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Unknown Input Observer Synthesis Method with Modified $\mathcal{H}_\infty$ Criteria for Nonlinear Systems Using Sobolev Norms

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Abstract: In this paper, we present an unknown input observer (UIO) design method for a class of nonlinear systems in the presence of disturbances in both the dynamics of the system and the output. The main idea lies in the introduction of Sobolev norms to define a new criteria to study robustness of the observer. Contrarily to the classical $\mathcal{H}_\infty$ criteria, this new criteria, called the *modified* $\mathcal{H}_\infty$ *criteria*, allows to solve the problem of unknown input observer design in the presence of disturbances. Based on the Lyapunov stability theory and the modified $\mathcal{H}_\infty$ criteria, new sufficient synthesis conditions are given in terms of Linear Matrix Inequalities (LMIs). To show the performances of the proposed method, we considered the problem of simultaneous synchronization and decryption in chaotic communication systems.

Keywords: Nonlinear systems, Unknown input Observers, $\mathcal{H}_\infty$ analysis, Sobolev spaces, Banach spaces, Weak derivatives (derivatives in the sense of distributions), LMIs approach, Chaos synchronization.

1. INTRODUCTION

The problem of estimating unknown inputs is motivated by certain applications such as fault detection, fault diagnostic, control system design and synchronization and decryption in chaotic communication systems.
Observer design for estimating the state of a system and the unknown inputs in the linear case has received considerable attention in the past. However, little research has been paid toward nonlinear case. Additionally, most of the existing results for nonlinear systems concerns only the estimation of the state of the system subject to unknown inputs, see Chen and Saif [2006], Pertew et al. [2005] for instance. Very few works have been carried out on estimating the unknown inputs. We cite here some of the available results in a synchronization and input recovery context. In Huijberts et al. [2000], the problem of unknown, constant or slowly time-varying input estimation using an Extended Kalman Filter (EKF) is discussed. In Boutayeb et al. [2002], the authors proposed an approach to estimate simultaneously the state of the system and the unknown inputs using a generalized state space observer. This approach is extended recently, to a more general class of nonlinear systems in Trinh et al. [2004], and to discrete-time case using the EKF in Boutayeb [2004].
In certain applications, such as chaos communication, the quality of the input reconstruction plays a very important role. Indeed, in chaos communication systems, the transmitted signal through a transmission channel is often corrupted by noise. Therefore, proposing a new unknown input observer synthesis method that take into account the disturbances that affect the output signal is necessary.

This paper deals with unknown input observer design method for a class of nonlinear systems. The proposed method takes into account the presence of disturbances in both the dynamics of the system and the output. The main result lies in the use of Sobolev norms to define a new criteria of robustness, called the *modified* $\mathcal{H}_\infty$ *criteria*. Contrarily to the standard $\mathcal{H}_\infty$ method, the modified $\mathcal{H}_\infty$ criteria offers the possibility to solve the unknown input observer synthesis problem in a noisy context. Based on the Lyapunov stability theory and the differential mean value theorem, new sufficient synthesis conditions are given in terms of Linear Matrix Inequalities (LMIs) easily tractable using standard convex optimization algorithms. It should be noticed that for more details on the Sobolev norms, we refer the reader to Bourles and Colledani [1995], Alessandri [2007].
The rest of this note is arranged as follows. In section 2, we present the motivating problem to show the significance of this work. In section 3, we introduce the modified $\mathcal{H}_\infty$ criteria, which is based on Sobolev norms. The proposed unknown input observer synthesis method is given is section 4. The validity of the theoretical work is shown through a numerical example in section 5. Finally, we end this note by a conclusion in section 6.

Notations : The following notations will be used throughout this paper.

- $\|.\|$ is the usual Euclidean norm;
- $(\star)$ is used for the blocks induced by symmetry;
- A^T represents the transposed matrix of A;
- I_r represents the identity matrix of dimension r;

- for a square matrix S, $S > 0$ ($S < 0$) means that this matrix is positive definite (negative definite);
- the notation $\|x\|_{\mathcal{L}_p^r} = \left(\int_0^{+\infty} \|x(t)\|^p \mathrm{d}t\right)^{\frac{1}{p}}$ is the $\mathcal{L}_p^r$ norm of the vector $x(.) \in \mathbb{R}^r$. The set $\mathcal{L}_p^r$ is defined by
$$\mathcal{L}_p^r = \left\{x(.) \in \mathbb{R}^r \ : \ \|x\|_{\mathcal{L}_p^r} < +\infty\right\}$$
and then $(\mathcal{L}_p^r, \|.\|_{\mathcal{L}_p^r})$ is called the *Lebesgue space*;
- $e_s(i) = \big(\underbrace{0, ..., 0, \overbrace{1}^{i \text{ th}}, 0, ..., 0}_{s \text{ components}}\big)^T \in \mathbb{R}^s, s \geq 1$ is a vector of the canonical basis of $\mathbb{R}^s$;
- the set $Co(x, y)$ is the convex hull of the set $\{x, y\}$, i.e.
$$Co(x, y) = \left\{\lambda x + (1 - \lambda)y, \lambda \in [0, 1]\right\}.$$

2. MOTIVATING PROBLEM

In chaotic communication systems, the transmitted signal through a transmission channel is generally disturbed by noises or bounded disturbances. For this reason the $\mathcal{H}_\infty$ performance analysis plays an important role on the quality of the reconstruction of the useful signal (the original message to be decrypted). There are many chaotic communication techniques. See Yang [2004] for an overview on all the existing methods. Among these techniques, we mention that based on unknown input observers. This method, called simultaneous synchronization and decryption technique, is represented in Figure 1.

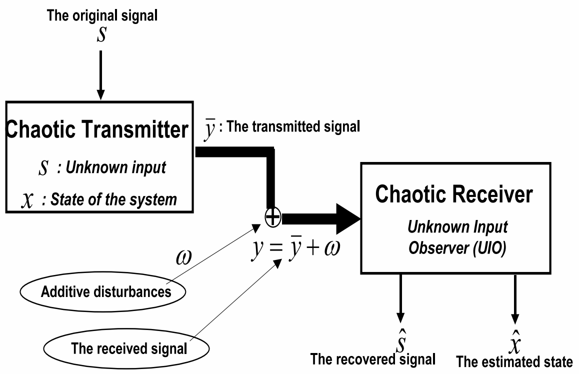


Fig. 1. Simultaneous synchronization and decryption technique based on UIO.

2.1 Analytic Study of the Problem

To study analytically the problem represented in Figure 1, we prefer to consider the general problem of $\mathcal{H}_\infty$ unknown input observer design for a class of nonlinear systems. Let us consider the class of systems described by the following set of equations :

$$\begin{cases} \dot{x} = A_x x + A_u u + Bf(x, u) + E_\omega \omega \\ y = Cx + Du + D_\omega \omega \end{cases} \tag{1}$$

where $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}^m$ denotes the unknown input to be estimated, $y \in \mathbb{R}^p$ is the output vector and ω is the vector of disturbances. At this stage, we assume only that $\omega \in \mathcal{L}_2^s([0, +\infty])$. A_x, A_u, B, C, D and D_ω are constant matrices of adequate dimensions. D is of full column rank. The nonlinear function f: $\mathbb{R}^n \times \mathbb{R}^m \mapsto \mathbb{R}^q$ is assumed to be differentiable with respect to x and u and satisfies the following inequality :

$$a_{ij} \leq \frac{\partial f_i}{\partial \zeta_j}(\zeta) \leq b_{ij}, \quad \forall\, \zeta \in \mathbb{R}^{n+m} \tag{2}$$

The inequality (2) implies that the differentiable function f is γ-Lipschitz where

$$\gamma = \sqrt{\sum_{i=1}^{i=q} \sum_{j=1}^{j=n+m} \max\left(|a_{ij}|^2, |b_{ij}|^2\right)}.$$

Note that the reformulation of the Lipschitz condition for differentiable functions as in (2) leads to less restrictive synthesis conditions and avoids *high gain* as shown in Zemouche et al. [2007].
Before presenting the observer, we need to define the bounded convex domain

$$\mathcal{H}_{q,n,m} = \left\{\nu = (\nu_{11}, ..., \nu_{q(n+m)}) \ : \ a_{ij} \leq \nu_{ij} \leq b_{ij}\right\} \tag{3}$$

of which the set of vertices is defined by :

$$\mathcal{V}_{\mathcal{H}_{q,n,m}} = \left\{\alpha = (\alpha_{11}, ..., \alpha_{q(n+m)}) \ : \ \alpha_{ij} \in \{a_{ij}, b_{ij}\}\right\}. \tag{4}$$

At this stage of the paper, we consider the same structure of the observer as in Boutayeb et al. [2002]. Before introducing the unknown input observer equations, we need the following notations for simplicity of the presentation :

$$E = [I_n \ \ 0] \tag{5}$$
$$M = [A_x \ \ A_u] \tag{6}$$
$$H = [C \ \ D] \tag{7}$$
$$\zeta = \begin{bmatrix} x \\ u \end{bmatrix} \tag{8}$$

Since D is of full column rank, then also $\begin{bmatrix} E \\ H \end{bmatrix}$ is of full column rank and then the matrix $\left(\begin{bmatrix} E \\ H \end{bmatrix}^T \begin{bmatrix} E \\ H \end{bmatrix}\right)^{-1}$ exists.
Now, we set

$$[P \ \ Q] = \left(\begin{bmatrix} E \\ H \end{bmatrix}^T \begin{bmatrix} E \\ H \end{bmatrix}\right)^{-1} \begin{bmatrix} E \\ H \end{bmatrix}^T \tag{9}$$

where P and Q are real matrices of dimension $(n + m).n$ and $(n + m).p$ respectively.
Therefore, we deduce from (9) that :

$$PE + QH = I_{n+m}. \tag{10}$$

The unknown input observer structure is of the form :

$$\begin{cases} \dot{z} = Nz + Ly + PBf(z + Qy) \\ \hat{\zeta} = z + Qy \end{cases} \tag{11}$$

where $\hat{\zeta}$ denotes the estimate of ζ.
The aim is to determine the matrices N and L such that the estimation error $\varepsilon = \hat{\zeta} - \zeta$ converges $\mathcal{H}_\infty$ asymptotically towards zero, i.e. :

$$\|\varepsilon\|_{\mathcal{L}_2^{n+m}} \leq \lambda \|\omega\|_{\mathcal{L}_2^s} \tag{12}$$

where $\lambda > 0$ is the disturbance attenuation level to be minimized

2.2 $\mathcal{H}_\infty$ Filtering Design Problem Li and Fu [1997]

Given the system (1) and the unknown input observer (11), then the problem of $\mathcal{H}_\infty$ filtering design is to determine the matrices N and L so that

$$\lim_{t\to\infty} \varepsilon(t) = 0 \;\text{ for }\; \omega(t) = 0; \tag{13}$$

$$\|\varepsilon\|_{\mathcal{L}_2^{n+m}} \le \lambda \|\omega\|_{\mathcal{L}_2^s} \;\forall\, \omega(t) \neq 0;\;\; \varepsilon(0) = 0. \tag{14}$$

The problem of $\mathcal{H}_\infty$ filtering design (13)-(14) is reduced to find a Lyapunov function $V(\varepsilon)$ such that

$$W(\varepsilon) \triangleq \dot{V} + \varepsilon^T\varepsilon - \lambda^2\omega^T\omega < 0. \tag{15}$$

It is easy to show that (15) implies (13) and (14). Indeed, if $\omega(t) = 0$, then (15) implies that $\dot{V} < 0$. Thus, from the Lyapunov theory, we deduce that the estimation error converges asymptotically to zero, and then we have (13). Now, if $\omega(t) \neq 0$ and $\varepsilon(0) = 0$, then (15) implies that

$$V(t) + \int_0^t \varepsilon^T(\theta)\varepsilon(\theta)d\theta - \lambda^2 \int_0^t \omega^T(\theta)\omega(\theta)d\theta < 0.$$

Since $V(\varepsilon(t)) \ge 0$ for all $t \ge 0$, then for $t \to +\infty$, we obtain

$$\int_0^\infty \varepsilon^T(\theta)\varepsilon(\theta)d\theta \le \lambda^2 \int_0^\infty \omega^T(\theta)\omega(\theta)d\theta$$

that leads to (14).

2.3 Formal $\mathcal{H}_\infty$ Convergence Analysis

From the fact that $y = H\zeta + D_\omega\omega$, we obtain :

$$\varepsilon = z + \Big(QH - I_{n+m}\Big)\zeta + QD_\omega\omega \tag{16}$$

and then from (10), the error vector becomes

$$\varepsilon = z - PE\zeta + QD_\omega\omega. \tag{17}$$

After completing the calculations, we obtain the following dynamics of the error vector :

$$\begin{aligned}\dot{\varepsilon} = N\varepsilon + \Big(N + FH - PM\Big)\zeta + PB\delta f \\ + \Big(FD_\omega - PE_\omega\Big)\omega + QD_\omega\dot{\omega}\end{aligned} \tag{18}$$

where

$$\delta f = f(\hat{\zeta}) - f(\zeta) \tag{19}$$

and

$$F = L - NQ \tag{20}$$

If we set

$$N = PM - FH \tag{21}$$

then, the error dynamics becomes

$$\begin{aligned}\dot{\varepsilon} = \Big(PM - FH\Big)\varepsilon + PB\delta f \\ + \Big(FD_\omega - PE_\omega\Big)\omega + QD_\omega\dot{\omega}\end{aligned} \tag{22}$$

Note that the term $\dot{\omega}$ is introduced formally, because at this stage ω belongs only to $\mathcal{L}_2^s([0, +\infty])$.
Now, if we use a quadratic Lyapunov function

$$V(\varepsilon) = \varepsilon^T S\varepsilon$$

where $S = S^T > 0$, we obtain, after detailing the calculations, the following expression :

$$\begin{aligned}\dot{V} = \varepsilon^T\Big[\Big(PM - FH\Big)^T S + S\Big(PM - FH\Big)\Big]\varepsilon \\ + 2\varepsilon^T S\Big(FD_\omega - PE_\omega\Big)\omega + 2\varepsilon^T SPB\delta f \\ + 2\varepsilon^T SQD_\omega\dot{\omega}\end{aligned} \tag{23}$$

2.4 Motivating Obstacle

In this subsection, we show that if we use the classical $\mathcal{H}_\infty$ criteria (12), we are unable to obtain a suitable synthesis method of the gains N and L. Indeed, this is due to the presence of the term $\dot{\omega}$.
Using the classical $\mathcal{H}_\infty$ criteria (12) or equivalently (15), we have :

$$\begin{aligned}W(\varepsilon) = \varepsilon^T\Big[\Big(PM - FH\Big)^T S + S\Big(PM - FH\Big) \\ + I_{n+m}\Big]\varepsilon + 2\varepsilon^T S\Big(FD_\omega - PE_\omega\Big)\omega \\ + 2\varepsilon^T SPB\delta f + 2\varepsilon^T SQD_\omega\dot{\omega} - \lambda^2\omega^T\omega\end{aligned} \tag{24}$$

In order to obtain a suitable condition under which

$$W(\varepsilon) < 0,\; \forall\, \varepsilon \neq 0$$

we must dominate (or eliminate) the terms δf and $\dot{\omega}$ from (24).
Since f satisfies (2), then we know that the problem of the presence of δf can be solved using the differential mean value theorem as in Zemouche et al. [2007] or the classical Cauchy-Schwartz inequality as in Rajamani [1998], Boutayeb et al. [2002]. However the presence of the term $\dot{\omega}$ in (24) is a true obstacle. Indeed, we cannot eliminate it from (24). Then, to solve this problem, we must add a negative term depending of $\dot{\omega}^T\dot{\omega}$. This solution need to introduce a modified $\mathcal{H}_\infty$ criteria. This is the subject of the next section.

3. MODIFIED $\mathcal{H}_\infty$ CRITERIA

Before introducing the modified $\mathcal{H}_\infty$ criteria that we propose in this paper, we start by defining the Sobolev spaces and Sobolev norms that we use later.

3.1 Sobolev Space and Sobolev Norm

Sobolev Space In mathematics, a Sobolev space is a vector space of functions equipped with a norm that is a combination of $\mathcal{L}_p$ norms of the function itself as well as its derivatives up to a given order. The derivatives are understood in a suitable weak sense [1] to make the space complete, thus a Banach space. It is named after *Sergei L. Sobolev.* The Sobolev space that we use in this paper can be defined by :

$$\begin{aligned}\mathcal{W}_r^{k,p}([0, +\infty]) = \Big\{z \;:\; [0, +\infty] \to \mathbb{R}^r \text{ such that} \\ \frac{\partial^i z}{\partial t^i} \in \mathcal{L}_p^r([0, +\infty]), i = 0, ..., k\Big\}\end{aligned} \tag{25}$$

All the derivatives in (25) are understood in a suitable weak sense.

[1] A *weak derivative* is a generalization of the concept of the derivative of a function (strong derivative) for functions not assumed differentiable, but only integrable, i.e. to lie in the Lebesgue space $\mathcal{L}_1([0, +\infty])$.
Let u be a function in the Lebesgue space $\mathcal{L}_1([0, +\infty])$. We say that v in $\mathcal{L}_1([0, +\infty])$ is a weak derivative of u if,

$$\int_0^{+\infty} u(t)\dot{\varphi}(t)\mathrm{d}t = -\int_0^{+\infty} v(t)\varphi(t)\mathrm{d}t$$

for all continuously differentiable functions φ with $\varphi(0) = \varphi(\infty) = 0$.

Sobolev Norm With the above definition, the Sobolev space $\mathcal{W}_r^{k,p}([0,+\infty])$ admits a natural norm defined as follows :

$$\begin{aligned}\left\|z\right\|_{k,p}^r &= \left[\sum_{i=0}^{i=k}\left(\left\|\frac{\partial^i z}{\partial t^i}\right\|_{\mathcal{L}_p^r}\right)^p\right]^{\frac{1}{p}} \\ &= \left(\sum_{i=0}^{i=k}\int_0^{+\infty}\left\|\frac{\partial^i z(t)}{\partial t^i}\right\|^p \mathrm{d}t\right)^{\frac{1}{p}}.\end{aligned} \tag{26}$$

$\mathcal{W}_r^{k,p}([0,+\infty])$ equipped with the norm $\|.\|_{k,p}^r$ is a Banach space.
Sobolev spaces with $p = 2$ are especially important because of their direct connection with the Lebesgue space $\mathcal{L}_2^r$, which is often used to analyze the $\mathcal{H}_\infty$ performance of estimators.

3.2 New Criteria to Study Robustness

Hereafter, we present a new robustness criteria which allows to solve the problem underlined in Section 2.4. First of all, we assume now that $\omega \in \mathcal{W}_r^{1,2}([0,+\infty])$. Then, this new criteria, called a *modified $\mathcal{H}_\infty$ criteria*, is defined rigorously as follows :

$$\|\varepsilon\|_{\mathcal{L}_2^{n+m}} \le \gamma_{1,2}\|\omega\|_{1,2}^r. \tag{27}$$

It is obvious that the criteria (27) allows to add to (24) a negative term depending of $\dot{\omega}^T\dot{\omega}$.

3.3 Modified $\mathcal{H}_\infty$ Filtering Design Problem

Given the system (1) and the unknown input observer (11), then the modified $\mathcal{H}_\infty$ filtering design problem consists to compute the matrices N and L so that

$$\lim_{t\to\infty}\varepsilon(t) = 0 \ \text{ for } \ \omega(t) = 0; \tag{28}$$

$$\|\varepsilon\|_{\mathcal{L}_2^{n+m}} \le \gamma_{1,2}\|\omega\|_{1,2}^r \ \forall\ \omega(t) \neq 0; \ \ \varepsilon(0) = 0. \tag{29}$$

Then, to satisfy (28)-(29) it is sufficient to find a Lyapunov function $V(\varepsilon)$ so that

$$W(\varepsilon) \triangleq \dot{V} + \varepsilon^T\varepsilon - \gamma_{1,2}^2\omega^T\omega - \gamma_{1,2}^2\dot{\omega}^T\dot{\omega} < 0. \tag{30}$$

We can show easily that (30) implies (28) and (29). Indeed, if $\omega(t) = 0$ (then $\dot{\omega}(t) = 0$), then (30) implies $\dot{V} < 0$ and then the estimation error converges asymptotically to zero as in (28). Nevertheless, if $\omega(t) \neq 0$ and $\varepsilon(0) = 0$, then (30) implies

$$\begin{aligned}V(t) &+ \int_0^t \varepsilon^T(\theta)\varepsilon(\theta)d\theta \\ &- \gamma_{1,2}^2\int_0^t \omega^T(\theta)\omega(\theta)d\theta - \gamma_{1,2}^2\int_0^t \dot{\omega}^T(\theta)\dot{\omega}(\theta)d\theta < 0.\end{aligned}$$

Since $V(\varepsilon(t)) \ge 0$ for all $t \ge 0$, then for $t \to +\infty$, we have

$$\int_0^\infty \varepsilon^T(\theta)\varepsilon(\theta)d\theta \le \gamma_{1,2}^2\left(\int_0^\infty \omega^T(\theta)\omega(\theta)d\theta - \int_0^t \dot{\omega}^T(\theta)\dot{\omega}(\theta)d\theta\right)$$

that leads to (29).

4. UNKNOWN INPUT OBSERVER SYNTHESIS METHOD

In this section, we present mainly the observer synthesis method used to design the matrices N and L, which solve the modified $\mathcal{H}_\infty$ filtering design problem with an optimal disturbance attenuation level $\gamma_{1,2}$.

4.1 Sufficient Synthesis Conditions

Hereafter, we state a theorem which provides sufficient conditions ensuring (30) with an optimal disturbance attenuation level $\gamma_{1,2}$.

Theorem 1. The modified $\mathcal{H}_\infty$ unknown input observer design problem corresponding to the system (1) and the observer (11) is solvable if there exist matrices $S = S^T > 0$ and R of adequate dimensions so that the following convex optimization problem is feasible :

$$\min(\mu) \ \text{ subject to } \ \Gamma(\alpha,\mu) < 0, \ \forall\alpha \in \mathcal{V}_{\mathcal{H}_{q,n,m}} \tag{31}$$

where

$$\Gamma(h,\mu) = \begin{bmatrix}\mathbb{M}(h) & RD_\omega - SPE_\omega & SQD_\omega \\ (\star) & -\mu I_s & 0 \\ (\star) & (\star) & -\mu I_s\end{bmatrix} \tag{32}$$

$$\mathbb{M}(h) = \mathcal{A}(h)^T S + S\mathcal{A}(\alpha) - H^T R^T - RH + I_{n+m} \tag{33}$$

$$\mathcal{A}(h) = P\Big[M + B\sum_{i=1}^{i=q}\sum_{j=1}^{j=n+m} h_{ij}e_q(i)e_{n+m}^T(j)\Big] \tag{34}$$

After solving (31), we can compute the matrix F as $F = RS^{-1}$. Then, we deduce the observer matrices and the minimum disturbance attenuation level as follows :

$$N = PM - FH$$
$$L = F + NQ$$
$$\gamma_{1,2} = \sqrt{\mu}$$

Proof. After detailing the calculations in (30), we obtain

$$\begin{aligned}W(\varepsilon) = \varepsilon^T&\Big[\Big(PM - FH\Big)^T S + S\Big(PM - FH\Big) \\ &+ I_{n+m}\Big]\varepsilon + 2\varepsilon^T S\Big(FD_\omega - PE_\omega\Big)\omega \\ &+ 2\varepsilon^T SPB\delta f + 2\varepsilon^T SQD_\omega\dot{\omega} \\ &- \gamma_{1,2}^2\omega^T\omega - \gamma_{1,2}^2\dot{\omega}^T\dot{\omega}\end{aligned} \tag{35}$$

If we use the differential mean value theorem Zemouche et al. [2007], we deduce that there exist $\eta_i \in Co(\zeta,\hat{\zeta})$ such that

$$\delta f = \left[\sum_{i=1}^{i=q}\sum_{j=1}^{j=n+m}\frac{\partial f_i}{\partial\zeta_j}(\eta_i)e_q(i)e_{n+m}^T(j)\right]\varepsilon. \tag{36}$$

By setting

$$h_{ij}(\eta_i) = \frac{\partial f_i}{\partial\zeta_j}(\eta_i), \ H_{ij} = e_q(i)e_{n+m}^T(j)$$

$$h = \Big(h_{11},...,h_{1(n+m)},h_{2(n+m)},...,h_{q(n+m)}\Big)$$

the equation (35) becomes

$$\begin{aligned}W(\varepsilon) = \varepsilon^T&\Bigg[\Bigg(\overbrace{P\Big[M + B\sum_{i=1}^{i=q}\sum_{j=1}^{j=n+m}h_{ij}(\eta_i)H_{ij}\Big]}^{\mathcal{A}(h)} - FH\Bigg)^T S \\ &+ S\Bigg(P\Big[M + B\sum_{i=1}^{i=q}\sum_{j=1}^{j=n+m}h_{ij}(\eta_i)H_{ij}\Big] - FH\Bigg) \\ &+ I_{n+m}\Bigg]\varepsilon + 2\varepsilon^T S\Big(FD_\omega - PE_\omega\Big)\omega \\ &+ 2\varepsilon^T SQD_\omega\dot{\omega} - \gamma_{1,2}^2\omega^T\omega - \gamma_{1,2}^2\dot{\omega}^T\dot{\omega}\end{aligned} \tag{37}$$

If we use the change of variable $R = FS$, then (37) will be rewritten as follows :

$$W(\varepsilon) = \begin{bmatrix} \varepsilon \\ \omega \\ \dot{\omega} \end{bmatrix}^T \Gamma\Big(h, \gamma_{1,2}^2\Big) \begin{bmatrix} \varepsilon \\ \omega \\ \dot{\omega} \end{bmatrix} \tag{38}$$

where $\Gamma(.)$ is defined in (32).

Since $h \in \mathcal{H}_{q,n,m}$ and $\Gamma\Big(h, \gamma_{1,2}^2\Big)$ is affine (then convex) in h, then from the convexity principle Boyd and Vandenberghe [2001], we deduce that

$$\Gamma\Big(h, \gamma_{1,2}^2\Big) < 0, \ \forall\ h \in \mathcal{H}_{q,n,m}$$

if

$$\Gamma\Big(\alpha, \gamma_{1,2}^2\Big) < 0, \ \forall\ \alpha \in \mathcal{V}_{\mathcal{H}_{q,n,m}}. \tag{39}$$

Therefore, (39) implies $W(\varepsilon) < 0$, which means that the modified $\mathcal{H}_\infty$ unknown input observer design problem is solvable. Then, obtaining the *minimum* value of the disturbance attenuation level $\gamma_{1,2}$ consists to solve (31) with $\mu = \gamma_{1,2}^2$. This ends the proof of Theorem 1.

4.2 Case where Rank(D) < m

In this case, we cannot use y to estimate simultaneously x and u. However, we must use another output which depends of the derivative of y. To simplify the study of this case, we assume without loss of generality that $D = 0$. This new output is under the form :

$$y_1 = T\dot{y} \tag{40}$$

where T is a matrix of appropriate dimension to be determined later. Then, we have

$$y_1 = \bar{C}x + \bar{D}u + \bar{E}_\omega \omega + \bar{D}_\omega \dot{\omega} + (TCB)f(x,u) \tag{41}$$

where

$$\bar{C} = TCA_x, \quad \bar{D} = TCA_u$$
$$\bar{E}_\omega = TCE_\omega, \quad \bar{D}_\omega = TD_\omega$$

with ω belongs now to $\mathcal{W}_r^{2,2}([0,+\infty])$.

Assuming that the matrix $\bar{D}$ is of full column rank, then the corresponding unknown input observer is :

$$\begin{cases} \dot{z} = Nz + Ly_1 + \bar{P}Bf(z + \bar{Q}y_1) \\ \hat{\zeta} = z + \bar{Q}y_1 \end{cases} \tag{42}$$

where

$$\left[\bar{P}\ \bar{Q}\right] = \left(\begin{bmatrix} E \\ \bar{H} \end{bmatrix}^T \begin{bmatrix} E \\ \bar{H} \end{bmatrix}\right)^{-1} \begin{bmatrix} E \\ \bar{H} \end{bmatrix}^T \tag{43}$$

$$\bar{H} = \left[\bar{C}\ \bar{D}\right] \tag{44}$$

Since $\ddot{\omega}$ will be appeared in the dynamics of the error vector $\varepsilon = \hat{\zeta} - \zeta$, then the Sobolev norm of order two must be used in order to obtain an adequate synthesis method. Therefore, we have the following new modified $\mathcal{H}_\infty$ criteria of order two :

$$\|\varepsilon\|_{\mathcal{L}_2^{n+m}} \le \gamma_{2,2} \|\omega\|_{2,2}^r. \tag{45}$$

In the following theorem, we state sufficient conditions under which the problem of modified $\mathcal{H}_\infty$ filtering design of order two, corresponding to the system (1) and the observer (42), is solvable.

Theorem 2. The modified $\mathcal{H}_\infty$ filtering design problem (45) corresponding to (1) and (42), is solvable if the following conditions hold :

- $T \in \ker(CB) \setminus \{0\}$, i.e. $T \neq 0$ and $TCB = 0$;
- $\text{Rank}\Big(\bar{D} = TCA_u\Big) = m$ (for the existence of (42));
- there exist matrices $S = S^T > 0$ and R of adequate dimensions so that the following convex optimization problem is feasible :

$$\min(\mu) \ \text{ subject to } \ \Gamma(\alpha, \mu) < 0, \ \forall \alpha \in \mathcal{V}_{\mathcal{H}_{q,n,m}} \tag{46}$$

where

$$\Gamma(h,\mu) = \begin{bmatrix} \mathbb{M}(h) & \mathbb{N} & \mathbb{L} & \mathbb{P} \\ (\star) & -\mu I_s & 0 & 0 \\ (\star) & (\star) & -\mu I_s & 0 \\ (\star) & (\star) & (\star) & -\mu I_s \end{bmatrix} \tag{47}$$

$$\mathbb{M}(h) = \mathcal{A}(h)^T S + S\mathcal{A}(\alpha) - \bar{H}^T R^T - R\bar{H} + I_{n+m} \tag{48}$$

$$\mathcal{A}(h) = \bar{P}\Big[M + B \sum_{i=1}^{i=q} \sum_{j=1}^{j=n+m} h_{ij} H_{ij}\Big] \tag{49}$$

$$\mathbb{N} = R\bar{E}_\omega - S\bar{P}E_\omega \tag{50}$$

$$\mathbb{L} = R\bar{D}_\omega + S\bar{Q}\bar{E}_\omega \tag{51}$$

$$\mathbb{P} = S\bar{Q}\bar{D}_\omega \tag{52}$$

The observer gains N, L and the minimum disturbance attenuation level $\gamma_{2,2}$ will be computed by :

$$N = \bar{P}M - RS^{-1}\bar{H}$$
$$L = RS^{-1} + N\bar{Q}$$
$$\gamma_{2,2} = \sqrt{\mu}$$

Proof. The proof uses the same tools as that of Theorem 1. Moreover, here we use the Lyapunov function $V(\varepsilon) = \varepsilon^T S \varepsilon$ to show that under the conditions of Theorem 2, we have always

$$W(\varepsilon) \triangleq \dot{V} + \varepsilon^T\varepsilon - \gamma_{2,2}^2\Big(\omega^T\omega - \dot{\omega}^T\dot{\omega} - \ddot{\omega}^T\ddot{\omega}\Big) < 0. \tag{53}$$

Remark 3. Note that if the derivative $y^{(k)}$ is needed to estimate simultaneously x and u, then we must use a modified $\mathcal{H}_\infty$ criteria of order $k+1$, i.e. :

$$\|\varepsilon\|_{\mathcal{L}_2^{n+m}} \le \gamma_{k+1,2} \|\omega\|_{k+1,2}^r. \tag{54}$$

Remark 4. If $\ker(CB) = \{0\}$, then we must consider as unknown input to estimate, the vector $\bar{u} = \begin{bmatrix} u \\ f(x,u) \end{bmatrix}$ and then, we estimate simultaneously x and $\bar{u}$, i.e. $\zeta = \begin{bmatrix} x \\ \bar{u} \end{bmatrix}$.

5. NUMERICAL EXAMPLE

In this section, we give a numerical example to illustrate the analytic results of the proposed method. We consider the problem of synchronization and decryption in chaotic communication systems represented in Figure 1. For this, we consider the case where only the transmitted signal is affected by disturbances. In order to well illustrate the performance and usefulness of the proposed theoretical results, we have chosen to use a picture as an unknown signal (unknown input) to be reconstructed (an encrypted picture to be decrypted). This picture represented in Figure 2, is transformed into a one dimensional signal u using the Matlab function "*reshape*".

The dynamic model of the transmitter is the chaotic system of Rössler. The parameters of the system are given as follows :

$$A_x = \begin{bmatrix} 0 & -1 & -1 \\ 1 & a & 0 \\ 0 & 0 & -c \end{bmatrix}, \ A_u = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \ B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

Fig. 2. Original picture

$$C = [0\ 1\ 0],\ D = 1,\ D_\omega = 1,\ E_\omega = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

and

$$f(x,u) = x_1 x_3 + x_1 u + b$$

where $a = 0.398, b = 2, c = 4$, values for which the system exhibits a chaotic behavior.

Matrices P and Q are

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix},\ Q = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Since the system is chaotic, then its states are bounded. Thus, we can compute numerically the bounds a_{ij} and b_{ij}. After simulating the system by using Matlab-Simulink, we obtained the following values :

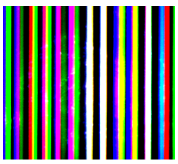
$$a_{11} = 0,\ a_{12} = -1.5,\ a_{13} = -1.5,$$

$$b_{11} = 5,\ b_{12} = 1,\ b_{13} = 1.$$

Using Matlab toolbox LMI, we obtain the following solutions of the convex optimization problem of Theorem 1 :

$$N = \begin{bmatrix} 0 & 0.04 & -1 & 2.04 \\ 1 & -0.90 & 0 & 0.70 \\ 0 & 1.32 & -4 & 1.32 \\ -1 & -1.26 & 0 & -2.86 \end{bmatrix},\ L = \begin{bmatrix} 1 \\ 2 \\ 0 \\ -2 \end{bmatrix},\ \ \gamma_{1,2} = 2.91$$

For numerical simulations, we have used as disturbance $\omega(t)$ a gaussian distributed random signal with mean zero. For $\omega(t)$ with a standard deviation $\sigma = 0.032$, the received and decrypted pictures corresponding to the signals y and $\hat{u}$, respectively, are represented in Figure 3. We observe that the picture is well decrypted in spite of the presence of weak disturbance.



(a) Received picture

(b) Decrypted picture

Fig. 3. Received and decrypted pictures

6. CONCLUSION

The main result of this paper lies in the definition of a new criteria to study robustness. This latter, called the *modified $\mathcal{H}_\infty$ criteria* is based on the use of Sobolev norms. Indeed, the derivative of the disturbances appears naturally in the dynamic error equation, and then the use of Sobolev norms is required because the disturbances must belong to the Sobolev space. Thanks to a Lyapunov quadratic function and the differential mean value theorem, new synthesis conditions, expressed in terms of LMIs, are given. The validity of the proposed method is shown through a numerical example. We have considered the problem of simultaneous synchronization and decryption in chaotic communication systems.

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