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Observers and State Reconstructions for Linear Neutral Time-Delay Systems

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Abstract The problem of state reconstruction is discussed for linear neutral systems with a finite number of commensurable point delays. Using as models systems with coefficients in a ring and following a geometric point of view, feasible and constructing procedures are proposed for the construction of observers of increasing complexity. Conditions are given to characterize neutral systems with delays for which linear observers exist not depending on the derivatives of the state. Some examples illustrating the results are worked out in details.

Keywords: Systems with time-delays, neutral systems, observer design, geometric approach

1. INTRODUCTION

Neutral systems with time delays are used to describe dynamic processes where the evolution of the state depends on the past values of both the state and its derivatives. Processes of this type are those including steam or water pipes, heat exchanges, fluctuation of voltage and current in transmission lines and in chemical engineering. The literature on this kind of systems is very reach and mainly devoted to the important issue of stability (see for instance Kolmanovskii and Myshkis [1999], Fridman [2001] and the references therein).
Here we consider the state reconstruction problem for neutral systems with delays, since several fundamental design procedure are based on the knowledge of the state of the system but in most practical cases, either the states of the delay system is not physically available for direct measurement or the cost of the measurement is prohibitively hight.
The problem of state reconstruction for delay systems, under various assumption, has been investigated by several authors that considered observers of neutral type (see, for instance, Wang et al. [2002], Fan et al. [2003] and Darouach [2005]).
In this work we present a geometric approach to the design of observers for neutral type systems with delays, based on the use of systems over rings, that provides a deep insight on the structural aspects and easy computable solutions (see Perdon et al. [2006]). In particular, conditions are given for the existence of observers whose state evolution does not depend on the past state derivatives for a class of neutral systems with delays. Several examples are worked out in detail.

2. NEUTRAL TIME DELAY SYSTEMS AND NEUTRAL SYSTEMS OVER RINGS

A linear, time invariant neutral system Σ_d with a finite number of commensurable point delays, is described by the equations

$$\Sigma_d = \begin{cases} \dot{x}(t) = \sum_{i=0}^{a} A_i x(t-ih) + \sum_{i=0}^{b} B_i u(t-ih) + \\ \qquad - \sum_{i=1}^{e} E_i \dot{x}(t-ih) \\ y(t) = \sum_{i=0}^{c} C_i x(t-ih) \\ x(t) = \varphi(t),\ t \in [-\alpha\, h, 0] \quad \alpha > 0 \end{cases} \tag{1}$$

where, denoting by $\mathbb{R}$ the field of real numbers, $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $y(t) \in \mathbb{R}^p$, $h \in \mathbb{R}^+$ is the delay, E_i, A_i, B_i, and C_i are matrices of suitable dimensions with entries in $\mathbb{R}$ and $\alpha = \max(e,\ a,\ b,\ c)$, $\varphi(t)$ is a consistent initial condition.
By introducing the delay operator δ defined, for any time function $f(t)$, by $\delta f(t) := f(t-h)$, Σ_d can be written as

$$\begin{cases} \dot{x}(t) = \sum_{i=0}^{a} A_i \delta^i x(t) + \sum_{i=0}^{b} B_i \delta^i u(t) - \sum_{i=1}^{e} E_i \delta^i\ \dot{x}(t) \\ y(t) = \sum_{i=0}^{c} C_i \delta^i x(t). \end{cases}$$

By formally replacing the delay operator δ with the algebraic indeterminate Δ, define

$$\begin{aligned} \tilde{E} &= \sum_{i=1}^{e} E_i \Delta^i, \quad A = \sum_{i=0}^{a} A_i \Delta^i, \\ B &= \sum_{i=0}^{b} B_i \Delta^i, \quad C = \sum_{i=0}^{c} C_i \Delta^i, \end{aligned} \tag{2}$$

Then, we can associate to (1) an abstract discrete time system defined by the equations

$$\Sigma = \begin{cases} x(t+1) = Ax(t) + Bu(t) - \tilde{E}x(t+1) \\ y(t) = Cx(t). \end{cases} \tag{3}$$

Definition 1. A *neutral linear system* over the ring in $\mathcal{R} = \mathbb{R}[\Delta]$ is a system Σ defined by equation (3) where, by abuse of notation, we denote by $x(\cdot)$, $u(\cdot)$ and $y(\cdot)$ elements of the finitely generated free $\mathcal{R}$-modules $\mathcal{X} = \mathcal{R}^n$, $\mathcal{U} = \mathcal{R}^m$

and $\mathcal{Y} = \mathcal{R}^p$ respectively and $\tilde{E}$, A, B and C are $\mathcal{R}$-linear maps.

Remark that, writing $E = I_n + \tilde{E}$, the system (3) is a particular case of system in descriptor form

$$\begin{cases} Ex(t+1) = Ax(t) + Bu(t) \\ y(t) = Cx(t). \end{cases} \tag{4}$$

over the ring $\mathcal{R}$ as described in Perdon and Anderlucci [2006].

Many control problems concerning systems with commensurable delays as (1) can be studied by means of an abstract object of the form (3). In fact, even if they are very different objects from a dynamical point of view, they share the structural properties that depend only on the defining matrices $(E,\ A,\ B,\ C)$. Therefore, problems concerning the input/output behavior of Σ_d can be naturally formulated in terms of the input/output behavior of Σ and solved in the framework of systems over rings. Substituting back the delay operator δ to the algebraic indeterminate Δ, the solutions can be interpreted in the original delay-differential framework (see, for instance Conte and Perdon [1995, 1998, 2005]).

Proposition 2. Assume that $E = I_n + \tilde{E}$, where $\tilde{E}$ is a nonzero polynomial matrix multiple of Δ, then $\ker E = \{0\}$.

Proof. Suppose $\ker E \neq \{0\}$ and write $\tilde{E} = \Delta\bar{E}$, with $\bar{E} = \sum_{i=1}^{e} E_i\Delta^{i-1}$. Then there exists an element $0 \neq v \in \mathcal{X}$ such that $Ev = 0$. This means $(I_n + \Delta\bar{E})v = v + \Delta\bar{E}v = 0$ and $v = -\Delta\bar{E}v$. Thus, v is divisible by Δ, i.e. there exists $\bar{v} \neq 0$ such that $v = \Delta\bar{v}$. Then $0 = Ev = E\Delta\bar{v} = \Delta E\bar{v}$ and $E\bar{v} = 0$. Computing $(I_n + \Delta\bar{E})\bar{v} = 0$, we have again $\bar{v} = -\Delta\tilde{E}\bar{v}$ and $\bar{v}$ is divisible by Δ, so that v is divisible by Δ^2. Going on with this reasoning, we have that every element of $\ker E$ is divisible by Δ^k for all k, which is an absurd.

Remark 3. Equivalence of generalized pencils is defined by $U(sE - A)V$ for unimodular U and V. Therefore different transformations are used in the domain and codomain of the maps E and A. Therefore domain and codomain modules cannot be considered as the same module.
Following Kuijper and Schumacher [1993], $\mathcal{X}$ will be denoted $\mathcal{X}_d$ when considered as domain of the maps E, A, C (descriptor space), and $\mathcal{X}_c$ when considered as codomain of the maps E, A, B (equation space).

2.1 Formal stability

When dealing with systems over rings, stability must be defined in a formal way, since a ring cannot, in general, be endowed with a natural metric structure. We are using systems over rings as models for delay systems, therefore we will adopt the following formal definition of stability, Hautus and Sontag [1980].

Definition 4. The *Hurwitz polynomials* are polynomials in s with coefficients in $\mathcal{R}$ that belong to the set:
$\mathcal{H} = \{p(s,\Delta) \text{ monic},\ p(\bar{s}, e^{-\bar{s}h}) \neq 0\ \forall\ \bar{s} \in \mathbb{C}, Re(\bar{s}) \geq 0\}$.

Definition 5. A system (3) over the ring $\mathcal{R}$ is *(formally) stable* if $\det\left(s(I_n + \tilde{E}) - A\right) \in \mathcal{H}$, i.e. is an Hurwitz polynomial.

2.2 Impulse eliminability

Proposition 6. The neutral system defined by equations (3) can be transformed into a regular system by a suitable change of basis if and only if

$$Im\ E = \mathcal{R}^n \tag{5}$$

Proof. Over an Hermite ring a pair (E, A) is *algebraically solvable* if and only if there exist unimodular matrices P and Q such that the pencil $sE - A$ can be put in standard canonical form

$$P(sE - A)Q = s\begin{bmatrix} I & 0 \\ 0 & J \end{bmatrix} - \begin{bmatrix} A_s & 0 \\ 0 & I \end{bmatrix}.$$

A necessary condition for algebraic solvability is that any one of the following conditions holds (see Cobb [2006]).
i) $\quad \text{Im } E + A \ker\ E = \mathcal{R}^n$
ii) $\quad \text{Im } E \cap A \ker\ E \neq \{0\}$
iii) $\quad \ker\ E \cap \ker \mathcal{A} \neq \{0\}$
For a neutral system we have $E = I_n + \Delta\bar{E}$ and, in virtue of Proposition 2 we have that *ii)* and *iii)* never hold, while *i)* can be replaced by $\text{Im } E = \mathcal{R}^n$.
Then, *i)* holds if and only if E is unimodular. In this case E has a polynomial inverse E^{-1}. Denoting by $\hat{A} = E^{-1}A$ we have that $E^{-1}(sE - A) = sI_n - \hat{A}$ and *i)* is also a sufficient condition for algebraic solvability. ∎

Corollary 7. Let Σ be the neutral system defined by equations (3). Assume that the matrix $E = I_n + \Delta\bar{E}$ is unimodular, then Σ may be rewritten as a regular delay system.

Proof. Denoting by $\hat{A} = E^{-1}A$ and $\hat{B} = E^{-1}B$ we have that Σ can be defined by equations

$$\begin{cases} x(t+1) = \hat{A}x(t) + \hat{B}u(t) \\ y(t) = Cx(t). \end{cases} \tag{6}$$

Most of existing works on observers for descriptor delay systems with unknown inputs, such as Koenig et al. [2004], assume that the systems is impulse-free or impulse-eliminable, namely that (5) holds. On the contrary, we will consider the case really interesting when $\det\ E$ is a nonzero polynomial. In this case, even if $\ker\ E = \{0\}$, $\text{Im } E$ is strictly contained in $\mathcal{X}_c = \mathcal{R}^n$ and *i)* is not verified.
In Picard [1996] the observability and state reconstruction properties of neutral systems of the form (3) when $\det\ E$ is a nonzero polynomial were investigated by considering the associated regular system of the form (6) with $\hat{A} = E^{-1}A$ and $\hat{B} = E^{-1}B$ matrices with elements in the ring of realizable fractions

$$\mathcal{R}_u(\Delta) = \{r(\Delta) = p(\Delta)/q(\Delta) | p, q \in \mathcal{R} \text{ and } q(0) \neq 0\}$$

Here, we consider a neutral system as a particular descriptor systems over the ring $\mathcal{R} = \mathbb{R}[\Delta]$ and we investigate the observer problem from a geometric point of view.

Definition 8. Let Σ be the neutral system defined by equation (3) over the ring $\mathcal{R} = \mathbb{R}[\Delta]$. The r-dimensional regular system

$$\Sigma_O : \begin{cases} z(t+1) = \tilde{A}z(t) + \tilde{B}u(t) + \tilde{G}y(t) \\ \hat{w}(t) = Lz(t) + Ky(t) \end{cases} \quad (7)$$

is an *observer* for Σ if the output $\hat{w}(t)$ asymptotically converges to $Hx(t)$, where $H \in \mathcal{R}^{s\times n}$ is a given matrix. Σ_O is called a *full-order* (*reduced-order*) observer if $r = n$ ($r < n$).

3. CONSTRUCTION OF THE OBSERVER

We will design observers of increasing complexity depending on the system's properties.

3.1 Full order observer

Proposition 9. Let Σ be a system over the ring $\mathcal{R} = \mathbb{R}[\Delta]$ defined by the equations (3). Assume that there exist a polynomial matrix G such that the equation

$$A + GC = \tilde{A}E \quad (8)$$

has a polynomial stable solution $\tilde{A}$. Then the output of the n-dimensional system

$$\Sigma_{O'} : \begin{cases} z(t+1) = \tilde{A}z(t) + Bu(t) - Gy(t) \\ w(t) = z(t) \end{cases} \quad (9)$$

asymptotically converges to $Ex(t)$, i.e. $\Sigma_{O'}$ is a (full order) observer for Σ not of neutral type.

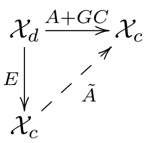
Proof. Equation $A + GC = \tilde{A}E$, i.e.

$$[\tilde{A} \;\; -G] \begin{bmatrix} E \\ C \end{bmatrix} = A \quad (10)$$

has a (possibly rational) solution $[\tilde{A} \;\; -G]$ if the following condition is satisfied:

$$\ker \begin{bmatrix} E \\ C \end{bmatrix} \subseteq \ker \; A \quad (11)$$

We have $\ker \; [E^t \; C^t]^t = \ker \; E \bigcap \ker \; C = \{0\} \subseteq \ker \; A$, then equation (10) is solvable. Assume that there exists a polynomial G and a polynomial stable solution $\tilde{A}$ so that the following diagram commute:



The tracking error $e(t) = w(t) - Ex(t) = z(t) - Ex(t)$ satisfies the following equation: $z(t+1) - Ex(t+1) =$

$$= \tilde{A}z(t) + Bu(t) - Gy(t) - Ax(t) - Bu(t)$$
$$= \tilde{A}z(t) - (A + GC)x(t) = \tilde{A}z(t) - \tilde{A}Ex(t) =$$
$$= \tilde{A}\,(z(t) - Ex(t))$$

Being $\tilde{A}$ stable, the tracking error asymptotically goes to zero and $w(t)$ asymptotically approaches $Ex(t)$.

Remark 10. The conditions on G and $\tilde{A}$ reduce to the well known necessary condition of detectability for a linear systems over a field, i.e. when dealing with systems without delays.

A polynomial solution to equation (10) exists if and only if the module generated by the columns of $A^\top$ is contained in the module generated by the columns of $[E^\top, \; C^\top]$ or, assuming G polynomial, if the module generated by the columns of $(A + GC)^\top$ is contained in the module generated by the columns of $E^\top$. When this does not happen, denote by φ is the G.C.D. of the entries of the matrix G and $\tilde{A}$, so that $\bar{A} = \varphi\tilde{A}$ and φG are polynomial solution of equation

$$\varphi A = \bar{A}E - \varphi GC$$

Remark that, since $\tilde{A} = (A + GC)E^{-1}$, φ is necessarily a divisor of *det* E, therefore a polynomial with non zero constant term.

Proposition 11. With the above hypotheses and notations, assume that $s\varphi I_n - \bar{A}$ is stable. Then, the output of the neutral system

$$\Sigma_{O''} : \begin{cases} \varphi I_n z(t+1) = \bar{A}z(t) + \varphi Bu(t) - \varphi Gy(t) \\ w(t) = z(t) \end{cases} \quad (12)$$

asymptotically converges to $Ex(t)$, i.e. $\Sigma_{O''}$ is a (full order) neutral observer for Σ.

Proof. The tracking error $e(t) = w(t) - Ex(t)$ satisfies the dynamic equations $\varphi I_n e(t+1) = \bar{A}z(t) - \varphi GCx(t) - \varphi Ax(t) = \bar{A}z(t) - \bar{A}Ex(t) = \bar{A}e(t)$. The result follows from stability of $s\varphi I_n - \bar{A}$

A fundamental tool in the geometric construction of reduced order observers is the notion of conditioned invariant submodule and the classical definitions have been extended in Perdon and Anderlucci [2006] to a class of singular systems over a ring that contains neutral systems of the form (3). Let us now briefly recall a few definitions and results that we'll use in the following.

Definition 12. (Perdon and Anderlucci [2006]) Given the system defined by equations (3), a submodule $\mathcal{S}$ of $\mathcal{X}_c$ is called

i) $(E, \; A, \; C)$-invariant or *conditioned invariant* if

$$A(E^{-1}\mathcal{S} \cap \ker C) \subseteq \mathcal{S} \quad (13)$$

ii) *injection invariant* if there exists a linear map G, called a *friend*, such that

$$(A + GC)E^{-1}\mathcal{S} \subseteq \mathcal{S} \quad (14)$$

When a G satisfying (14) has rational elements but GC is polynomial, then G is called a *generalized friend*.

Proposition 13. (Perdon and Anderlucci [2006]) Any closed [1] conditioned invariant submodule $\mathcal{S}$ has a friend, possibly generalized, such that (14) is satisfied.

Proposition 14. (Perdon and Anderlucci [2007]) Given a system defined by equations (3), if $\mathcal{S} \subseteq \mathcal{X}_c$ is a conditioned invariant submodule, then its closure $\bar{\mathcal{S}}$ is conditioned invariant too.

Since the output measures linear combinations of part of the states, it seems natural that only a subset of the states need to be estimated through the observer. To eliminate this redundancy, a reduced-order observer can be build.

Proposition 15. Let Σ be a system defined by equations (3), let $\mathcal{W}$ be a closed submodule of $\mathcal{X}_d$ such that

$$\mathcal{X}_d = \mathcal{W} \oplus \ker \; C$$

and denote $\mathcal{S} = E\mathcal{W}$. Then, $\mathcal{S}$ is a conditioned invariant submodule of $\mathcal{X}_c$ and $E^{-1}(\mathcal{S}) = \mathcal{W}$. If $\mathcal{S}$ is not closed, we can consider its closure $\bar{\mathcal{S}}$ and we still have $E^{-1}(\bar{\mathcal{S}}) = \mathcal{W}$.

[1] The *closure* of a submodule $\mathcal{S}$ is the submodule $\bar{\mathcal{S}} = \{x \in \mathcal{X}_c$ for which there exists a non zero $a \in \mathcal{R}$, such that $ax \in \mathcal{S}\}$. $\mathcal{S}$ is closed if and only if $\mathcal{S} = \bar{\mathcal{S}}$. $\bar{\mathcal{S}}$ is the smallest closed submodule containing $\mathcal{S}$.

Proof. We have $E^{-1}\mathcal{S} = \mathcal{W} + \ker\ E = \mathcal{W}$, then $A(E^{-1}\mathcal{S} \bigcap \ker\ C) = \{0\} \subseteq \mathcal{S}$. Furthermore, $E^{-1}\bar{\mathcal{S}} = \{x \in \mathcal{X}_d$ such that $Ex \in \bar{\mathcal{S}}$, namely $aEx \in \mathcal{S}$ for $0 \neq a \in \mathcal{R}\}$. Then, $Eax \in \mathcal{S}$ and x belongs to the closure of $E^{-1}\mathcal{S} = \mathcal{W}$.

Assuming that C has rank p we have that $\ker C$ has dimension $n-p$ and the direct summand $\mathcal{W}$ has dimension p. As proved in Proposition 2, E has full row rank, therefore the submodule $\mathcal{S} = E\mathcal{W}$ has again dimension p.

In the following we will assume that $\mathcal{S}$ is a closed conditioned invariant submodule, then, by Proposition 14, $\mathcal{S}$ is injection invariant, i.e. there exists a possibly generalized friend G such that $(A + GC)E^{-1}(\mathcal{S}) \subseteq \mathcal{S}$.
We can write $\mathcal{S} = \ker\ T$ for a suitable matrix T that can be chosen of full row rank (e.g. the canonical projection $\mathcal{X}_c \to \mathcal{X}_c/\mathcal{S}$) and we have $E^{-1}\mathcal{S} = \ker\ TE$ (see Perdon and Anderlucci [2006]). Then, the following diagram commutes

$$\begin{array}{ccc} \mathcal{X}_d & \xrightarrow{A+GC} & \mathcal{X}_c \\ {\scriptstyle TE}\downarrow & & \downarrow{\scriptstyle T} \\ \mathcal{X}_d/E^{-1}\mathcal{S} & \xrightarrow{\tilde{A}} & \mathcal{X}_c/\mathcal{S} \end{array} \tag{15}$$

Proposition 16. With the above notations, assume that there exist polynomial matrices G and $\tilde{A}$ such that the diagram (15) commutes, i.e.

$$T(A + GC) = \tilde{A}TE \tag{16}$$

and that $\tilde{A}$ is stable. Then, the system

$$\Sigma_{O'} : \begin{cases} z(t+1) = \tilde{A}z(t) + TBu(t) - TGy(t) \\ w(t) = z(t) \end{cases} \tag{17}$$

is a not neutral reduced order observer for Σ , i.e. its output asymptotically converges to $TEx(t)$.

Proof. Assume that the Lyapunov equation (16) has a stable polynomial solution $\tilde{A}$ and denote by $e(t) = w(t) - TEx(t)$ the tracking error. Then, we have
$e(t+1) = z(t+1) - TEx(t+1) =$
$= \tilde{A}z(t) + TBu(t) - TGy(t) - T\,(Ax(t) - Bu(t)) =$
$= \tilde{A}z(t) - T(A + GC)x(t) = \tilde{A}\,(z(t) - TEx(t))$.
The stability of $\tilde{A}$ implies that the tracking error goes asymptotically to zero and that $w(t)$ converges to $TEx(t)$.

Equation (16) has a polynomial solution $\tilde{A}$ if and only if the module generated by the columns of $(A + GC)^{\top}T^{\top}$ is contained in the module generated by the columns of $E^{\top}T^{\top}$.

When the condition a polynomial solution does not exists, as we did above, we can still construct a reduced order observer of neutral type for Σ as follows.

Proposition 17. Assume that the matrix $\tilde{A}$ that solves equation (16 has rational elements, denote by φ the G.C.D. of entries in $\tilde{A}$ and denote $\bar{A} = \varphi\tilde{A}$. Assume that $s\varphi I_r - \bar{A}$ is stable. Then, the following system

$$\Sigma_{O"} : \begin{cases} \varphi I_r z(t+1) = \bar{A}z(t) + \varphi TBu(t) - \varphi TGy(t) \\ w(t) = z(t) \end{cases} \tag{18}$$

is a reduced order observer for Σ, i.e. $w(t)$ asymptotically converges to $TEx(t)$.

Proof. $\bar{A}$ is a polynomial matrix satisfying the relation Then, the following relation holds

$$\varphi T(A + GC) = \bar{A}TE \tag{19}$$

Then, the tracking error $e(t) = z(t) - TEx(t)$ satisfies the equation $\varphi I_r e(t+1) = \varphi z(t+1) - \varphi TEx(t+1) = \bar{A}z(t) - \varphi TGCx(t) - \varphi TAx(t) = \bar{A}(z(t) - TEx(t)) = \bar{A}e(t)$
The stability of $s\varphi I_r - \bar{A}$ assures that the tracking error asymptotically goes to zero.

Remark 18. As showed in diagram (15), $z(t)$ belongs to a quotient module of dimension $n - p$, so that $n - p \leq r$ depending on the number of rows of matrix $T \in \mathcal{R}^{r \times n}$.

3.2 State reconstruction

Proposition 19. Let Σ be a system over the ring $\mathcal{R}$ described by the equations (3) and assume we want to reconstruct $Hx(t)$, for a given polynomial matrix H. Assume that an observer for Σ exists of the form (16, in particular equation (16) has a polynomial stable solution $\tilde{A}$. Then, there exist a couple of matrices $\tilde{L}$ and $\tilde{K}$ such that the output of the system

$$\Sigma_O : \begin{cases} z(t+1) = \tilde{A}z(t) + Bu(t) - Gy(t) \\ \hat{w}(t) = \tilde{L}z(t) + \tilde{K}y(t) \end{cases} \tag{20}$$

asymptotically converges to $Hx(t)$.

Proof. With the notations of Proposition 15, since $\ker\ TE \bigcap \ker\ C = W \bigcap \ker\ C = \{0\}$, we have that $\ker\ \begin{bmatrix} TE \\ C \end{bmatrix} \subseteq \ker\ H$ for every H. Then, equation

$$LTE + KC = H \tag{21}$$

is solvable for every H. If $[L\ K]$ has polynomial entries, $\tilde{L} = L$ and $\tilde{K} = K$. Let us introduce the new variable $\zeta(t) := z(t) - TEx(t)$. By Proposition 17 or by Proposition 18, $z(t)$ asymptotically converges to $TEx(t)$, then $\zeta(t)$ asymptotically converges to zero. Then, we have equation: $\hat{w}(t) - Hx(t) = Lz(t) + Ky(t) - Hx(t) = L\zeta(t) + (LTE + KC)x(t) - Hx(t) = L\zeta(t)$. Then, $\hat{w}(t)$ asymptotically converges to $Hx(t)$. We can chose $H = I_n$, so that the whole state can be reconstructed. ∎

Proposition 20. With the above notations and hypotheses, assume that the solution $[L\ K]$ of equation (21), with $H = I_n$, has rational entries, and that ψ is the G.C.D. of its elements. Then,

$$\begin{cases} z(t+1) = \tilde{A}z(t) + Bu(t) - Gy(t) \\ \hat{w}(t) = \psi Lz(t) + \psi Ky(t) \end{cases} \tag{22}$$

asymptotically converges to $\psi I_n x(t)$.

Proof. We have that $\psi LTE + \psi KC = \psi I_n$, then

$$\begin{aligned} \hat{w}(t) - \psi x(t) &= \psi Lz(t) + \psi Ky(t) - \psi x(t) = \\ &= \psi L\zeta(t) + \psi(LTE + KC)x(t) - \psi x(t) = \\ &= \psi L\zeta(t). \end{aligned}$$

Then, since $\zeta(t)$ asymptotically converges to zero, $\hat{w}(t)$ asymptotically converges to $\psi x(t)$. ∎

We can state propositions analogous to Proposition 19 and 20 where the observer dynamics is neutral, i.e.

Example 21. Let be given a linear time-invariant descriptor delay-differential system Σ_d with unknown disturbances described by the following equations:

$$\begin{cases} \dot{x}_1(t) = 1.5x_1(t) + 2x_1(t-h) + 0.5x_2(t) + x_2(t-h) + \\ \quad +x_2(t-2h) - \dot{x}_1(t-h) - \dot{x}_2(t-2h) \\ \dot{x}_2(t) = -2x_1(t-h) - 2x_2(t) + 2x_2(t-h) - \dot{x}_2(t-h) \\ y_1(t) = x_1(t) \end{cases}$$

where h represents a delay.

By introducing the delay operator δ defined, for any time function $f(t)$, by $\delta f(t) := f(t-h)$, and then by formally replacing it with the algebraic indeterminate Δ, we can associate to Σ_d the system $\Sigma = (E,\ A,\ C)$ over the ring $\mathcal{R} = \mathbb{R}[\Delta]$ of real polynomials in one indeterminate:

$$\Sigma : \begin{cases} Ex(t+1) = Ax(t) \\ y(t) = Cx(t) \end{cases}$$

where

$$E = I_2 - \Delta \cdot \begin{pmatrix} 1 & \Delta \\ 0 & 1 \end{pmatrix},$$
$$A = \begin{pmatrix} 3/2 + 2\Delta & -1/2 + \Delta + \Delta^2 \\ -2\Delta & -2 + 2\Delta \end{pmatrix},$$
$$C = (1\ 0)$$

In this case, $\det(E) = \Delta^2 - 2\Delta + 1$ which is not a real constant, so that E does not have a polynomial inverse. The Lyapunov equation

$$[\tilde{A}\ -G] \cdot \begin{bmatrix} E \\ C \end{bmatrix} = A$$

admits solutions $[\tilde{A}\ -G]$ with elements in $\mathcal{R}$ if the module generated by the columns of $A^\top$ is contained in the module generated by the columns of $[E^\top,\ C^\top]$.

In this case, $< A^\top >= \langle \begin{pmatrix} \Delta \\ 1-\Delta \end{pmatrix}, \begin{pmatrix} 1.5 \\ -2.5 + 3\Delta + \Delta^2 \end{pmatrix} \rangle$ is certainly contained in

$$< [E^\top\ C^\top] >= \langle \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \rangle = \mathcal{R}^2.$$

Remark that the module generated by the columns of $A^\top$ is not contained in the module generated by the columns of only $E^\top$. Following the proposed construction, solving the equation

$$A = \tilde{A}E - CG$$

with respect to $\tilde{A}, G$, we obtain the ordinary time-delay observer over rings $\Sigma_{O'}$:

$$\begin{cases} z(t+1) = \begin{pmatrix} -1.5 & -0.5 + 0.5\Delta \\ 0 & -2 \end{pmatrix} z(t) - \begin{pmatrix} -3 - 0.5\Delta \\ 2\Delta \end{pmatrix} y(t) \\ \hat{w}(t) = z(t) \end{cases}$$

which asymptotically observes $Ex(t)$.

Going back to the original framework, the system

$$\begin{cases} \dot{z}_1(t) = -1.5z_1(t) - 0.5z_2(t) + 0.5z_2(t-h) + \\ \quad -3y(t) - 0.5y(t-h) \\ \dot{z}_2(t) = -2z_2(t) + 2y(t-h) \\ w_1(t) = z_1(t) \\ w_2(t) = z_2(t) \end{cases}$$

is such that its output asymptotically observes

$$\begin{pmatrix} x_1(t) - x_1(t-h) - x_2(t-2h) \\ x_2(t) - x_2(t-h) \end{pmatrix}.$$

Now, applying the step described in Subsection 3.2, we can reconstruct not only $Ex(t)$ but every $Hx(t)$ where H is a polynomial matrix such that the columns of its transposed generate a submodule of the module generated by the columns of $[E^\top,\ C^\top]$ that is, in this case, the whole $\mathcal{R}^2$. Then we can reconstruct the whole state. Solving $LE + KC = I_2$ with respect to (L, K), we find

$$L = \begin{pmatrix} 0 & 0 \\ -1 & 1+\Delta \end{pmatrix}, \quad K = \begin{pmatrix} 1 \\ 1-\Delta \end{pmatrix}.$$

Now the ordinary time-delay observer over rings Σ_O:

$$\begin{cases} z(t+1) = \begin{pmatrix} -1.5 & -0.5 + 0.5\Delta \\ 0 & -2 \end{pmatrix} z(t) - \begin{pmatrix} -3 - 0.5\Delta \\ 2\Delta \end{pmatrix} y(t) \\ \hat{w}(t) = \begin{pmatrix} 0 & 0 \\ -1 & 1+\Delta \end{pmatrix} z(t) + \begin{pmatrix} 1 \\ 1-\Delta \end{pmatrix} y(t) \end{cases}$$

asymptotically observes $x(t)$.

Going back to the original framework, the system

$$\begin{cases} \dot{z}_1(t) = -1.5z_1(t) - 0.5z_2(t) + 0.5z_2(t-h) + \\ \quad -3y(t) - 0.5y(t-h) \\ \dot{z}_2(t) = -2z_2(t) + 2y(t-h) \\ w_1(t) = y(t) \\ w_2(t) = -z_1(t) + z_2(t) + z_2(t-h) + y(t) - y(t-h) \end{cases}$$

is such that its output asymptotically observes $\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$.

Example 22. This example was introduced in Wang et al. [2002] and then solved in other way by Darouach [2005]. Consider the linear neutral delay system defined by equations (3) with

$$E = I_2 - 0.1I_2 = \begin{pmatrix} 1 - 0.1\Delta & 0 \\ 0 & 1 - 0.1\Delta \end{pmatrix}$$
$$A = \begin{pmatrix} 2.5 + 0.1\Delta & -0.5 - 0.05\Delta \\ 0.03\Delta & -3 + 0.1\Delta \end{pmatrix} \quad C = (1\ 0)$$

In this case the module generated by the columns of $A^\top$:

$$< A^\top >= \langle \begin{pmatrix} -150 \\ 23\Delta - 570 \end{pmatrix}, \begin{pmatrix} 23\Delta + 500 \\ -400 \end{pmatrix} \rangle$$

has dimension 2 but it is not contained in the module generated by the columns of $[E^\top,\ C^\top]$:

$$< [E^\top,\ C^\top] >= \langle \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -10 + \Delta \end{pmatrix} \rangle.$$

Thus an ordinary system cannot be used to observe this neutral system. We have that ker C is a closed submodule in $\mathcal{X}_d = \mathcal{R}^n$ and therefore a direct summand (see Conte and Perdon [1982]).We have then ker $C \oplus \mathcal{W} = \mathcal{X}_d$, where

$$\ker C = \left\langle \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle \qquad W = \left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\rangle$$

$E\mathcal{W} = \langle [(1 - 0.1\Delta) \quad 0]^\top \rangle$ is trivially a conditioned invariant because $E^{-1}(E\mathcal{W}) = \mathcal{W} + \ker E = \mathcal{W}$ therefore condition $A\left(\ker C \cap E^{-1}(E\mathcal{W})\right) \subseteq E\mathcal{W}$ can be written as $A(\ker C \cap \mathcal{W}) \subseteq E\mathcal{W}$, i.e. $\{0\} \subseteq E\mathcal{W}$. By Proposition 13 its closure $S = \overline{E\mathcal{W}} = \left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\rangle$ is also a conditioned invariant, therefore , by Proposition 14 it is injection invariant, i.e. there exists a linear map G with elements in $\mathbb{R}(\Delta)$, such that $(A + GC)E^{-1}(S) \subseteq S$.

In this case all friends are polynomials of the form $G = \begin{pmatrix} g_1 \\ -0.03\Delta \end{pmatrix}$ with arbitrary $g_1 \in \mathbb{R}[\Delta]$. $\mathcal{S}$ is closed, then it can be seen as the kernel of the a linear map T. In this case we can take $T = (0\ 1) \in \mathcal{R}^{1\times 2}$ and $\mathcal{W} = E^{-1}\mathcal{S} = \ker\ TE$.

In this case, a polynomial solution $\tilde{A}$, with $\tilde{A}$ stable, of the reduced-order equation $T(A+GC) = \tilde{A}TE$ does not exist; the solution has rational elements:

$$\begin{pmatrix} \frac{0.1\Delta - 3}{1 - 0.1\Delta} \end{pmatrix}$$

Define $\varphi(\Delta) := 1 - 0.1\Delta$ so that $\bar{A} = (0.1\Delta - 3)$ is a polynomial solution of $\varphi(\Delta)T(A+GC) = \bar{A}TE$. Thus the dynamics of the neutral system that observes TEx will be:

$$\varphi(\Delta)z(t+1) = \bar{A}z(t) + \varphi(\Delta)TBu(t) - \varphi(\Delta)TGy(t)$$

The matrix pair

$$(\varphi(\Delta)I, \bar{A}) = ((-0.1\Delta + 1), (0.1\Delta - 3))$$

is stable because $\det(s\varphi(\Delta)I - \bar{A}) = 3 + s - 0.1\Delta - 0.1s\Delta$ has root

$$s = -\frac{3 - 0.1\Delta}{1 - 0.1\Delta} = -\frac{\Delta - 30}{\Delta - 10}.$$

If we want to estimate $Hx(t) = (1 \;\; 1)x(t)$ (as in Darouach [2005]), we have to solve the polynomial matrix equation

$$LTE + KC = H,$$

with respect to L and K. In this case equation

$$L(0 \;\; 1 - 0.1\Delta) + K(1 \;\; 0) = (1 \;\; 1)$$

has rational solutions:

$$L = \begin{pmatrix} \frac{1}{1 - 0.1\Delta} \end{pmatrix}, \; K = (1)$$

Define $\psi(\Delta) := 1 - 0.1\Delta$, then $\bar{L} = (1)$, $\bar{K} = (1 - 0.1\Delta)$ are polynomial solution of th eequation

$$\bar{L}TE + \bar{K}C = \psi(\Delta)H$$

This means that the neutral system

$$\begin{cases} \varphi(\Delta)z(t+1) = \bar{A}z(t) + \varphi(\Delta)TBu(t) - \varphi(\Delta)TGy(t) \\ \hat{w} = z(t) + (1 - 0.1\Delta)y(t) \end{cases}$$

will observe $\psi(\Delta)Hx(t) = (1 - 0.1\Delta)(1 \; 1)x(t)$, i.e. $Hx(t)$ with some delays.

If we want observe all the state, the best we can do is to observe $Hx(t) = \begin{pmatrix} 1 & 0 \\ 0 & 1 - 0.1\Delta \end{pmatrix} x(t)$, obtaining

$$\begin{cases} \varphi(\Delta)z(t+1) = \bar{A}z(t) + \varphi(\Delta)TBu(t) - \varphi(\Delta)TGy(t) \\ \hat{w}(t) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} z(t) + \begin{pmatrix} 1 \\ 0 \end{pmatrix} y(t) \end{cases}$$

4. CONCLUSION

The state observation the problem for neutral systems with delays was considered. Using as models neutral systems over a ring of polynomials and adopting a geometric approach, observers with delays not necessarily of neutral type were constructed. Computational aspects were also considered.

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