Proceedings of the 17th World Congress
The International Federation of Automatic Control
Seoul, Korea, July 6-11, 2008

Passivity of infinite–dimensional linear systems with state, input and output delays ⋆

S. Hadd and Q.-C. Zhong

Department of Electrical Engineering & Electronics
The University of Liverpool
Brownlow Hill, L69 3GJ, Liverpool, United Kingdom
(E-mails: S.Hadd@liv.ac.uk, Q.Zhong@liv.ac.uk).

Abstract: In this note we present sufficient conditions to guarantee the passivity of linear systems with state, input and output delays in Hilbert spaces. Our approach is mainly based on the transformation of such systems into distributed parameter systems.

1. INTRODUCTION

Passivity theory is one of the most important ones in control theory. It can be applied to a large class of systems. Roughly speaking, passivity means that the system does not have internal energy sources. The importance of the passivity comes from the fact that it is closely related to stability and can be used to solve stabilization problems. The conditions assuring the passivity are normally only based on standard estimations of certain quadratic functions, also called Lyapunov functions, which does not require structural mathematical arguments.

The stability analysis of infinite dimensional linear systems is difficult since the localization of the spectrum in the left half plane is not sufficient for the stability. Although there is a sophistical mathematical tool to handle the stability of such systems, see e.g. Engel & Nagel (2000), passivity is introduced to avoid structural mathematical tools and has been used by many researchers, see e.g. Desoer & Vidyasagar (1975), Niculescu (2001), Staffans (2002a), Staffans (2005), Tucsnak & Weiss (2007), and Van der Schaft (1996).

One of the fields in which passivity plays a key role is delay systems, which are infinite dimensional. By an infinite dimensional system we mean one whose corresponding state space (the space in which the solution lies) is infinite dimensional. Previous works on passivity are mainly addressed to systems with state delay, see Niculescu (2001), Niculescu & Lozano (2001), Kharitonov (2006), Hale & Lunel (1993). However, as mentioned in many papers, e.g. the survey paper by Richard (2003), the passivity of linear systems with state, input and output delays seems not well investigated. As shown in Hadd & Idrissi (2005), Hadd *et al.* (2006), the difficulty to study control properties of such systems comes from the fact that they are well-posed infinite dimensional linear systems in the sense of Staffans (2005), Salamon (1987), and Weiss (1994), where the control and observation operators of the distributed parameter system associated with the state, input and output delay system are unbounded. It is to be noted that passivity of well-posed linear systems has mainly been investigated by Staffans (2002a), Staffans (2002b), Staffans (2002c), and for boundary systems by Malinen & Staffans (2007).

It is our aim in this paper to bridge the gap between the passivity for state-delay systems and that for state–input–output delay systems. We will consider partial functional systems with state, input and output delays in Hilbert spaces. We will use the transformation of such delay systems into distributed parameter systems and some properties of adjoint generators to deduce conditions assuring the passivity. These conditions generalize those introduced in Niculescu & Lozano (2001) for systems with finite dimensional state spaces and state delays. Some results on the passivity of systems with state delays can be found in the recent work Kharitonov (2006) based on the notion of fundamental matrices.

Notation: Throughout this note, we use the following notation. Let $(Z, \langle\cdot,\cdot\rangle)$ be a Hilbert space with norm $\|z\| = \sqrt{\langle z, z\rangle}$. For another Hilbert space Y, we denote by $\mathcal{L}(Z, Y)$ the space of all linear bounded operators from Z to Y, and we set $\mathcal{L}(Z) := \mathcal{L}(Z, Z)$. Let $G : \mathcal{D}(G) \subset Z \to Z$ be a densely defined linear operator. The adjoint operator G^* of G is defined as

$$\mathcal{D}(G^*) = \big\{ z \in Z : \ \exists \gamma_z \geq 0, \quad |\langle Gx, z\rangle| \leq \gamma_z \|x\|, \ \forall\, x \in \mathcal{D}(G) \ \big\},$$
$$\langle Gx, z\rangle = \langle x, G^* z\rangle, \qquad \forall\, x \in \mathcal{D}(G), \quad \forall\, z \in \mathcal{D}(G^*).$$

If $G : \mathcal{D}(G) \to Z$ is a generator of a strongly continuous semigroup on Z, then $\mathcal{D}(G)$ endowed with the norm

$$\|z\|_1 = \sqrt{\langle z, z\rangle + \langle Gz, Gz\rangle}$$

is a Hilbert space. On the other hand, G^* is a generator of a strongly continuous semigroup $\mathfrak{T} = (\mathfrak{T}(t))_{t\geq 0}$ on Z. Now if we denote by $[\mathcal{D}(G^*)]'$ the strong dual of $\mathcal{D}(G^*)$ then

$$\mathcal{D}(G) \subset Z \subset [\mathcal{D}(G^*)]'$$

with continuous embedding, and $\mathfrak{T}$ can be extended to a strongly continuous semigroup on $[\mathcal{D}(G^*)]'$.

We let $L^2([-r, 0], Z)$ denote the Hilbert space of square integrable functions $f : [-r, 0] \to Z$. Its associated Sobolev

⋆ This work was supported by the EPSRC, UK under grant No. EP/C005953/1.

space is denoted by $W^{1,2}([-r,0],Z)$. We define

$$Q_Z f = \frac{\partial}{\partial\theta} f, \qquad \mathcal{D}(Q_Z) = \{f \in W^{1,2}([-r,0],Z) : f(0) = 0\}. \tag{1}$$

It is known that Q_Z is the generator of the left shift semigroup on $L^2([-r,0],Z)$.

For any $t \geq 0$ and a function $g : [-r,0] \to Z$, we denote

$$g(t+\cdot) : [-r,0] \to Z, \quad \theta \mapsto g(t+\theta).$$

2. THE CASE WITH DISCRETE DELAYS

In this section we are interested in giving conditions guaranteeing the passivity of the following delay system

$$\begin{cases} \dot{x}(t) = Ax(t) + A_1x(t-r) + Bu(t) + B_1u(t-r), & t \geq 0, \\ x(0) = x, \quad x(t) = \varphi(t), \quad u(t) = \psi(t), & -r \leq t \leq 0, \\ y(t) = Cx(t), \ t \geq 0. \end{cases} \tag{2}$$

Here A is the generator of a strongly continuous semigroup $T := (T(t))_{t\geq 0}$ on a (state) Hilbert space X, and $A_1 : X \to X,\ B, B_1 : U \to X,\ C : X \to U$ are linear bounded operators, where U is a Hilbert space. We assume that the initial state $x \in X$ and the initial state and control history functions $\varphi : [-r,0] \to X$ and $\psi : [-r,0] \to U$, respectively, are square integrable functions.

In order to define and study the passivity for the system (2), we first reformulate this system into a distributed parameter system. Consider the operator

$$\mathcal{A}_0 = \begin{pmatrix} A & A_1\delta_{-r} \\ 0 & \frac{\partial}{\partial\theta} \end{pmatrix},$$
$$\mathcal{D}(\mathcal{A}_0) = \left\{ \left(\begin{smallmatrix} x \\ \varphi \end{smallmatrix}\right) \in \mathcal{D}(A) \times W^{1,p}([-r,0],X) : \varphi(0) = x \right\},$$

which is closely related to the system (2) with $B_1 = 0$. It is known (see e.g. Bátkai & Piazzera (2005)) that $\mathcal{A}_0$ is a generator of a strongly continuous semigroup on $\mathcal{X}_0 := X \times L^2([-r,0],X)$.

The idea of the reformulation of the delay system (2) to a distributed parameter system is based on the fact that one could introduce a new state gathering the initial state $x(t)$, its history function $x(t+\cdot)$ and the history of the control $u(t+\cdot)$, see Bensoussan *et al.* (2007), Hadd & Idrissi (2005), and the references therein. We consider the new sate

$$\xi : [0,\infty) \longrightarrow \mathcal{X} := \mathcal{X}_0 \times L^2([-r,0],U), \qquad t \mapsto \xi(t) = (x(t), x(t+\cdot), u(t+\cdot))^\top. \tag{3}$$

With this choice, it is shown in Hadd & Idrissi (2005) that a system like (2) can be rewritten as

$$\begin{cases} \dot{\xi}(t) = \mathcal{A}\xi(t) + \mathcal{B}u(t), & t \geq 0, \\ y(t) = \mathcal{C}\xi(t), & t \geq 0, \\ \xi(0) = (x,\varphi,\psi)^\top \in \mathcal{X}, \end{cases} \tag{4}$$

with the generator

$$\mathcal{A} = \left(\begin{array}{cc|c} \multicolumn{2}{c|}{\mathcal{A}_0} & \begin{matrix} B_1\delta_{-r} \\ 0 \end{matrix} \\ \hline 0 & 0 & Q_U \end{array}\right), \qquad \mathcal{D}(\mathcal{A}) = \mathcal{D}(\mathcal{A}_0) \times \mathcal{D}(Q_U), \tag{5}$$

the control operator

$$\mathcal{B} = \begin{pmatrix} B \\ 0 \\ B_U \end{pmatrix}, \tag{6}$$

where $B_U \in \mathcal{L}\big(U, [\mathcal{D}((Q_U)^*)]'\big)$ satisfies $B_U^* f = f(0)$ for $f \in W^{1,2}([-r,0],U)$, and the observation operator

$$\mathcal{C} : \mathcal{X} \to U, \quad \mathcal{C} = [C \ \ 0 \ \ 0]. \tag{7}$$

Note that $\mathcal{A}$ generates a strongly continuous semigroup $\mathcal{T} = (\mathcal{T}(t))_{t\geq 0}$ on $\mathcal{X}$, see Hadd & Idrissi (2005). We denote the extension of $\mathcal{T}$ by $\mathcal{T}_{-1} = (\mathcal{T}_{-1}(t))_{t\geq 0}$, which is a strongly continuous semigroup on $[\mathcal{D}(\mathcal{A}^*)]'$. According to Hadd & Idrissi (2005), $\mathcal{B}$ is an admissible control operator for $\mathcal{A}$ in the sense of Weiss (1989), that is

$$\Phi(t)u := \int_0^t \mathcal{T}_{-1}(t-s)\mathcal{B}u(s)\,ds \in X \tag{8}$$

for $t \geq 0$ and $u \in L^2_{loc}(\mathbb{R}_+, U)$. On the other hand, since the observation operator $\mathcal{C}$ is bounded, the system (4) is well-posed in the sense of Weiss (1994) and its state trajectory is

$$\xi(t) = \mathcal{T}(t)\xi(0) + \int_0^t \mathcal{T}_{-1}(t-s)\mathcal{B}u(s)\,ds \tag{9}$$

for $t \geq 0$ and $\xi(0) \in \mathcal{X}$.

We can now investigate the passivity of the system (2). There are mainly two classes of passive well-posed linear systems, i.e., scattering passive systems and impedance passive systems, see Staffans (2002a) and Tucsnak & Weiss (2007). We will follow the definitions given in Staffans (2002a) and Tucsnak & Weiss (2007) about impedance passive systems.

Definition 1. Let $\mathcal{P} \in \mathcal{L}(\mathcal{X})$ be a self–adjoint positive operator. The delay system (2) (or (4)) is called *impedance $\mathcal{P}$–passive* if for all $t > 0$, the solution (ξ, y) of the system (4) satisfies

$$2\int_0^t \langle y(s), u(s)\rangle\,ds \geq \langle \mathcal{P}\xi(t), \xi(t)\rangle - \langle \mathcal{P}\xi(0), \xi(0)\rangle. \tag{10}$$

Definition 2. Let $\mathcal{P} \in \mathcal{L}(\mathcal{X})$ be a self–adjoint positive operator. The delay system (2) (or (4)) is called *output-strictly impedance $\mathcal{P}$–passive* if there exists $\varepsilon > 0$ such that for all $t \geq 0$, the solution (ξ, y) of the system (4) satisfies

$$2\int_0^t \langle y(s), u(s)\rangle\,ds \geq \langle \mathcal{P}\xi(t), \xi(t)\rangle - \langle \mathcal{P}\xi(0), \xi(0)\rangle + \varepsilon\int_0^t \|y(\tau)\|^2 d\tau. \tag{11}$$

Remark 3. This definitions correspond to Definition 3.1 in Staffans (2002a) with

$$J = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$$

and

$$J = \begin{pmatrix} -\varepsilon I & I \\ I & 0 \end{pmatrix},$$

respectively.

Theorem 4. If there exist positive and self-adjoint operators $P, S \in \mathcal{L}(X)$ and $R \in \mathcal{L}(U)$, and $\varepsilon > 0$ such that

$$\begin{aligned} &A^*P + PA + PA_1S^{-1}A_1^*P + PB_1R^{-1}B_1^*P + S < \varepsilon C^*C \\ &C^* = PB \end{aligned} \tag{12}$$

then the delay system (2) is output-strictly impedance $\mathcal{P}$–passive with the self-adjoint positive operator $\mathcal{P} \in \mathcal{L}(\mathcal{X})$ defined by

$$\mathcal{P} = \begin{pmatrix} P & 0 & 0 \\ 0 & \mathcal{S} & 0 \\ 0 & 0 & \mathcal{R} \end{pmatrix}. \tag{13}$$

Here $\mathcal{S} \in \mathcal{L}(L^2([-r,0],X))$ and $\mathcal{R} \in \mathcal{L}(L^2([-r,0],U))$ are positive and self-adjoint multiplicative operators defined by

$$\mathcal{S} : L^2([-r,0],X) \to L^2([-r,0],X), \qquad (\mathcal{S}f)(s) = Sf(s)$$

and

$$\mathcal{R} : L^2([-r,0],U) \to L^2([-r,0],U), \qquad (\mathcal{R}g)(s) = Rg(s).$$

Proof. Let the operators $P, S \in \mathcal{L}(X)$ and $R \in \mathcal{L}(U)$ be self-adjoint, positive and satisfy (12). Let $\mathcal{P} \in \mathcal{L}(\mathcal{X})$ be defined by (13). We now consider the Lyapunov function

$$V(\xi(t)) = \langle \mathcal{P}\xi(t), \xi(t) \rangle, \qquad t \geq 0.$$

We assume that $\xi(0) = (x, \varphi, \psi)^\top \in \mathcal{D}(\mathcal{A})$ and $u \in W^{1,2}_{loc}([-r,\infty),U)$. Then we have

$$\frac{d}{dt}V(\xi(t))_{|t=0} = 2\langle \mathcal{P}\dot{\xi}(0), \xi(0) \rangle.$$

Let $\Phi(t)$ be given by (8). Due to Weiss (1989) we have

$$\lim_{t\to 0} \frac{1}{t}\Phi(t)u = \mathcal{B}u(0).$$

Now by (9) we have

$$\begin{aligned} \dot{\xi}(0) &= \lim_{t\to 0} \frac{\xi(t)-\xi(0)}{t} \\ &= \lim_{t\to 0} \frac{\mathcal{T}(t)\xi(0)-\xi(0)}{t} + \lim_{t\to 0}\frac{1}{t}\Phi(t)u \\ &= \mathcal{A}\xi(0) + \mathcal{B}u(0). \end{aligned}$$

This shows that

$$\frac{d}{dt}V(\xi(t))_{|t=0} = 2\langle \mathcal{P}(\mathcal{A}(x,\varphi,\psi)^\top + \mathcal{B}u), (x,\varphi,\psi)^\top \rangle,$$

where we set $\xi(0) = (x,\varphi,\psi)^\top \in \mathcal{D}(\mathcal{A})$ and $u := u(0) \in U$. The inequality (11) is then equivalent to

$$\begin{aligned} &2\langle \mathcal{A}(x,\varphi,\psi)^\top, \mathcal{P}(x,\varphi,\psi)^\top \rangle + 2\langle \mathcal{B}u, \mathcal{P}(x,\varphi,\psi) \rangle \\ &\quad \leq 2\langle \mathcal{C}(x,\varphi,\psi)^\top, u \rangle + \varepsilon\langle \mathcal{C}(x,\varphi,\psi)^\top, \mathcal{C}(x,\varphi,\psi)^\top \rangle \\ &\quad = 2\langle Cx, u \rangle + \varepsilon\langle Cx, Cx \rangle \end{aligned} \tag{14}$$

for all $(x,\varphi,\psi)^\top \in \mathcal{D}(\mathcal{A})$ and $u \in U$. Note that

$$\begin{aligned} \langle \mathcal{B}u, \mathcal{P}(x,\varphi,\psi)^\top \rangle &= \langle u, \mathcal{B}^*\mathcal{P}(x,\varphi,\psi)^\top \rangle \\ &= \langle u, B^*Px + R\psi(0) \rangle \\ &= \langle u, B^*Px \rangle, \end{aligned}$$

where $\psi(0) = 0$ because $\psi \in \mathcal{D}(Q_U)$ and (1). Now since $C = B^*P$, (14) is equivalent to

$$2\langle \mathcal{A}(x,\varphi,\psi)^\top, \mathcal{P}(x,\varphi,\psi)^\top \rangle \leq \varepsilon\langle Cx, Cx \rangle \tag{15}$$

for all $(x,\varphi,\psi)^\top \in \mathcal{D}(\mathcal{A})$. According to Cauchy–Schwarz inequality and (15), we have

$$\begin{aligned} |\langle \mathcal{A}(x,\varphi,\psi)^\top, \mathcal{P}(x,\varphi,\psi)^\top \rangle| &\leq \frac{\varepsilon}{2}\|C\|\,\|x\|^2 \\ &\leq \frac{\varepsilon}{2}\|C\|\|x\|\|(x,\varphi,\psi)^\top\|. \end{aligned}$$

This shows that

$$\mathcal{P}\begin{pmatrix} x \\ \varphi \\ \psi \end{pmatrix} \in \mathcal{D}(\mathcal{A}^*), \qquad \forall \begin{pmatrix} x \\ \varphi \\ \psi \end{pmatrix} \in \mathcal{D}(\mathcal{A}). \tag{16}$$

We note that the inequality (15) can be also written as

$$\begin{aligned} &\langle \mathcal{P}\mathcal{A}(x,\varphi,\psi)^\top, (x,\varphi,\psi)^\top \rangle + \langle \mathcal{A}^*\mathcal{P}(x,\varphi,\psi)^\top, (x,\varphi,\psi)^\top \rangle \\ &\qquad \leq \varepsilon\langle C^*Cx, x \rangle \end{aligned} \tag{17}$$

for all $(x,\varphi,\psi)^\top \in \mathcal{D}(\mathcal{A})$.

Let us now investigate the relation (16), which is the key in this proof. For this we first compute $\mathcal{A}^*$. Take $(x,\varphi,\psi)^\top \in \mathcal{D}(\mathcal{A})$ and $(z,f,g)^\top \in \mathcal{X}$. Then

$$\begin{aligned} &\langle \mathcal{A}(x,\varphi,\psi)^\top, (z,f,g)^\top \rangle \\ &\quad = \langle \mathcal{A}_0(x,\varphi)^\top, (z,f)^\top \rangle + \langle \psi(-r), B_1^*z \rangle + \langle \psi', g \rangle \\ &\quad = \langle \mathcal{A}_0(x,\varphi)^\top, (z,f)^\top \rangle + \langle \psi(-r), B_1^*z - g(-r) \rangle \\ &\qquad\qquad + \langle \psi, -g' \rangle. \end{aligned}$$

Then

$$\mathcal{A}^* = \left(\begin{array}{cc|c} \multicolumn{2}{c|}{\mathcal{A}_0^*} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \hline 0 & 0 & -\frac{d}{d\sigma} \end{array}\right),$$

$$\mathcal{D}(\mathcal{A}^*) = \left\{ \begin{pmatrix} x \\ \varphi \\ \psi \end{pmatrix} \in \mathcal{D}(\mathcal{A}_0^*) \times W^{1,2}([-r,0],U) : \psi(-r) = B_1^*x \right\}. \tag{18}$$

Moreover, we have

$$\mathcal{A}_0^* = \begin{pmatrix} A^* & \delta_0 \\ 0 & -\frac{d}{d\sigma} \end{pmatrix},$$

$$\mathcal{D}(\mathcal{A}_0^*) = \left\{ \begin{pmatrix} x \\ \varphi \end{pmatrix} \in \mathcal{D}(A^*) \times W^{1,2}([-r,0],X) : \varphi(-r) = A_1^*x \right\}. \tag{19}$$

Due to (16), we have

$$Px \in \mathcal{D}(A^*),\ S\varphi(-r) = A_1^*Px,\ R\psi(-r) = B_1^*Px. \tag{20}$$

By combining (17), (18), (19), and (20) we obtain

$$\begin{aligned} &\Big\langle \big(PA + A^*P + PA_1S^{-1}A_1^*P + PB_1R^{-1}B_1^*P + S\big)x, x \Big\rangle \\ &\qquad < \varepsilon\langle C^*Cx, x \rangle. \end{aligned} \tag{21}$$

This ends the proof. ∎

Remark 5. In Niculescu & Lozano (2001) it is assumed that $B_1 = 0$ (without input delay). In this case, the delay system can be reformulated as a linear distributed parameter system with generator $\mathcal{A}_0$, bounded control operator $\left(\begin{smallmatrix} B \\ 0 \end{smallmatrix}\right)$ and bounded observation operator $(C\ 0)$. The result given in Niculescu & Lozano (2001) can be obtained by setting $B_1 = 0$ in Theorem 4. However, in the presence of input delays, the state-input delay system (2) can be only transformed into a distributed parameter system with unbounded control operator. This is the obstacle why the technique of Niculescu & Lozano (2001) cannot be directly used. To overcome such a problem we have to use the properties of the adjoint operators associated with the delay system.

3. THE CASE WITH DISTRIBUTED DELAYS

In this section we investigate the following distributed–delay system

$$\begin{cases} \dot{x}(t) = Ax(t) + Lx(t+\cdot) + Bu(t) + B_1u(t-r), & t \geq 0, \\ y(t) = Cx(t) + Nx(t+\cdot), & t \geq 0, \end{cases} \tag{22}$$

with the operators L and N defined as

$$L\varphi = A_1\varphi(-r) + \int_{-r}^{0} A_2\varphi(\theta)\, d\theta, \quad N\varphi = \int_{-r}^{0} C_1\varphi(\theta)\, d\theta,$$

where $A_2 \in \mathcal{L}(X)$ and $C_1 \in \mathcal{L}(X, U)$. Define the operator

$$\mathcal{A}_1 = \begin{pmatrix} A & L \\ 0 & \frac{\partial}{\partial\theta} \end{pmatrix},$$

$$\mathcal{D}(\mathcal{A}_1) = \left\{ \left(\begin{smallmatrix} z \\ \varphi \end{smallmatrix}\right) \in \mathcal{D}(A) \times W^{1,p}([-r,0],X) : \varphi(0) = z \right\}.$$

Then $\mathcal{A}_1$ is the generator of a strongly continuous semigroup on $\mathcal{X}_0$ (see Bátkai & Piazzera (2005)).

Similarly, as in the previous section, the delay system (22) can be reformulated as the system (4) but with different generator and observation operator respectively given by

$$\mathcal{A} = \left(\begin{array}{cc|c} \multicolumn{2}{c|}{\mathcal{A}_1} & \begin{matrix} B_1\delta_{-r} \\ 0 \end{matrix} \\ \hline 0 & 0 & Q_U \end{array}\right), \qquad \mathcal{D}(\mathcal{A}) = \mathcal{D}(\mathcal{A}_1) \times \mathcal{D}(Q_U), \tag{23}$$

and

$$\mathcal{C} : \mathcal{X} \longrightarrow U, \quad \mathcal{C} = (C \;\; N \;\; 0). \tag{24}$$

The control operator $\mathcal{B}$ remains the same as given in (6). The state trajectory of the transformed distributed parameter system is given in (9), where $\mathcal{T}(t)$ is the strongly continuous semigroup on $\mathcal{X}$ generated by $\mathcal{A}$ in (23).

We denote by $\xi : \mathbb{R}_+ \to \mathcal{X}$ the state trajectory of the regular well-posed linear system defined by $\mathcal{A}, \mathcal{B}, \mathcal{C}$ as in (4).

Definition 6. The delay system (22) is called *output strictly impedance passive* if there exists a function $V : \mathcal{X} \to [0, \infty)$ such that the following inequality

$$2\int_0^t \langle y(s), u(s)\rangle\, ds \geq V(\xi(t)) - V(\xi(0)) + \varepsilon \int_0^t \|y(\tau)\|^2 d\tau. \tag{25}$$

for any $t \geq 0$ and for some $\varepsilon > 0$. It is called *impedance passive* if the ε in (25) is zero.

Definition 6 is more general than Definition 1 and Definition 2 where the function V has been specified.

The following result is an extension of Theorem 4.

Theorem 7. If there exist four positive and self-adjoint operators $P, S, K \in \mathcal{L}(X)$, $R \in \mathcal{L}(U)$, and an operator $M \in \mathcal{L}(X)$ with $\langle Mx, y\rangle \geq 0$, $\forall\, x, y \geq 0$, and $\varepsilon > 0$ such that

$$\begin{pmatrix} \Delta & 0 \\ 0 & H \end{pmatrix} < \varepsilon \begin{pmatrix} C^*C & 0 \\ 0 & N^*N \end{pmatrix} \tag{26}$$

$$C = B^*P, \quad C_1 = B^*M,$$

with

$$\Delta = A^*P + PA + (PA_1 - M)(S^{-1}A_1^*P + S^{-1}M - M) + PB_1R^{-1}B_1^*P + S + M^* + r\Big(\frac{1}{4}\Sigma^*K^{-1}\Sigma + K\Big),$$
$$H = \mathbf{1}(\cdot)M^*(A_1S^{-1}PM + A_2)\Psi, \tag{27}$$

$$\Sigma := M^*\Big(A + A_1S^{-1}(A_1^*P - M^*) + B_1R^{-1}B_1^*P\Big) + 2(A_1^*P - 2\varepsilon C_1^*C), \tag{28}$$

$$\Psi\varphi = \int_{-r}^{0} \varphi(\theta)\, d\theta,$$

and $\mathbf{1}(\cdot)$ is the constant function defined by $\mathbf{1}(\theta) = 1$ for all $\theta \in [-r, 0]$, then the delay system (22) is output-strictly impedance passive.

Proof. Let $\mathcal{M} : L^2([-r,0],X) \to X$ be the operator defined by

$$\mathcal{M}\varphi = \int_{-r}^{0} M\varphi(\theta)\, d\theta.$$

Let $\mathcal{S}$ and $\mathcal{R}$ be the multiplication operators defined as in Theorem 4 and define the operator $\mathcal{K} \in \mathcal{L}(\mathcal{X})$ by

$$\mathcal{K} = \begin{pmatrix} P & \mathcal{M} & 0 \\ \mathcal{M}^* & \mathcal{S} & 0 \\ 0 & 0 & \mathcal{R} \end{pmatrix}.$$

Define

$$\mathcal{P}_0 = \begin{pmatrix} P & \mathcal{M} \\ \mathcal{M}^* & \mathcal{S} \end{pmatrix}.$$

Since P and S are self–adjoint operators, $\mathcal{P}_0 \in \mathcal{L}(\mathcal{X}_0)$ is self–adjoint operator as well. Let $(x, \varphi) \in \mathcal{L}(\mathcal{X}_0)$. We have

$$\langle \mathcal{P}_0 \left(\begin{smallmatrix} x \\ \varphi \end{smallmatrix}\right), \left(\begin{smallmatrix} x \\ \varphi \end{smallmatrix}\right)\rangle = \langle Px, x\rangle + 2\int_{-r}^{0} \langle M\varphi(\theta), x\rangle\, d\theta + \langle \mathcal{S}\varphi, \varphi\rangle.$$

With the assumption $\langle Mx, y\rangle \geq 0$ for all $x, y \geq 0$, it follows that $\mathcal{K}$ is positive.

Let $K \in \mathcal{L}(X)$ be positive and self–adjoint. Define the following Lyapunov function

$$V(\xi(t)) = \langle \mathcal{K}\xi(t), \xi(t)\rangle + \int_{-r}^{0}\int_{t+\theta}^{t} \langle Kx(s), x(s)\rangle ds\, d\theta. \tag{29}$$

Here $\xi(t) = (x(t), x(t+\cdot), u(t+\cdot))^\top$, $t \geq 0$. Then

$$\frac{d}{dt}V(\xi(t)) = 2\langle \mathcal{K}\xi(t), \frac{d}{dt}\xi(t)\rangle + r\langle Kx(t), x(t)\rangle - \int_{-r}^{0} \langle Kx(t+\theta), x(t+\theta)\rangle\, d\theta. \tag{30}$$

As in the proof of Theorem 4, the definition of the passivity (25) is equivalent to the following algebraic equation

$$\begin{aligned} &2\langle \mathcal{A}(x,\varphi,\psi)^\top, \mathcal{K}(x,\varphi,\psi)^\top\rangle + 2\langle \mathcal{B}u, \mathcal{K}(x,\varphi,\psi)^\top\rangle \\ &\quad + r\langle Kx, x\rangle - \int_{-r}^{0}\langle K\varphi(\theta), \varphi(\theta)\rangle\, d\theta \\ &\leq 2\langle \mathcal{C}(x,\varphi,\psi)^\top, u\rangle + \varepsilon\langle \mathcal{C}(x,\varphi,\psi)^\top, \mathcal{C}(x,\varphi,\psi)^\top\rangle \\ &= 2\langle Cx, u\rangle + \langle N\varphi, u\rangle + 2\varepsilon\langle Cx, N\varphi\rangle + \varepsilon\langle Cx, Cx\rangle \\ &\quad + \varepsilon\langle N\varphi, N\varphi\rangle \end{aligned} \tag{31}$$

for all $(x,\varphi,\psi)^\top \in \mathcal{D}(\mathcal{A})$ and $u \in U$. As in the proof of Theorem 4 and by using the seconde condition in (26) one can see that

$$\begin{aligned}\langle \mathcal{B}u, \mathcal{K}(x,\varphi,\psi)^\top \rangle &= \langle B^*Px, u\rangle + \langle B^*M\varphi, u\rangle \\ &= \langle Cx, u\rangle + \langle N\varphi, u\rangle.\end{aligned}$$

Now (31) becomes

$$\begin{aligned}&2\langle \mathcal{A}(x,\varphi,\psi)^\top, \mathcal{K}(x,\varphi,\psi)^\top \rangle + r\langle Kx, x\rangle \\ &\quad - \int_{-r}^{0} \langle K\varphi(\theta), \varphi(\theta)\rangle \, d\theta \\ &\leq 2\varepsilon\langle Cx, N\varphi\rangle + \varepsilon\langle Cx, Cx\rangle + \varepsilon\langle N\varphi, N\varphi\rangle \quad (32)\end{aligned}$$

for all $(x,\varphi,\psi)^\top \in \mathcal{D}(\mathcal{A})$. As in the proof of Theorem 4, using Cauchy–Schwarz inequality and (32) it follows that

$$\mathcal{K}\begin{pmatrix} x \\ \varphi \\ \psi \end{pmatrix} \in \mathcal{D}(\mathcal{A}^*), \qquad \forall \begin{pmatrix} x \\ \varphi \\ \psi \end{pmatrix} \in \mathcal{D}(\mathcal{A}). \quad (33)$$

Note that (32) can be also rewritten as

$$\begin{aligned}&\langle \mathcal{K}\mathcal{A}(x,\varphi,\psi)^\top, (x,\varphi,\psi)^\top \rangle + \langle \mathcal{A}^*\mathcal{K}(x,\varphi,\psi)^\top, (x,\varphi,\psi)^\top \rangle \\ &\quad + r\langle Kx, x\rangle - \int_{-r}^{0} \langle K\varphi(\theta), \varphi(\theta)\rangle \, d\theta \\ &\leq 2\varepsilon\langle Cx, N\varphi\rangle + \varepsilon\langle Cx, Cx\rangle + \varepsilon\langle N\varphi, N\varphi\rangle\end{aligned} \quad (34)$$

for all $(x,\varphi,\psi)^\top \in \mathcal{D}(\mathcal{A})$.

Let us now compute the adjoint operator of $\mathcal{A}_1$. Take $(x,\varphi)^\top \in \mathcal{D}(\mathcal{A}_1)$ and $(z,f)^\top \in \mathcal{X}_0$, then

$$\begin{aligned}&\langle \mathcal{A}_1\begin{pmatrix} x \\ \varphi \end{pmatrix}, \begin{pmatrix} z \\ f \end{pmatrix}\rangle \\ &= \langle Ax, z\rangle + \langle A_1\varphi(-r), z\rangle + \int_{-r}^{0} \langle A_2\varphi(\theta), z\rangle d\theta + \langle \dot\varphi, f\rangle \\ &= \langle x, A^*z + f(0)\rangle + \langle \varphi(-r), A_1^*z - f(-r)\rangle \\ &\quad + \langle \varphi, \mathbf{1}(\cdot)A_2^*z - \dot f\rangle\end{aligned}$$

Then

$$\begin{aligned}\mathcal{A}_1^* &= \begin{pmatrix} A^* & \delta_0 \\ \Gamma & -\frac{d}{d\sigma} \end{pmatrix}, \\ \mathcal{D}(\mathcal{A}_1^*) &= \Big\{ \begin{pmatrix} x \\ \varphi \end{pmatrix} \in \mathcal{D}(A^*) \times W^{1,2}([-r,0], X) : \\ &\qquad \varphi(-r) = A_1^*x \Big\}.\end{aligned} \quad (35)$$

Here the operator $\Gamma : X \to L^2([-r,0], X)$ is such that $(\Gamma x)(\theta) = A_2^*x$ for all $x \in X$ and $\theta \in [-r,0]$. Then

$$\begin{aligned}\mathcal{A}^* &= \begin{pmatrix} A^* & \delta_0 & 0 \\ \Gamma & -\frac{d}{d\sigma} & 0 \\ 0 & 0 & -\frac{d}{d\sigma} \end{pmatrix}, \\ \mathcal{D}(\mathcal{A}^*) &= \Big\{ \begin{pmatrix} x \\ \varphi \\ \psi \end{pmatrix} \in \mathcal{D}(\mathcal{A}_0^*) \times W^{1,2}([-r,0], U) : \\ &\qquad \psi(-r) = B_1^*x \Big\}.\end{aligned} \quad (36)$$

On the other hand, from (33), (35) and (36) we have

$$\begin{aligned}S\varphi(-r) &= A_1^*Px - M^*x + A_1^*P\mathcal{M}\varphi, \\ R\psi(-r) &= B_1^*Px.\end{aligned} \quad (37)$$

Due to (36) and (37), the inequality (34) is equivalent to

$$\begin{aligned}&\langle \Lambda x, x\rangle + \int_{-r}^{0} \langle \Sigma x, \varphi(\theta)\rangle d\theta - \int_{-r}^{0} \langle K\varphi(\theta), \varphi(\theta)\rangle \, d\theta \\ &\quad + r\langle Kx, x\rangle + \langle H\varphi, \varphi\rangle \leq \varepsilon\langle Cx, Cx\rangle + \varepsilon\langle N\varphi, N\varphi\rangle \quad (38)\end{aligned}$$

with

$$\begin{aligned}\Lambda := &PA + A^*P + (PA_1 - M)(S^{-1}A_1^*P + S^{-1}M - M) \\ &+ PB_1R^{-1}B_1^*P + S + M^*\end{aligned}$$

and Φ and Σ are given by (27) and (28). On the other hand

$$\begin{aligned}&\int_{-r}^{0} \langle \Sigma x, \varphi(\theta)\rangle d\theta - \int_{-r}^{0} \langle K\varphi(\theta), \varphi(\theta)\rangle \, d\theta \\ &= \frac{r}{4}\langle K^{-1}\Sigma x, \Sigma x\rangle \\ &\quad - \int_{-r}^{0} \langle K(\frac{1}{2}K^{-1}\Sigma x - \varphi(\theta)), \frac{1}{2}K^{-1}\Sigma x - \varphi(\theta)\rangle \, d\theta. \quad (39)\end{aligned}$$

Now we set

$$\Delta = \Lambda + r\Big(\frac{1}{4}\Sigma^*K^{-1}\Sigma + K\Big).$$

From the positivity of K we have that the last integral in (39) is positive and the condition (38) becomes

$$\langle \Delta x, x\rangle + \langle H\varphi, \varphi\rangle \leq \varepsilon\langle Cx, Cx\rangle + \varepsilon\langle N\varphi, N\varphi\rangle.$$

This ends the proof.

■

Remark 8. In the case of no input–output delays, R disappears and $M = 0$. Then we obtain the passivity condition of infinite dimensional state delay systems which is an extension of the result of Niculescu & Lozano (2001).

4. CONCLUSION

In this paper, sufficient conditions in terms of Riccati inequalities is proposed to guarantee the passivity of infinite dimensional linear systems with state, input and output delays. The approach used here is based on linear distributed parameter systems and the explicit computation of the adjoint of their generators. This paper extends the results of Niculescu & Lozano (2001) to systems with state, input and output delays in infinite dimensional spaces.

REFERENCES

A. Bensoussan, G. Da Prato, M. C. Delfour and S. K. Mitter, *Representation and Control of Infinite–Dimensional Systems*, Birkhäuser, Boston, Basel, Berlin, 2007.

A. Bátkai and S. Piazzera, *Semigroups for Delay Equations*, Research Notes in Mathematics, 10. A K Peters, Ltd., Wellesley, MA, 2005.

R. Bellman and K.L. Cooke, *Differential–Difference Equations*, Academic Press, New York, 1963.

C. A. Desoer and M. Vidyasagar, *Feedback Systems: Input-Output Properties*, New York: Academic, 1975.

K.-J. Engel and R. Nagel, *One-Parameter Semigroups for Linear Evolution Equations*, Springer-Verlag, 2000.

S. Hadd, Unbounded perturbations of $\mathcal{C}_0$–semigroups in Banach spaces and applications, *Semigroup Forum*, 70: 451–465, 2005.

S. Hadd and A. Idrissi, Regular linear systems governed by systems with state, input and output delays, *IMA Journal Mathematical Control and Information*, 22:423–439, 2005.

S. Hadd, A. Idrissi and A. Rhandi, The regular linear systems associated to the shift semigroups and application to control delay systems, *Mathematical Control Signals Systems*, 18:272–291, 2006.

J.K. Hale and S.M. Verduyn Lunel, *Introduction to Functional Differential Equations*, Applied Mathematical Sciences, vol. 99, Springer-Verlag, New York, 1993.

J. Malinen and O. J. Staffans, Impedance Passive and Conservative Boundary Control Systems, *Complex Analysis and Operator Theory* 1:279–300, 2007.

S–I. Niculescu, *Delay Effects on Stability: A Robust Control Approach*, Lecture Note in Control and Information Sciences, Springer–Verlag London Berlin Heidelberg, 2001.

S–I. Niculescu and R. Lozano, On the passivity of linear delay systems, *IEEE Trans. Aut. Control*, 46:460–464, 2001.

V.L. Kharitonov Lyapunov matrices for a class of time delay systems, *Systems & Control Letters*, 55:610–617, 2006.

V.B. Kolmanovskii, S–I. Niculescu and J–P. Richard, On the Liapunov–Krasovskii fonctionals for stability analysis for linear delay systems, *Int. J. Control*, 72: 374–384, 1999.

R. Lozano, B. Brogliato, O. Egeland and B. Maschke, *Dissipative Systems Analysis and Control. Theory and Application*, London, U.K: CES, Spronger, 2000.

J–P. Richard, Times–delay systems: an overview of some recent advances and open problems, *Automatica*, 39: 1667–1694, 2003.

D. Salamon, Infinite-dimentional linear system with unbounded control and observation: a functional analytic approach, *Transaction American Mathematical Society*, 300:383–431, 1987.

O. J. Staffans, Passive and conservative continuous-time impedance and scattering systems. Part I: Well-posed systems, *Mathematics of Control, Signals, and Systems*, 15:291–315, 2002.

O. J. Staffans, *Passive and conservative infinite-dimensional impedance and scattering systems (from a personal point of view)*, in Mathematical Systems Theory in Biology, Communication, Computation, and Finance, J. Rosenthal and D. S. Gilliam, eds, IMA Volumes in Mathematics and its Applications 134, Springer-Verlag, New Yor, 375–413, 2002.

O. J. Staffans, *Stabilization by collocated feedback*, in Directions in Mathematical Systems Theory and Optimization, A. Rantzer and C. I. Byrns, eds., Lecture Notes in Control and Information Sciences 286, Springer-Verlag, Berlin, 261-278, 2002.

O. J. Staffans, *Well-Posed Linear Systems,* Cambridge Univ. Press, 2005.

M. Tucsnak and G. Weiss, *Passive and Conservative Linear Systems*, Book preprint, 2007: available at www.ee.imperial.ac.uk/gweiss/personal/book.pdf

A. Van der Schaft, *L_2–gain Stability and Passivity Techniques in Nonlinear Control*, London, Springer, 1996.

G. Weiss, Admissibility of unbounded control operators *SIAM Journal Control Optimization*, 27:527–545, 1989.

G. Weiss, Regular linear systems with feedback, *Mathematical Control Signals Systems*, 7:23–57, 1994.

J. Willems, Dissipative dynamical systems–Part I: General theory, *Arch. Ration. Mech. Anal,* 45:321–351, 1972.

J. Willems, Dissipative dynamical systems–Part II: Linear systems with quadratic supply rates, *Arch. Ration. Mech. Anal,* 45:322–393, 1972.