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# Continuous Time-Varying Pure Feedback Control for Chained Nonholonomic Systems with Exponential Convergent Rate

**Hongliang Yuan** [*] **Zhihua Qu** [**]

[*] *University of Central Florida, Orlando, FL 32826 USA (Tel: 407-882-2209; e-mail: hyuan@ucf.edu).*
[**] *University of Central Florida, Orlando, FL 32826 USA (Tel: 407-823-5976;e-mail: qu@mail.ucf.edu).*

**Abstract:** In this paper, feedback stabilization problem of nonholonomic chained system is studied. A new continuous and time-varying design approach is proposed and it is a pure state feedback control. By injecting an exponential decaying disturbance scaled by the norm of the states, $u_1$ drives the initial states away from the singular manifold $\{x|x_1 = 0, \|x\| \neq 0\}$ that causes singularity. In addition to the continuity, such a control has the exponential convergent rate that discontinuous control has. The enclosed simulation results verified the effectiveness of the proposed approach. Comparisons made with other existing continuous and discontinuous controls show that the proposed control has superior performance.

## 1. INTRODUCTION

Feedback stabilization of nonlinear systems has been one of the most important subjects in the study of nonlinear control problems. As early as 1980's, feedback linearization technique has been prevailing, and sufficient and necessary conditions for exact feedback linearization of large classes of affine nonlinear systems were explicitly established with the adoption of differential geometry methods [5][14]. Later on, the renewed interests on Lyapunov methods become dominant with the invention of the notion of control Lyapunov function and recursive designs such as backstepping [7][9] in order to deal with more large classes of nonlinear systems with unmatched and/or generalized matched uncertainties [16]. While those conventional nonlinear control designs are broadly applicable, there exist some classes of inherently nonlinear systems, such as nonlinear systems with uncontrollable linearization [2], which do not admit any smooth (or even continuous) pure state feedback controls as observed in the seminal paper [4] and therefore make the standard feedback linearization technique and Lyapunov direct method no longer straightforwardly applicable. A typical such class of nonlinear systems are nonholonomic mechanical systems [8], which are not feedback linearizable and their feedback stabilization problem is challenging due to Brockett's necessary condition [4].

It is well known that chained systems are of canonical forms and many nonholonomic mechanical systems such as car-like mobile robots [17] can be transformed into the chained form by state and input transformations. Apparently, chained system does not satisfy Brockett's necessary condition, discontinuous and/or time-varying feedback controls have to be sought for its stabilization. During the past decades, extensive studies have been performed and a great deal of solutions have been obtained following the lines of using discontinuous control method and/or time-varying control method [8]. In general, discontinuous controls can render exponential stability [3][6][11][12], while time-varying controls lead to asymptotic stability [15][21][20]. More recent study has also seen the results of $\rho-$exponential stability of chained system using time-varying homogeneous feedback controls [13]. While the existing controls provide elegant solutions, there is still a desire for seeking a global singular-free transformation that maps the chained system into a controllable linear system. The motivation comes from the simple discontinuous controls proposed in [6][11][1] in which $\sigma$-process based state scaling transformation is used. In such a method, a state scaling transformation

$$\xi_i = \frac{z_i}{x_1^{n-i}}, \quad 1 \leq i \leq n-1$$

is defined on a non-singular subspace $\Omega = \{x \in \Re^n : x_1 \neq 0\}$. The obvious shortcoming is that the resulting controls are discontinuous by nature, and a separate switching control law is required to keep the state off this singularity hyperplane of $x_1 = 0$. Improvements were made in [22][10], in which dynamic extension for control component $u_1$ was introduced to bypass the possible singularity due to singular initial conditions. The proposed methods are quasi-smooth and achieve quasi-exponential stability.

In this paper, we present a new design of time-varying and continuous feedback control, which globally asymptotic stabilize the chained nonholonomic systems while avoiding singularity problem and have exponential convergent rate.

## 2. PROBLEM FORMULATION

The objective of this paper is to design time-varying and continuous pure state-feedback controls $u(x,t) = [u_1(x,t) \;\; u_2(x,t)]^T$ such that the chained system is stabilized with an exponential convergent rate. It is straightforward to extend the proposed results to $m$-input nonholonomic systems that can be transformed into the chained systems.

Let's consider the following chained system with the initial condition $x(t_0)$:

$$\dot{x}_1 = u_1, \quad \dot{x}_2 = x_3 u_1, \cdots, \dot{x}_{n-1} = x_n u_1 \quad \dot{x}_n = u_2, \tag{1}$$

where $x = [x_1, \cdots, x_n]^T \in \Re^n$ is the state, $u = [u_1, \ u_2]^T \in \Re^2$ is the control input.

System (1) can be partitioned into the following two subsystems:

$$\dot{x}_1 = u_1, \tag{2}$$

and

$$\dot{z} = u_1 Az + Bu_2, \tag{3}$$

where $z = [z_1 \ z_2 \ \cdots \ z_{n-1}]^T \triangleq [x_2 \ x_3 \ \cdots \ x_n]^T$, and

$$A \triangleq \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & & 1 \\ 0 & 0 & \cdots & & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}.$$

As such, it is well recognized that the chained system has several nice properties:

- Subsystem (2) is linear, and $u_1$ can be easily found to stabilize $x_1$.
- Subsystem (3) is a LTV system, with the time varying components only in the matrix $A$. Specifically, it is a chain of integrators weighted by $u_1$.
- System (1) is nonlinearly controllable everywhere since the Lie brackets argument on its vector fields are of full rank.

Despite of these properties, stabilization of the chained system (1) remains to be difficult due to the following technical issues: (i) Topologically, the chained system cannot be stabilized by any continuous static feedback control $u = u(x)$ due to its nonlinear characteristics [4]; (ii) Though the system is nonlinearly controllable everywhere, it is not globally feedback linearizable (local feedback incarnation is possible such as the $\sigma$-process but singularity manifold remains in all the neighborhoods around the origin), and nonlinear controllability does not necessarily translate into systematic control design; (iii) System (1) is not linearly controllable around the origin. The apparent dichotomy between nonlinear and linear controllability properties is of particular importance as it characterizes both difficulty of control design and need of having systematic design and improving control performance.

## 3. CONTROL DESIGN

In this section, we propose a feedback control design of component $u_1$ with certain structure and properties. Based on the design, a global state-scaling transformation is introduced to overcome the singularity problem of the existing scaling transformations. This new design enables the designer to recover uniform complete controllability for the chained system and to design a class of continuous, time-varying, and pure state feedback controls which make the system states converge to the origin exponentially.

### *3.1 A General Design Scheme*

We propose the following control for component $u_1(t)$:

$$u_1(t) = -\alpha x_1 + g(x)e^{-\beta t}, \tag{4}$$

where $\alpha > \beta > 0$ and $g(x)$ has the following properties:

- It is continuous, monotone increases with respect to $t$ and is bounded between zero and some positive number.
- If $\|x(t_0)\| = 0$, there should be $g(x(t)) \equiv 0$.
- If $x(t_0)$ is on the hyperplane $\{x | x_1 = 0, \|x\| \neq 0\}$, $g(x(t_0)) \neq 0$.

*Remark 1.* In the first property, $g(x)$ is automatically bounded from below since it is continuous, monotone increase and $t \geq t_0$. By stating explicitly, we emphasize that the lower bound should not be less than zero. The second property tries to maintain the states if they are initially at origin. The third property ensures if only $x_1$ is at origin, the other states are still controllable.

For the subsystem (3), the following state transformation is performed:

$$\xi_i = \begin{cases} 0 & if \ \|x(t_0)\| = 0 \\ \dfrac{z_i}{e^{-(n-1-i)\beta t}} & if \ else \end{cases}, \tag{5}$$

In the case that $\|x(t_0)\| \neq 0$, for $i = 1, \cdots, n-2$

$$\begin{aligned} \dot{\xi}_i &= \frac{\dot{z}_i}{e^{-(n-1-i)\beta t}} - \frac{-\beta(n-1-i)e^{-\beta t}}{e^{-(n-i)\beta t}} z_i \\ &= \frac{u_1}{e^{-\beta t}} \xi_{i+1} + \beta(n-1-i)\xi_i. \end{aligned} \tag{6}$$

For $i = n-1$, since $\xi_i = z_i$, it follows that:

$$\dot{\xi}_{n-1} = u_2. \tag{7}$$

Combine (6) and (7) into the matrix form and put together with the case that $\|x(t_0)\| = 0$,

$$\dot{\xi} = \begin{cases} 0 & if \ \|x(t_0)\| = 0 \\ F(t)\xi + Bu_2 & if \ else \end{cases}, \tag{8}$$

where

$$F(x,t) = diag\{\beta(n-2), \beta(n-3), \cdots, \beta, 0\} + h(x,t)A$$

with

$$h(x,t) = \frac{u_1}{e^{-\beta t}} = g(x) - \alpha \frac{x_1}{e^{-\beta t}}.$$

*Theorem 1.* Under the state transformation (5), the pair $\{F(x,t), \ B\}$ is uniformly completely controllable.

**Proof:** By simple derivation, it shows that,

$$\frac{d}{dt}\frac{x_1(t)}{e^{-\beta t}} = -(\alpha - \beta)\frac{x_1}{e^{-\beta t}} + g(x). \tag{9}$$

Since the function $g(x)$ monotone increases with respect to $t$, and is bounded between zero and some positive number, there must be a supremum that is greater than zero. Thus

$$\lim_{t->\infty} g(x(t)) = c > 0,$$

hence,

$$\lim_{t->\infty} \frac{x_1(t)}{e^{-\beta t}} = \frac{c}{\alpha - \beta}.$$

It follows that

$$\lim_{t->\infty} h(x,t) = \lim_{t->\infty} g(x(t)) - \alpha \lim_{t->\infty} \frac{x_1}{e^{-\beta t}} = -\frac{c\beta}{\alpha-\beta}.$$

Therefore,

$$F_\infty = \lim_{t->\infty} F(x,t) = diag\{\beta(n-2), \cdots, \beta, 0\} - \frac{c\beta}{\alpha-\beta} A.$$

It shows that the system matrix converges to a constant matrix, and the pair $\{F_\infty,\ B\}$ is controllable, which implies the system is uniformly completely controllable.

*3.2 Application of The Design Scheme*

As an example of applying the design scheme, we propose that:

$$g(x) = \frac{\max_{\tau\le t}\|x(\tau)\|}{1+\max_{\tau\le t}\|x(\tau)\|}, \tag{10}$$

Let the control component of the subsystem (3) be defined as:

$$u_2(t) = -r_2^{-1}(t)B^TP(t)\xi, \tag{11}$$

where $P(t)$ is the solution of the following differential Riccati equation with $P(\infty) > 0$ and any given $0 < \underline{q}_2 \le q_2(t) \le \overline{q}_2$, $0 < \underline{r}_2 \le r_2(t) \le \overline{r}_2$.

$$\begin{aligned}0 =& \dot{P}(t) + P(t)F(x,t) + F^T(x,t)P(t)\\ &+q_2(t)I - P(t)Br_2^{-1}(t)B^TP(t).\end{aligned}$$

Since the pairs $\{F(x,t), B\}$ is uniformly completely controllable, solution $P(t)$ exists and is uniformly bounded and $P(t) > 0$ for all $t$.

*Theorem 2.* The proposed $g(x)$ in (10) has the three properties listed in section 3.1.

**Proof.** The first property is straightforward to show since $\max_{\tau\le t}\|x(\tau)\|$ is an increasing function of $t$ and $g(x)$ is an increasing function of $\max_{\tau\le t}\|x(\tau)\|$. For the second property, by transformation (5), if $\|x(t_0)\| = 0$, then $u_1 = u_2 = 0$, hence $x(t) \equiv 0$, which means $g(x) \equiv 0$. The third property obviously holds since on the specified hyperplane, $\|x\| \ne 0$.

*Theorem 3.* Under the control (4) with (10) and (11), system (1) is asymptotic stable and has exponential convergent rate.

**Proof.** It is clear from (4), (10) and (11) that if $\|x(t_0)\| = 0$ implies $u_1 = u_2 = 0$ and hence $x(t) \equiv 0$.
Consider the case that $\|x(t_0)\| \ne 0$. Construct the positive definite functions $V_1(x_1) = \frac{1}{2}x_1^2$ and $V_2(\xi) = \xi^TP\xi$ as Lyapunov functions.

For the subsystem (2),

$$\begin{aligned}\dot{V}_1(x_1) =& x_1\dot{x}_1\\ =& -\alpha x_1^2 + \frac{x_1\max_{\tau\le t}\|x(\tau)\|}{1+\max_{\tau\le t}\|x(\tau)\|}e^{-\beta t}\\ <& -\alpha x_1^2 + e^{-\beta t_0}|x_1|.\end{aligned}$$

Here it is identified that $x_1$ is uniformly ultimately bounded by $\frac{e^{-\beta t_0}}{\alpha}$. Hence,

$$\dot{V}_1(x_1) < -2\alpha V_1 + \varepsilon e^{-\beta t},$$

where $\varepsilon = \frac{e^{-\beta t_0}}{\alpha}$. Thus the first subsystem is globally exponentially attractive [16]. From which, the asymptotic stability and exponential convergent rate of $x_1$ is concluded.

For the subsystem (3),

$$\begin{aligned}\dot{V}_2(\xi) =& \dot{\xi}^TP\xi + \xi^T\dot{P}\xi + \xi^TP\dot{\xi}\\ =& \xi^T(\dot{P} + F^TP + PF - 2r^{-1}PBB^TP)\xi\\ =& -\xi^T[q_2I + r^{-1}PBB^TP]\xi,\end{aligned}$$

which is negative definite, therefore the exponential stability of $\xi$ can be concluded [18]. It follows from (5) that exponential stability of $\xi$ implies that of $z$.

Combine the results for the subsystems (2) and (3), it can be concluded that the overall system has asymptotic stability with exponential convergent rate.

## 4. OPTIMAL PERFORMANCE

Before studying the optimality of the proposed control, notice the fact that $\|x(t)\|$ has been shown to be exponentially convergent, so there exists $\tau_{\max}$ such that

$$\|x(\tau_{\max})\| = \sup_{\tau\ge t_0}\|x(\tau)\|.$$

Therefore,

$$\max_{\tau\le\tau_{\max}}\|x(\tau)\| = \lim_{t->\infty}\max_{\tau\le t}\|x(\tau)\|.$$

Hence, after $t = \tau_{\max}$, $u_1$ becomes a linear control since $\max_{\tau\le t}\|x(\tau)\|$ becomes a constant, i.e.

$$g(x(t)) = c, \quad for\ t \ge \tau_{\max},$$

where $c \in (0,\ 1)$ is ensured by the design. It follows that,

$$\dot{x}_1 = -\alpha x_1 + ce^{-\beta t}, \quad for\ t \ge \tau_{\max}. \tag{12}$$

For $t \in [t_0,\ \tau_{\max}]$ subsystem (2) is equivalent to:

$$\dot{y} = -(\alpha-\beta)y + v, \tag{13}$$

with $y = \frac{x_1}{e^{-\beta t}}$ and $v = g(x)$.

To quantify performance of the proposed control, let us introduce performance index $J = J_1 + J_2$ where

$$J_1 = \int_{t_0}^{\tau_{\max}}[q_1(t)y^2 + r_1(t)v^2]dt + \int_{\tau_{\max}}^{\infty}\gamma(\frac{\beta x_1+u_1}{\alpha-\beta})^2dt \tag{14}$$

and

$$J_2 = \int_{t_0}^{\infty}[q_2(t)\xi^T\xi + r_2(t)u_2^2(t)]dt, \tag{15}$$

with $q_1(t)$, $r_1(t)$, $q_2(t)$, $r_2(t)$ positively-valued and uniformly bounded time functions and $\gamma$ a positive constant.

*Theorem 4.* The control (4) with $g(x)$ given by (10) and (11) is optimal with respect to performance index $J$.

**Proof.**

Consider (12), define $\eta = x_1 - \frac{c}{\alpha-\beta}e^{-\beta t}$, it follows that

$$\dot{\eta} = -\alpha\eta = w. \tag{16}$$

Now, consider performance index $J' = \frac{1}{2}\int_{\tau_{\max}}^{\infty}(\gamma\eta^2 + \rho w^2)dt$, for some constants $\gamma, \rho > 0$. We know that control $w$ is optimal with respect to $J'$ provided that $\alpha = p_1/\rho$ where $p_1 = \sqrt{\gamma\rho}$. It is straightforward to verify that differential equation (16) is equivalent to (12). Hence (12) is also optimal and the performance index $J'$ can be expressed as the second term in $J_1$ using original variable.

For $t \in [t_0,\ \tau_{\max}]$, consider equation (13), by procedure of Pontryagin minimum principle, the Hamiltonian is:

$$H = q_1(t)y^2 + r_1(t)v^2 + \lambda[-(\alpha-\beta)y + v].$$

the following partial differential equations should hold:

$$\frac{\partial H}{\partial v} = 0, \quad \frac{\partial H}{\partial \lambda} = \dot{y}, \quad -\frac{\partial H}{\partial y} = \dot{\lambda}.$$

It follows that:

$$\begin{cases} \lambda^* = -2r_1(t)g(x) \\ \dot{\lambda}^* = (\alpha-\beta)\lambda^* - 2q_1(t)y^*, \\ \dot{y}^* = -(\alpha-\beta)y^* + g(x) \end{cases} \tag{17}$$

where asterisk indicates the variables in optimal version. Let $r_1(t) = \frac{\delta(t)}{g(x)}$, where $\gamma(t)$ is a positive differentiable function. It follows that

$$\dot{\delta}(t) = (\alpha-\beta)\delta + q_1(t)y^*. \tag{18}$$

Without losing any generality, we can set $q_1(t) = 1$. From (17), the solution of $y^*$ is continuous and bounded since in nontrivial case, $g(x) \in (0,\ 1)$. It is obviously seen that by properly setting initial conditions of $\delta$ to be positive, the solution of (18) could be made positive in the period $t \in [t_0,\ \tau_{\max}]$. Hence $r_1(t)$ can be solved numerically.

For subsystem (3), recall the property of uniform complete controllability revealed in the proof theorem 3, we know that control (11) optimally stabilizes system (8) under performance index (15), which can be expressed in original state variables by an inverse state transformation.

## 5. SIMULATIONS AND COMPARISONS WITH OTHER EXISTING CONTROLS

In this section, simulation results are provided to illustrate the effectiveness of the proposed control. In section 5.1, 5.2 and 5.3, comparisons are made with classical discontinuous control, periodic time-varying feedback control, and aperiodic time-varying feedback control respectively.

The simulations is conducted for a 3rd order systems. Noting (4) and (9) require that $\alpha > \beta > 0$, the parameters are set as: $\alpha = 3,\ \beta = 1$, and to show the effectiveness of the proposed control in dealing with singularity, the initial states are put on the singular hyperplane, $x(t_0) = [0\ 0\ 1]^T$.

The state response and control signal of system (1) are illustrated in figure 1 and figure 2 respectively. Figure 1 shows that despite the initial states are on the singular hyperplane, control $u_1$ slightly drives it away from origin then converge to origin again with exponential rate. Hence the singularity problem in discontinuous control is overcame. Further, the response of states and controls are smooth with exponential convergent rate, and no oscillation was found.

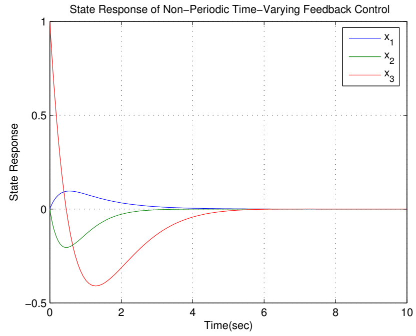


Fig. 1. State response of time-varying feedback controls

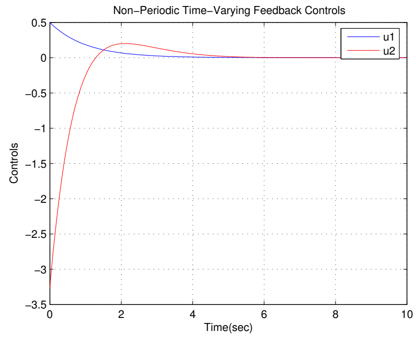


Fig. 2. Continuous time-varying feedback controls

### 5.1 Comparison with Discontinuous Control

The key idea of discontinuous control is to switch control laws after system states leave the singular manifold, hence it avoids the difficulty of designing a single continuous but time-varying control. The so called $\sigma$-process proposed by Astolfi [1] is a common representative of existing discontinuous designs. Consider nonholonomic systems in the chained form,

$$\dot{x}_1 = u_1, \quad \dot{x}_2 = u_2, \quad \dot{x}_i = x_{i-1}u_1, \quad i = 3, \cdots, n.$$

The following transformation is valid for all $x_1 \neq 0$,

$$\xi_1 = x_1, \quad \xi_2 = x_2, \quad \xi_i = \frac{x_i}{x_1^{i-2}}, \quad i = 3, \cdots, n.$$

If we let $u_1 = -k\xi_1$, the $\xi$-system becomes

$$\dot{\xi} = \begin{bmatrix} -k & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & -k & k & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & (n-2)k \end{bmatrix} \xi + \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} u_2.$$

It is a linear system with $u_2$ as the new input and is stabilizable, one can choose the following linear control law

$$u_2 = p_2\xi_2 + p_3\xi_3 + \cdots + p_n\xi_n$$

to place the eigenvalues in left half of the complex plane and make the closed-loop system (in $\xi$-coordinates) globally exponentially stable. However the linear control $u_2$ is not defined on the set $D = \{x \in R^n : x_1 = 0\}$, because the transformation is not valid in the set. This is often omitted and when treated, it is proposed to first apply an open-loop control during some a priori fixed time $t_s$ in order to steer the state away from the singularity and then switch to the linear feedback control law [1][23][24]. The simulation results for this approach is provided. Figure 3 and figure 4 illustrate the state response and control signal respectively.

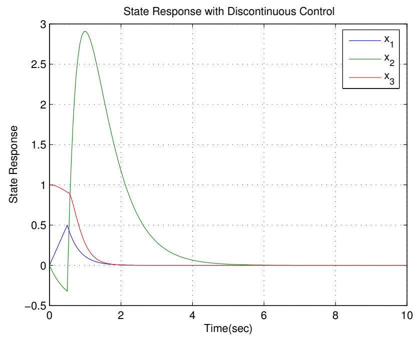


Fig. 3. State response of discontinuous control

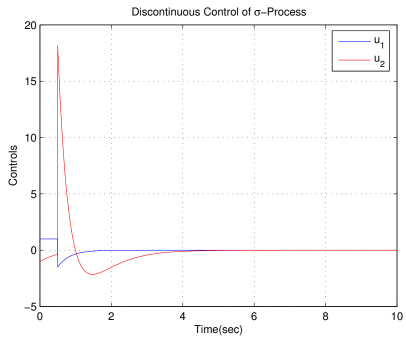


Fig. 4. Discontinuous control generated by $\sigma$-process

Figure 4 shows that the controls are discontinuous when it is switched at time $t_s$ (in the simulation, $t_s = 0.5$). Therefore the state response is not smooth at $t_s$ as can be seen in figure 3. From $t_s$, linear control law were applied, system states and control converge to origin exponentially. However, with the a priori determined $t_s$, the transitory period and the open-loop control remains important regardless of the closeness of the initial conditions to the origin, therefore the closed loop system is not stable in the Lyapunov sense and the performance is not guaranteed.

### *5.2 Comparison with Periodic Time-Varying Controls*

In contrast to the discontinuous control design, researchers also proposed various types of smooth time-varying feedback control, either periodic or aperiodic. [15][21] has proposed a common design of aperiodic time-varying control. E.g. for the system: $\dot{x}_1 = u_1, \quad \dot{x}_2 = u_2, \quad \dot{x}_3 = x_2 u_1$, [15] proposed the following periodic continuous control:

$$\begin{cases} u_1(t,x) = -x_1 + x_3\cos(t) \\ u_2(t,x) = -x_2 + x_3^2\sin(t) \end{cases}. \tag{19}$$

Asymptotic stability of the closed-loop system can be illustrated by using the following Lyapunov function:

$$V(t,x) = (x_1 - \frac{x_3}{2}(\cos(t)+\sin(t)))^2 + (x_2 - \frac{x_3^2}{2}(\sin(t)-\cos(t)))^2 + x_3^2.$$

Later on, to improve its convergence rate, [25] introduced so-called $\rho$-exponential stabilizer using homogeneous feedback, i.e. the control changes to:

$$\begin{cases} u_1(t,x) = -x_1 + \lambda x_3\cos(t) \\ u_2(t,x) = -x_2 + \lambda^3 x_3^2\sin(t) \end{cases}, \tag{20}$$

where $\lambda$ is obtained from

$$V(t, \Delta_\lambda x) = C, \tag{21}$$

with $\Delta_\lambda x = (\lambda x_1, \ \lambda x_2, \ \lambda^2 x_3)$ and $C$ is a constant.

The simulation results for these two controls are illustrated in figure 5 and figure 6. Figure 5(a) and figure 6(a) show that the convergent rate of both state response and control of (19) is unfavorably slow, while the $\rho$-exponential stabilizer (20) does much better in figure 5(b) and figure 6(b). However its setting time (around 10 sec) is still much larger than the proposed approach (around 4 sec) and has more oscillations before converging.

Another drawback of $\rho$-exponential stabilizer is its performance is critically determined by the level set value $C$ in equation (21), however there is no systematic way to determine what $C$ should be except numerical tests.

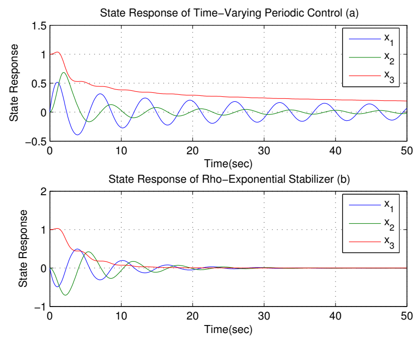


Fig. 5. (a) Original control (b) $\rho$-exponential stabilizer

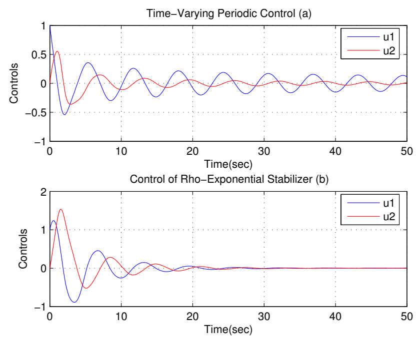


Fig. 6. (a) Original control (b) $\rho$-exponential stabilizer

### *5.3 Comparison with Aperiodic Time-Varying Controls*

On the other hand, design of aperiodic time-varying feedback control was explored in [19] and [22]. [19] adopted a dynamic control,

$$\dot{u}_1 = -(k_1+\zeta)u_1 - k_1\zeta x_1, \quad u_1(t_0) = c_u\|x(t_0)\|.$$

Based on the dynamic control, a virtual output was constructed $y_d = \frac{k_1 x_1 + u_1}{k_1 - \zeta}$. Using the property $\dot{y}_d = -\zeta y_d$, $y_d$ is applied in state scaling. Noting that the solution of $y_d$ is $y_d = ce^{-\zeta t}$, the undergoing transformation is similar to the transformation proposed in this paper.

[22] obtained $u_1$ by augmenting the first subsystem to:

$$\dot{x}_0 = x_1, \;\; \dot{x}_1 = u_1.$$

Let $\alpha$ be the greater eigenvalue and $\beta$ be the smaller one of the augmented system, then $u_1 = e^{-\beta t} f(t)$, where

$$f(t) = \beta^2 \frac{\alpha x_0(0) + x_1(0)}{\alpha - \beta} - \alpha^2 \frac{\beta x_0(0) + x_1(0)}{\alpha - \beta} e^{-(\alpha-\beta)t}.$$

And $z(t) = e^{-\beta t}$ is used in the state scaling transformation. The advantages of these two controls are that the state response and controls are smooth, exponentially converging fast (similar rate with the approach in this paper) with no oscillations. However their disadvantage is, as illustrated in the control equations, the successful control relies on proper tuning of some controller parameters that related to the system's initial conditions, making it fail to be a pure state feedback control, hence is less favorable.

## 6. CONCLUSION

In this paper, feedback stabilization of nonholonomic chained systems is studied. It is shown that linear controllability does not hold for stabilization of the chained system but can be recovered under a state scaling transformation. Based on this idea, we proposed a new design methodology and implemented one particular control. The procedure is shown to be systematic and straightforward. By simulations and comparisons with other existing controls, the proposed control is shown to be effective and exhibited advantages in continuity, convergent rate, oscillations, and being pure state feedback.

## REFERENCES

[1] A. Astolfi, Discontinuous control of nonholonomic systems. *Systems & Control Letters, 27:37-45,* 1996.

[2] A. Bacciotti, Local Stabilizability of Nonlinear Control Systems. World Scientific, Singapore, 1992.

[3] A. Bloch, M. Reyhanoglu and N.H. McClamroch, Control and stabilization of nonholonomic dynamic systems. *IEEE Trans. on Auto Contr., 37:1746-1757,* 1992.

[4] R. W. Brockett, Asymptotic stability and feedback stabilization. In *Differential Geometric Control Theory, pp. 181-191,* 1983.

[5] A. Isidori, Nonlinear Control Systems. Springer-Verlag, Berlin, 1995.

[6] Z.P. Jiang, Robust exponential regulation of nonholonomic systems with uncertainties. *Automatica, 36:189-209,* 2000.

[7] H. Khalil, Nonlinear Systems, 3rd ed., NJ: Prentice-Hall, Upper Saddle River, 2002.

[8] I. Kolmanovsky and N.H. McClamroch, Developments in nonholonomic control problems. *IEEE Contr. Sys. Mag., 15:20-36,* 1995.

[9] M. Krstic, I. Kanellakopoulos and P.V. Kokotovic, Nonlinear and Adaptive Control Design. Wiley, New York, 1995.

[10] M. Laiou and A. Astolfi, Quasi-smooth control of chained systems. In *Proceeding of the American Control Conference,* San Diego, pp. 3940-3944.

[11] J. Luo and P. Tsiotras, Control design for chained-form systems with bounded inputs. *Systems & Control Letters, 39:123-131* 2000.

[12] N. March and M. Alamir, Discontinuous exponential stabilization of chained form systems. *Automatica, 39:343-348* 2003.

[13] R.T. M'Closkey and R.M. Murray, Exponential stabilization of driftless nonlinear control system. *Systems & Control Letters, 39:123-131* 2000.

[14] H. Nijmeijer and A.J. Van der Schaft, Nonlinear Dynamical Control Systems. Springer, Berlin, 1990.

[15] J.B. Pomet, Explicit design of time-varying stabilizing control laws for a class of controllable systems without drift. *Systems & Control Letters, 18:147-158* 1992.

[16] Z. Qu, Robust Control of Nonlinear Uncertain Systems. Wiley-InterScience, 1998.

[17] Z. Qu, J. Wang and C. E. Plaisted, A new analytical solution to mobile robot trajectory generation in the presence of moving obstacles. *IEEE Transactions on Robotics, 20:978-993,* 2004.

[18] Z. Qu, J. Wang, C. E. Plaisted and R.A. Hull, Global-Stabilizing Near-Optimal Control Design for Nonholonomic Chained Systems. *IEEE Transactions on Automatic Control, 51:1440-1456,* 2006.

[19] Z. Qu, J. Wang, R. A. Hull, and Jeffrey Martin, Continuous and Inverse Optimal Control Design for Chained Systems. *Optimal Control Application and Methods,* to appear 2008.

[20] C. Samson, Control of chained systems: Application to path following and time-varying point-stabilization of mobile robots. *IEEE Trans. on Automatic Control, 40:64-77,* 1995.

[21] A. R. Teel, R. M. Murray and G. Walsh., Nonholonomic control systems: from steering to stabilization with sinusoids. In *Proceedings of the 31st IEEE Conference on Decision and Control,* Tucson, Arizona, 1992, pp. 1603-1609.

[22] Y. P. Tian and S. Li, Smooth exponential stabilization of nonholonomic systems via time-varying feedback. In *Proceedings of the 39th IEEE Conference on Decision and Control,* Sydney, 2000, pp. 1912-1917.

[23] C. Canudas de Witt and H. Khennouf, Quasi-continuous stabilizing controllers for nonholonomic systems: Design and robustness considerations. In *Proceedings of the European Control Conference,* 1995.

[24] M. Reyhanoglu, S. Cho, N. McClamroch and I. Kolmanovsky, Discontinuous feedback control a planar rigid body with an underactuated degree of freedom. In *Proceedings of the 37th Conference on Decision and Control,* 1998, pp. 433-438.

[25] R.T. M'Closkey and R.M. Murray, Exponential Stabilization of Driftless Nonlinear Control System Using Homogeneous Feedback. *IEEE Trans. on Automatic Control, 42:614-628*, 1997.