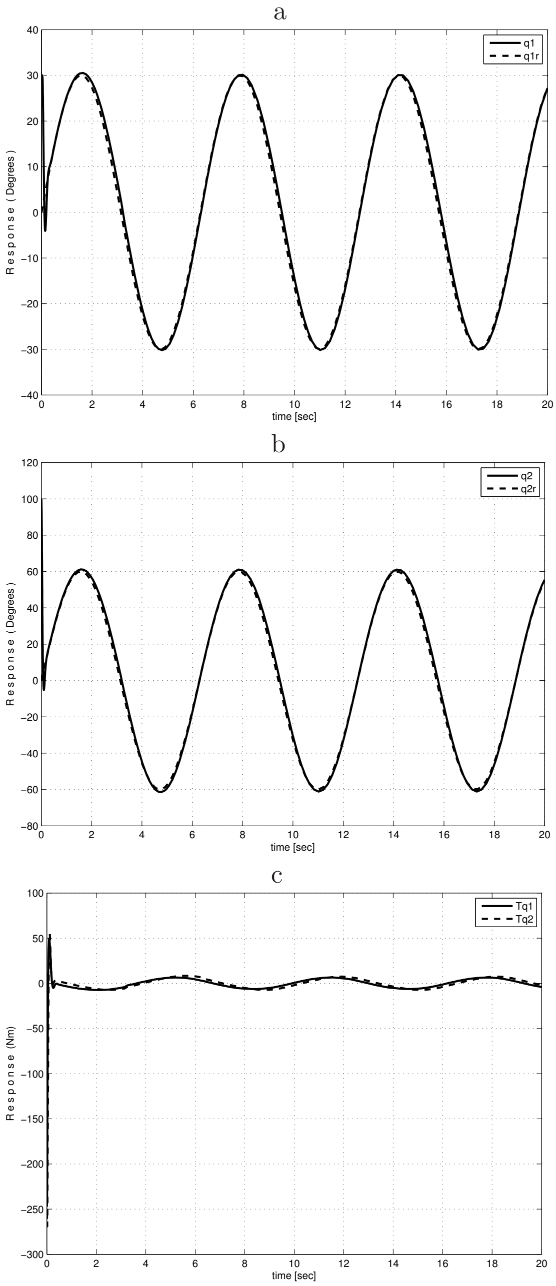

Fig. 1. Exact model: (a) Positions of link 1. (b) Positions of link 2. (c) Torques.

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Appendix A. PROPERTIES AND ASSUMPTIONS

A 1. The matrix $M(q)$ is symmetric positive definite and there exists some positive number σ_m and σ_M such that

$$\sigma_m I \le M(q) \le \sigma_M I \tag{A.1}$$

A 2. There exist some positive constants σ_1 and σ_2 such that

$$\sigma_1 \ge \sup_{q\in\Re^n} \|g(q)\| \tag{A.2}$$

$$\sigma_2 \ge \sup_{q\in\Re^n} \left\|\frac{\partial g(q)}{\partial q}\right\| \tag{A.3}$$

P 1. The representation of the matrix $C(q,\dot q)$ is unique and it can be obtained by the entries of the inertia matrix $M(q)$. Let the `ij-th` element of the inertia matrix $M(q)$ be denoted by m_{ij}, and let the `ik-th` element of the matrix $C(q,\dot q)$ be given by

$$C_{ik}(q,\dot q) = \sum_{j}^{n} c_{ijk}(q) \tag{A.4}$$

where

$$c_{ijk}(q) \equiv \frac{1}{2}\left(\frac{\partial m_{ik}(q)}{\partial q_j} + \frac{\partial m_{jk}(q)}{\partial q_i} - \frac{\partial m_{ij}(q)}{\partial q_k}\right) \tag{A.5}$$

are the Christoffel symbols of the first kind. Then the property

$$\dot M(q) = C(q,\dot q) + C^T(q,\dot q), \quad \forall q,\dot q \tag{A.6}$$

holds. (The proof can be found in [1]).

P 2. The matrix $C(v_1,v_2)$ is bounded in v_1 and linear in v_2, then

$$C(v_1,v_2)v_3 = C(v_1,v_3)v_2, \quad \forall v_1,v_2,v_3 \in \Re^n \tag{A.7}$$

$$\|C(v_1,v_2)\| \le \sigma_4\|v_2\|, \quad \texttt{for some } \sigma_4 > 0,\ \forall v_1,v_2 \tag{A.8}$$

P 3. The system can be parameterized as follows

$$M(q)\ddot q + C(q,\dot q) + H\dot q + G(q) = Y_1(q,\dot q,\ddot q)^T p \tag{A.9}$$

where $p \in \Re^p$ is a vector of constant parameters and $Y_1(q,\dot q,\ddot q) \in \Re^{p\times n}$ is called *regressor matrix* (see [1]).

P 4. Define

$$\Delta W = [M(q)-M(q_r)]\ddot q_r + [C(q,\dot q_r)-C(q_r,\dot q_r)]\dot q_r + [G(q)-G(q_r)], \quad e_1 = q - q_r \tag{A.10}$$

where $(q_r,\dot q_r,\ddot q_r)$ is the desired trajectory, which is bounded. Therefore, there is a positive number b_{mcg} such that $\|\Delta W\| \le b_{mcg}\|e_1\|, \forall e_1$, (see [2]).