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Stabilization of trajectories for systems on Lie groups. Application to the rolling sphere.

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Abstract: This paper addresses the stabilization of admissible reference trajectories generated with constant inputs for driftless systems on Lie groups. The general expression of the linear approximation of the tracking error system is derived from the system's constants of structure and a necessary condition for the controllability of this approximation is specified in terms of the growth of the filtration of the Lie Algebra generated by the system's vector fields. This condition is illustrated with examples of mechanical systems whose control inputs correspond to velocity variables. By contrast with nonholonomic mobile robots whose kinematic equations can be transformed into the chained form, the linearized system associated with the rolling sphere is never controllable. Consequences of this lack of controllability as for stabilization problems are discussed from a general viewpoint and addressed more specifically for the rolling sphere. Finally, a practical stabilizer for this system based on the transverse function approach is proposed.

Keywords: nonholonomic system, Lie group, controllability, stabilization, rolling sphere

1. INTRODUCTION

This paper addresses the control of driftless systems on Lie groups, i.e.

$$\dot{g} = \sum_{i=1}^{m} X_i(g)v_i \tag{1}$$

with g the system's state belonging to an n-dimensional connected Lie group, $X_1, \ldots, X_m$ some independent *left-invariant* vector fields (v.f.) (i.e. elements of the group's Lie algebra), and $v = (v_1, \ldots, v_m)$ the vector of control inputs. It is further assumed that i) $m < n$, since otherwise the control of System (1) is trivial, and ii) the system is controllable, i.e., it satisfies the Lie algebra rank condition at the group's unit element (and thus at any point due to the invariance of the control v.f.). The assumption $m < n$ is characteristic of nonholonomic mechanical systems (wheeled mobile robots, rolling spheres, etc) often met in robotic applications. As for the invariance of the v.f. X_i, this property, although restrictive, holds for many physical systems: unicycle-like mobile robots, cars and N-trailer systems whose kinematic equations can be transformed into the chained form (Sørdalen [1993]), rolling spheres [Jurdjevic, 1997, Ch. 14] also referred to as ball-plate systems. Moreover, it is known that any driftless system (not necessarily on a Lie group) which is locally controllable at some point can be immersed, via a dynamic extension, into a system which locally admits a controllable homogeneous approximation, left-invariant w.r.t. to a group product, of the form (1) (see e.g. [Morin and Samson, 2003, Sec. VI] for details). The possibility of approximating many systems by systems on Lie groups is one of the reasons why these latter systems play a prominent role in nonlinear system theory (Sussmann [1987], Jurdjevic [1997]).

The assumption $m < n$ renders the control of the class of systems (1) particularly challenging because the linearization of System (1) at any fixed point (or state) is not controllable, and also due to the non-satisfaction of Brockett's necessary condition (Brockett [1983]) for the existence of smooth pure-state feedbacks capable of stabilizing a fixed point asymptotically. These difficulties have motivated an intense research in the last fifteen years and a systematic investigation of alternate classes of feedback laws (continuous time-varying, discontinuous, hybrid,...) in order to derive fixed-point asymptotic stabilizers. The reader is referred to Morin and Samson [2006] for more details on this topic. Some of the results here presented can be used for the stabilization of fixed points, but the present paper is primarily devoted to the stabilization of *non-stationary* reference trajectories, i.e. solutions $g_r(.)$ of System (1) associated with non-zero control inputs $v_r(.)$. The literature on this subject is relatively meager on the theoretical side, compared to publications dedicated to the stabilization of fixed points. In fact, it is commonly believed that, for these systems, non-stationary reference trajectories are much easier to stabilize than fixed points. The reason is that the linearized system along such trajectories may be controllable, whereas the linearization of the system at a fixed point is never controllable (nor stabilizable). In particular, it follows from Sontag [1992] that, for any controllable system (1) and "almost all" reference control inputs $v_r(.)$, the linearized system along any associated reference trajectory $g_r(.)$ is "controllable" [1]. This is a very interesting result. It is however important to interpret it correctly and not underestimate the importance of the "small" set of inputs and trajectories for which the controllability of the linearized system does not

[1] The reader is referred to Sontag [1992] for a precise statement of this property.

hold. For example, in the case of a car for which v_1 denotes the vehicle's driving velocity, the difficulty is "moderately" acute because the controllability of the linearized system on any time interval $[t_i, t_f]$ (with $t_i < t_f$) is satisfied whenever the reference velocity $v_{r,1}$ is a smooth function not identically equal to zero on $[t_i, t_f]$. By contrast with this "favorable" example, a result established further in the paper indicates that, for controllable driftless systems, the controllability of the linearized system along a reference trajectory generated with constant inputs is far from constituting the general rule. In terms of control design, the importance of this property comes from that it ensures the existence, and much simplifies the derivation, of smooth feedback laws capable of stabilizing exponentially such a reference trajectory generated with constant inputs (i.e. constant velocities in the case of a mechanical system). Up to now, concerning driftless systems with less control inputs than state components, this trajectory stabilization problem has been essentially addressed for systems which can be transformed into (or can be locally approximated by) chained systems (Kanayama et al. [1990], Samson [1990]). This case constitutes a generalization of the car example evoked previously in the sense that the controllability of the linearized system is obtained under the same conditions upon $v_{r,1}$ with v_1 the control input associated with the non-constant control vector field. For constant reference inputs it thus suffices that $v_{r,1} \neq 0$.

The present work further investigates the controllability and stabilizability properties of the class of systems (1) along non-stationary reference trajectories. First, a necessary condition for the controllability of the linearized error system along reference trajectories associated with constant inputs v_r is provided. From a mathematical standpoint "most" systems of the form (1) do not verify this condition. This is illustrated with the rolling sphere example (see e.g. Mukherjee et al. [2002] for open-loop motion issues and the path-planning problem, and Date et al. [2004], Oriolo and Vendittelli [2005] for fixed-point stabilization results) which constitutes the simplest controllable physical system whose linear approximation along admissible trajectories generated with constant inputs is never controllable. This example, which contrasts with the case of classical nonholonomic mobile robots, advocates for a better understanding of the control possibilities for this class of systems. A first step is to show that despite the lack of controllability of the linearized error system, the asymptotic stabilization of reference trajectories associated with constant inputs can always be achieved. The next issue is the design of trajectory stabilizers for these systems, starting with the rolling sphere. The solution here considered consists in applying the Transverse Function approach to derive a *practical* stabilizer yielding ultimately bounded and arbitrarily small tracking errors.

Due to space limitations, the proofs of the results here presented are omitted. They can be obtained upon request to the authors.

2. NOTATION AND RECALLS

Special vectors and matrices Throughout the paper, the transpose of a vector x in $\mathbb{R}^n$ is denoted as x'. The i-th vector of the canonical basis of $\mathbb{R}^n$ is denoted as b_i, i.e. all components of b_i are equal to zero except for the i-th one which is equal to one. The cross product in $\mathbb{R}^3$ is denoted as $\times$ and $\hat{x}$ is the skew-symmetric matrix associated with this product, i.e. $\hat{x}y = x \times y$. The identity matrix associated with $\mathbb{R}^n$ is denoted as I_n.

Systems on Lie groups We recall hereafter standard definitions and notation about Lie groups (see e.g. Varadarajan [1984] for more details on this topic), and elementary properties of systems on Lie groups. Let G denote a connected Lie group of dimension n. The unit element of G is denoted as e, i.e. $\forall g \in G \;:\; ge = eg = g$. The inverse g^{-1} of $g \in G$ is the (unique) element in G such that $gg^{-1} = g^{-1}g = e$. The left (resp. right) translation operator on G is denoted as l (resp. r), i.e. $\forall(\sigma,\tau) \in G^2 \;:\; l_\sigma(\tau) = r_\tau(\sigma) = \sigma\tau$. A v.f. X on G is left-invariant iff $\forall(\sigma,\tau) \in G^2,\; dl_\sigma(\tau)X(\tau) = X(\sigma\tau)$, with df denoting the differential of the function f. The Lie algebra –of left-invariant v.f.– of the group G is denoted as $\mathfrak{g}$. If $X \in \mathfrak{g}$, $\exp(tX)$ is the solution at time t of $\dot{g} = X(g)$ with the initial condition $g(0) = e$. The adjoint representation of G is denoted as Ad, i.e. $\forall\sigma \in G,\; \mathrm{Ad}(\sigma) := dJ_\sigma(e)$, with $J_\sigma : G \to G$ defined by $J_\sigma(g) := \sigma g \sigma^{-1}$. The adjoint representation of $\mathfrak{g}$ is denoted as ad. One has $(\mathrm{ad}\, X)(Y) = [X, Y]$ with $[.,.]$ the Lie bracket operator, and also

$$\frac{d}{dt}_{|t=0} \mathrm{Ad}(\exp(tX))Y(e) = (\mathrm{ad}\, X)(Y)(e), \quad \forall X, Y \in \mathfrak{g} \tag{2}$$

By extension of the definition of ad, we define by induction $(\mathrm{ad}\, X)^k(Y) = (\mathrm{ad}\, X)((\mathrm{ad}\, X)^{k-1}(Y))$ for integers $k \geq 2$.

A driftless control system (1) on G is said to be left-invariant if the control v.f. X_i are left-invariant. Given a family $Y := \{Y_1, \ldots, Y_p\}$ of vector fields on G and a vector $v \in \mathbb{R}^p$, we denote by Yv the vector field $\sum_{i=1}^p Y_i v_i$, and by $Y(g)v$ its value at g.

Let $X = \{X_1, \ldots, X_n\}$ denote a basis of $\mathfrak{g}$. The constants of structure associated with the basis X are denoted as $c^r_{p,q}$ (i.e., for any (p, q), $[X_p, X_q] = \sum_{r=1}^n c^r_{p,q} X_r$. If $(g_a(t), v_a(t))$ and $(g_b(t), v_b(t))$ $(t \geq 0)$ are two solutions to $\dot{g} = X(g)v$, then (by omitting the time index)

$$\frac{d}{dt}(g_a g_b^{-1}) = X(g_a g_b^{-1})\mathrm{Ad}^X(g_b)(v_a - v_b) \tag{3}$$

with Ad^X the expression of the Ad operator in the basis X, i.e. the (invertible) matrix-valued function defined by $\forall\sigma \in G,\; \forall v \in \mathbb{R}^n,\; \mathrm{Ad}(\sigma)X(e)v = X(e)\mathrm{Ad}^X(\sigma)v$. According to this definition, $\mathrm{Ad}^X(e) = I_n$. We have also

$$\frac{d}{dt}(g_a^{-1} g_b) = X(g_a^{-1} g_b)(v_b - \mathrm{Ad}^X(g_b^{-1} g_a)v_a) \tag{4}$$

In a way similar to the definition of Ad^X, we denote by ad^X the expression of the ad operator in the basis X, i.e. $\forall v_1, v_2 \in \mathbb{R}^n,\; X(e)\mathrm{ad}^X(v_1)v_2 = (\mathrm{ad}\, Xv_1)(Xv_2)(e) = [Xv_1, Xv_2](e)$. Let us remark that for any vector v the matrix $\mathrm{ad}^X(v)$ can be expressed as a function of the constants of structure $c^r_{p,q}$ associated with the basis X. More precisely, one verifies that $\mathrm{ad}^X(v)$ can be decomposed in column vectors as follows:

$$\mathrm{ad}^X(v) = \Big((c^j_{k1})_{jk} v \mid \ldots \mid (c^j_{kn})_{jk} v\Big) \tag{5}$$

with $(c_{kp}^j)_{jk}$ $(p = 1, \ldots, n)$ denoting the matrix with element c_{kp}^j at row j and column k.

3. CONTROLLABILITY ALONG REFERENCE TRAJECTORIES

3.1 A necessary condition

Let $X = \{X_1, \ldots, X_n\}$ denote a basis of $\mathfrak{g}$. To simplify the notation, we rewrite System (1) as $\dot{g} = X(g)Cv$, with $C = (I_m \mid 0_{m\times(n-m)})'$. Let g_r denote an admissible reference trajectory, i.e. a solution to the equation $\dot{g}_r = X(g_r)Cv_r$. Then it follows from (4) that the "tracking error" $\tilde{g} = g_r^{-1}g$ satisfies the equation

$$\begin{aligned}\dot{\tilde{g}} &= X(\tilde{g})(Cv - \mathrm{Ad}^X(\tilde{g}^{-1})Cv_r)\\ &= X(\tilde{g})(C\tilde{v} - (\mathrm{Ad}^X(\tilde{g}^{-1}) - I_n)Cv_r)\end{aligned} \tag{6}$$

with $\tilde{v} = v - v_r$. In order to linearize this (tracking) error equation, one needs first to define a system of coordinates around the unit element $e \in G$. We choose coordinates $\tilde{x}$ of the first kind defined by the equality $\tilde{g} = \exp(\tilde{x}_1X_1 + \ldots + \tilde{x}_nX_n) = \exp(X\tilde{x})$.

Lemma 1. The linearization of System (6) at the equilibrium $(\tilde{g}, \tilde{v}) = (e, 0)$ is given, in the coordinates $\tilde{x}$, by

$$\dot{\tilde{x}} = -\mathrm{ad}^X(Cv_r)\tilde{x} + C\tilde{v} \tag{7}$$

It follows from this lemma that when v_r is constant, the linear system (7) is controllable if and only if the pair $(\mathrm{ad}^X(Cv_r), C)$ is controllable. Note that, from (5) and (7), the linearized error system and its controllability properties can be deduced from the sole knowledge of the constants of structure. It is also important to remark at this point that these properties are independent of the choice of local coordinates $\tilde{x}$.

The second main result of this section follows. It establishes a necessary condition for controllability, in terms of the growth of the filtration of $\mathfrak{g}$ associated with the system's control vector fields.

Proposition 2. Let $\mathfrak{u}^k$ $(k = 1, \ldots, K)$ denote the subspaces of $\mathfrak{g}$ defined recursively by $\mathfrak{u}^1 = \mathrm{span}\{X_1, \ldots, X_m\}$ and $\mathfrak{u}^k = \mathfrak{u}^{k-1} + [\mathfrak{u}, \mathfrak{u}^{k-1}]$ for $k = 2, \ldots, K$, with K the smallest integer such that $\mathfrak{u}^K = \mathfrak{g}$. Also, let d_k $(k = 1, \ldots, K)$ denote the dimension of $\mathfrak{u}^k$. Then, when v_r is constant, a necessary condition for the controllability of System (7) is that

$$\forall k = 1, \ldots, K-1, \qquad d_{k+1} - d_k \leq d_k - d_{k-1} \tag{8}$$

with $d_0 = 0$. In particular, if $m = 2$, a necessary condition for the controllability of System (7) is that

$$\forall k = 2, \ldots, K, \qquad d_k = k + 1 \tag{9}$$

Note that the necessary conditions (8) and (9) for controllability can be checked out without computing the expression of the matrix $\mathrm{ad}^X(Cv_r)$. These conditions are clearly restrictive and one can thus infer that "most" controllable driftless systems do not satisfy them. For example, Condition (8) is never satisfied by *free* systems (see e.g. Kawski [1992] for a definition) unless $m \in \{2, 3\}$ and $n = m + 1$. Of course, the non-controllability of the linearized system (7) does not imply that the nonlinear error system (6) is not controllable. As a matter of fact, it follows from the controllability of System (1) and from e.g. [Sussmann, 1987, Prop. 7.4] that for any constant v_r, System (6) is Small Time Locally Controllable at $\tilde{g} = e$.

3.2 Examples

Chained systems: The control v.f. of the classical n-dimensional chained system with two control inputs are defined by $X_1(x) = (1, 0, x_2, \ldots, x_n)'$ and $X_2(x) = (0, 1, 0, \ldots, 0)'$. This is a system on a Lie group, with the group operation defined by

$$xy = \begin{cases} x_i + y_i & \text{if } i = 1, 2 \\ x_i + y_i + \displaystyle\sum_{j=2}^{i-1} \frac{y_1^{i-j}}{(i-j)!}x_j & \text{otherwise} \end{cases}$$

A basis $X = \{X_1, \ldots, X_n\}$ of the associated Lie algebra is obtained by setting

$$\forall i = 3, \ldots, n, \quad X_i = (\mathrm{ad}\, X_1)^{i-2}(X_2) = (-1)^i b_i$$

with b_i the i-th canonical vector in $\mathbb{R}^n$. It follows that Condition (9) of Proposition 2 is satisfied so that the controllability of the linearized error system cannot be ruled out. Proceeding thus further, one verifies that the constants of structure $c_{p,q}^r$ $(p, q, r = 1, \ldots, n)$ associated with this basis are

$$c_{p,q}^r = \begin{cases} 1 & \text{if } p = 1, q \neq 1, r = q+1 \\ -1 & \text{if } q = 1, p \neq 1, r = p+1 \\ 0 & \text{otherwise} \end{cases}$$

From these expressions and (5)

$$\mathrm{ad}^X(Cv_r) = \begin{pmatrix} 0 & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & 0 & 0 & \cdots & \cdots & 0 \\ -v_{r,2} & v_{r,1} & 0 & 0 & \cdots & 0 \\ 0 & 0 & v_{r,1} & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & v_{r,1} & 0 \end{pmatrix}$$

One deduces from this expression that, when v_r is constant, the linearized system (7) is controllable if and only if $v_r \neq 0$ when $n = 3$, and $v_{r,1} \neq 0$ when $n > 3$.

Rolling sphere: This case physically corresponds to a sphere rolling on a horizontal plane (without slipping or twisting) under the action of dry-friction forces produced by an actuated moving plate placed on top of it (see Fig. 1). The (idealized) kinematic equations of this system are given by (see e.g. [Jurdjevic, 1997, Ch. 14])

$$\begin{cases} \dot{x} = v \\ \dot{R} = R(v_2\hat{b}_1 - v_1\hat{b}_2) \end{cases} \tag{10}$$

with $x \in \mathbb{R}^2$ the first two coordinates of the sphere's center C in the inertial frame $\{0, \boldsymbol{i}_0, \boldsymbol{j}_0, \boldsymbol{k}_0\}$, $v = (v_1, v_2)'$ the vector of control inputs, and $R \in SO(3)$ the matrix representing the orientation of the sphere (the column vectors of which correspond to the components of the inertial frame vectors $\boldsymbol{i}_0, \boldsymbol{j}_0, \boldsymbol{k}_0$ expressed in the body frame $\{C, \boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$).

System (10) is a system on the Lie group $\mathbb{R}^2 \times SO(3)$ with the group operation defined by $(x, R)(\bar{x}, \bar{R}) = (x + \bar{x}, R\bar{R})$. The system's control v.f. are

$$X_1(x, R) = \begin{pmatrix} 1 \\ 0 \\ -R\hat{b}_2 \end{pmatrix}, \quad X_2(x, R) = \begin{pmatrix} 0 \\ 1 \\ R\hat{b}_1 \end{pmatrix} \tag{11}$$

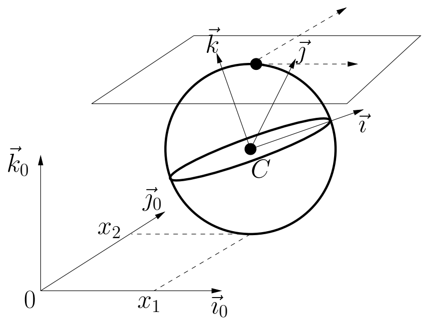

Fig. 1. The rolling sphere actuated by a moving plate

A direct calculation of $X_3 = [X_1, X_2]$, $X_4 = [X_3, X_1]$, and $X_5 = [X_3, X_2]$ yields

$$X_3(R) = \begin{pmatrix} 0 \\ 0 \\ R\hat{b}_3 \end{pmatrix}, \; X_4(R) = \begin{pmatrix} 0 \\ 0 \\ R\hat{b}_1 \end{pmatrix}, \; X_5(R) = \begin{pmatrix} 0 \\ 0 \\ R\hat{b}_2 \end{pmatrix} \tag{12}$$

It follows from the above equalities that $d_1 = 2$, $d_2 = 3$, and $d_3 = 5$, so that $d_3 - d_2 > d_2 - d_1$. By Proposition 2, the associated linearized system (7) is never controllable when v_r is constant.

4. STABILIZATION OF REFERENCE TRAJECTORIES

Proposition 2, illustrated by the rolling sphere example, shows that for many systems of the form (1), the linearized system along reference trajectories with constant velocities is not controllable. This raises a certain number of issues about the possibility and means of stabilizing such trajectories. First question: can these trajectories be asymptotically stabilized by continuous feedback? The next result, which essentially follows from the results of Coron [1995], provides a positive answer to this question.

Proposition 3. For any constant vector v_r, the equilibrium point $\tilde{g} = e$ of System (6) is locally asymptotically stabilizable by continuous time-varying (periodic) feedback.

Second question: can these trajectories be asymptotically stabilized (at least in some cases) by continuous or even smooth *pure state* feedback? By analogy with the case $v_r = 0$, it is tempting to give a negative answer to this question (by application of Brockett's theorem). However, due to the presence of the drift term in (6), a general proof of this assertion does not seem straightforward, so that the issue remains open. Third question: can these trajectories be *exponentially* stabilized by *smooth*, possibly time-varying, feedback? Again, one would guess a negative answer to this question but this remains to be shown (or invalidated).

Although general answers to these last two questions are not yet available, answers in the specific case of the rolling sphere example are stated in the following proposition.

Proposition 4. Consider the tracking error system (6) associated with the rolling sphere kinematic model (10), with v_r a constant vector. Then, the equilibrium $(\tilde{g}, \tilde{v}) = (e, 0)$

(1) cannot be asymptotically stabilized by continuous pure state feedback,
(2) cannot be exponentially stabilized by lipschitz time-varying (periodic) feedback.

Coming back to the general error system (6), since asymptotic stabilization of admissible trajectories generated with constant inputs is possible (by Proposition 3) the next issue is the *design* of stabilizing control laws. When the reference trajectory is a fixed point, (6) is a driftless system and there exist general methods and/or algorithms to synthesize local asymptotic stabilizers (see e.g. Morin et al. [1999]). Ad hoc solutions have also been proposed by Date et al. [2004] and Oriolo and Vendittelli [2005]. In the case of non-stationary trajectories, System (6) has a drift term, and no general method for the design of *asymptotic* stabilizers is known. As a matter of fact, to our knowledge, no such stabilizer has even been proposed for the rolling sphere example, i.e. the simplest of the controllable driftless systems whose linear approximation along admissible non-stationary trajectories generated with constant inputs is never controllable. On the other hand, a general method to design *practical* stabilizers for systems in the form (6), yielding ultimately bounded and arbitrarily small (but non-zero) tracking errors, is proposed in Morin and Samson [2003]. This method is based on the use of so-called Transverse Functions and one of its most noticeable feature is that it also applies to non-admissible reference trajectories (i.e. such that $\dot{g}_r$ does not always belong to $\mathrm{span}\{X_1(g_r), \ldots, X_m(g_r)\}$) with the same results. By a proper extension, the control solutions derived with this method may also be asymptotic stabilizers of admissible trajectories (see e.g. Morin and Samson [2004]).

The remainder of this section is devoted to the particularization of the Transverse Function approach to the rolling sphere example. This is a preliminary work and the possibility of obtaining asymptotic stabilizers of non-stationary trajectories with this approach will be addressed in subsequent studies. A brief recall of the approach is now given before detailing its application to the rolling sphere. Let

- $\mathbb{T}^k$ denote the k-dimensional torus, with $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$,
- $X = \{X_1, \ldots, X_n\}$ denote a basis of $\mathfrak{g}$,
- f denote a smooth function from $\mathbb{T}^{n-m}$ to a neighborhood $\mathcal{U} \subset G$ of e.

Then, there exists a matrix-valued function $A(\alpha)$ such that, along any differentiable path $\alpha(t)$ on $\mathbb{T}^{n-m}$, one has

$$\begin{aligned} \dot{f}(\alpha) &= X(f(\alpha))A(\alpha)\dot{\alpha} \\ &= X^1(f(\alpha))A^1(\alpha)\dot{\alpha} + X^2(f(\alpha))A^2(\alpha)\dot{\alpha} \end{aligned} \tag{13}$$

with $X^1 = \{X_1, \ldots, X_m\}$ and $X^2 = \{X_{m+1}, \ldots, X_n\}$. The function f is said to be *transversal to the v.f.* $X_1, \ldots, X_m$ if (and only if) $A^2(\alpha)$ is invertible $\forall \alpha \in \mathbb{T}^{n-m}$. One can verify that this condition is equivalent to the invertibility, $\forall \alpha \in \mathbb{T}^{n-m}$, of the $n \times n$ matrix

$$\bar{C}(\alpha) = (C \mid -A(\alpha)) \tag{14}$$

The transverse function theorem given in Morin and Samson [2003] asserts the existence of such functions, *whatever the size of* $\mathcal{U}$, provided that the Lie algebra generated by the family X^1 is equal to $\mathfrak{g}$. It also provides a general expression for a family of such functions. Given a function f transverse to the control v.f. of System (1), it follows from (4), (6), and (14), that the variable $z = \tilde{g}f(\alpha)^{-1}$ satisfies the equality

$$\dot{z} = X(z)\mathrm{Ad}^X(f(\alpha))(\bar{C}(\alpha)\bar{v} - \mathrm{Ad}^X(\tilde{g}^{-1})Cv_r) \tag{15}$$

with $\bar{v} = (v_1, \ldots, v_m, \dot{\alpha}_{m+1}, \ldots, \dot{\alpha}_n)'$ and $\alpha_{m+1}, \ldots, \alpha_n$ the $n - m$ components of α. In view of the invertibility of the matrix $\bar{C}(\alpha)$, it is not difficult to find feedback laws that make $z = e$ locally asymptotically stable. Set e.g.

$$\bar{v} = \bar{C}(\alpha)^{-1}(\mathrm{Ad}^X(\tilde{g}^{-1})Cv_r + \mathrm{Ad}^X(f(\alpha)^{-1})K(z)) \quad (16)$$

with $K(z)$ any feedback which asymptotically stabilizes $z = e$ for the *fully actuated* (thus much more easily controlled) system $\dot{z} = X(z)u$. The convergence of z to e implies the convergence of $\tilde{g}$ to the (bounded) image set $f(\mathbb{T}^{n-m}) \subset \mathcal{U}$. Note that v_r can take any value and does not need to be constant. An important step in the application of this control design method is the calculation of the transverse functions intervening in the control expression. This can be done in several ways with significant variations of the closed-loop system performance. As for now, we will just indicate a possible choice for the rolling sphere, following [Morin and Samson, 2003, Sec. IV].

Transverse functions f for the rolling sphere can be defined as the group product of three elementary functions of one variable, i.e.

$$f(\alpha) = f_5(\alpha_5)f_4(\alpha_4)f_3(\alpha_3) \quad (17)$$

with the functions f_3, f_4, f_5 defined as follows:

$$\begin{aligned} f_3(\alpha_3) &= \exp(\varepsilon_3 \sin\alpha_3 X_1 + \varepsilon_3 \cos\alpha_3 X_2) \\ f_4(\alpha_4) &= \exp(\varepsilon_4^2 \sin\alpha_4 X_3 + \varepsilon_4 \cos\alpha_4 X_1) \\ f_5(\alpha_5) &= \exp(\varepsilon_5^2 \sin\alpha_5 X_3 + \varepsilon_5 \cos\alpha_5 X_2) \end{aligned} \quad (18)$$

for some strictly positive numbers $\varepsilon_3, \varepsilon_4, \varepsilon_5$, where exp denotes the exponential function on $\mathbb{R}^2 \times SO(3)$, and $X_1, \ldots, X_5$ denote the basis vectors defined by (11)–(12). It follows from (10) that the exp function can be decomposed as

$$\exp(v, R\hat{\omega}) = \begin{pmatrix} v \\ \mathrm{Exp}(\hat{w}) \end{pmatrix}$$

with Exp the matrix exponential (i.e. $\mathrm{Exp}(\hat{w})$ is the rotation of angle $|w|$ about the axis w). Therefore, from (18) and (11)–(12)

$$f_3(\alpha_3) = \begin{pmatrix} \varepsilon_3 \sin\alpha_3 \\ \varepsilon_3 \cos\alpha_3 \\ \mathrm{Exp}(\varepsilon_3 \cos\alpha_3 \hat{b}_1 - \varepsilon_3 \sin\alpha_3 \hat{b}_2) \end{pmatrix}$$

$$f_4(\alpha_4) = \begin{pmatrix} \varepsilon_4 \cos\alpha_4 \\ 0 \\ \mathrm{Exp}(\varepsilon_4^2 \sin\alpha_4 \hat{b}_3 - \varepsilon_4 \cos\alpha_4 \hat{b}_2) \end{pmatrix} \quad (19)$$

$$f_5(\alpha_5) = \begin{pmatrix} 0 \\ \varepsilon_5 \cos\alpha_5 \\ \mathrm{Exp}(\varepsilon_5^2 \sin\alpha_5 \hat{b}_3 + \varepsilon_5 \cos\alpha_5 \hat{b}_1) \end{pmatrix}$$

The analytical expressions of the matrices Exp(.) in the above expressions are then obtained by application of Rodrigue's formula (see e.g. [Marsden and Ratiu, 1999, Sec. 9.2]). This yields analytical expressions of both the function f (from (17)) and the matrix $A(\alpha)$ in (13)-(14). There remains to address the choice of the parameters ε_3, ε_4, and ε_5, knowing that the property of transversality (i.e. the inversibility, for any α, of the 3×3 matrix $A^2(\alpha)$ defined by (13)) does not hold for all values of these parameters. The proof of the main theorem in Morin and Samson [2003] suggests choosing ε_5 "small enough", then ε_4 "small enough" w.r.t. ε_5 and, finally, ε_3 "small enough" w.r.t. ε_4. The determination of sufficient conditions on these parameters in order to ensure the property of transversality of f will be addressed in a subsequent study. This control strategy is now illustrated by simulation results.

The quaternion parameterization of $SO(3)$ is particularly well suited to the present framework because the set $\mathbb{Q}^1$ of unitary quaternions is also a Lie group. More precisely, System (10) can be written as

$$\begin{cases} \dot{x} = v \\ \dot{q} = \dfrac{1}{2} q.(v_2, -v_1, 0)^T \end{cases} \quad (20)$$

with $q = (q_0, q_p') = (q_0, q_1, q_2, q_3) \in \mathbb{Q}^1$ the quaternion associated with the rotation matrix R. Recall that when $R = \mathrm{Exp}(\theta\hat{b})$ with b a unitary vector, then $q_0 = \cos\theta/2$ and $q_p = (q_1, q_2, q_3)' = b \sin\theta/2$. The product $q.(v_2, -v_1, 0)^T$ in the above equation is defined via the more general expression

$$q.\omega = \begin{pmatrix} q_p'\omega \\ q_0\omega + \hat{q}_p\omega \end{pmatrix}$$

System (20) defines a system on the Lie group $\mathbb{R}^2 \times \mathbb{Q}^1$ with the group product $(x, q).(\bar{x}, \bar{q}) = (x + \bar{x}, q\bar{q})$ where $q\bar{q}$ denotes the product of q and $\bar{q}$ in $\mathbb{Q}^1$. Therefore, all calculations involved in the determination of the control law (16) can be directly carried out in this Lie group. Finally, given a function $f = (f_x, f_R) \in \mathbb{R}^2 \times SO(3)$ transversal to the control v.f. of System (10), $\bar{f} = (f_x, f_q) = (f_x, \mathrm{quat}(f_R)) \in \mathbb{R}^2 \times \mathbb{Q}^1$ is a function transversal to the control v.f. of System (20), with $\mathrm{quat}(R) = (q_0, q_p')$ the quaternion associated with the matrix R, defined by

$$q_0 = \frac{1}{2}\sqrt{1 + \mathrm{tr}(R)}\,, \quad \hat{q}_p = \frac{1}{2\sqrt{1 + \mathrm{tr}(R)}}(R - R^T)$$

with $\mathrm{tr}(R)$ the trace of R.

The simulation results presented now have been obtained by applying the control law (16) to System (20), with the transverse function $\bar{f}$ deduced from the function f defined by (17) as described above. The parameters ε_i $(i = 3, 4, 5)$ have been chosen as $\varepsilon_3 = 0.15, \varepsilon_4 = 0.3, \varepsilon_5 = 0.5$. Numerous simulations suggest that the property of transversality of f is satisfied with this choice of parameters. The feedback law $K(z)$ has been chosen as

$$K(z) = -0.05(z(1), z(2), 2z(4) - z(2), 2z(5) + z(1), z(6))'$$

Note that $z = (\tilde{x} - f_x(\alpha), \tilde{q}f_q(\alpha)^{-1})$ with $\tilde{x} = x - x_r$ the position tracking error, and $\tilde{q}$ the quaternion associated with the orientation error w.r.t. the reference trajectory. The velocity v_r associated with the reference motion has been chosen as follows:

$$v_r = \begin{cases} (0,0)' & \text{for } t \in [0, 80) \\ (0.1, 0)' & \text{for } t \in [80, 120) \\ (0, -0.1)' & \text{for } t \in [120, 160) \\ (-0.1, 0.1)' & \text{for } t \in [160, 190) \\ (0,0)' & \text{for } t \in [190, 200] \end{cases}$$

The initial conditions have been set as $x(0) = (0.3, 0.5)'$, $x_r(0) = (0, 0)'$, $\tilde{q}_p(0) = (0.3, -0.15, 0)'$, and $\alpha_i(0) = 0$ $(i = 3, 4, 5)$. The position error $\tilde{x}$ is represented on Fig. 2, and the three components of $\tilde{q}_p$, characterizing the orientation error are represented on Fig. 3. Finally, the motion in the plane of both the sphere's center (solid lines) and reference point (dashed lines) are represented on Fig. 4. As predicted by the theory, the tracking errors remain bounded during all motion phases, by values commensurable with the parameters ε_i of the transverse function. This is an important asset of the transverse function approach.

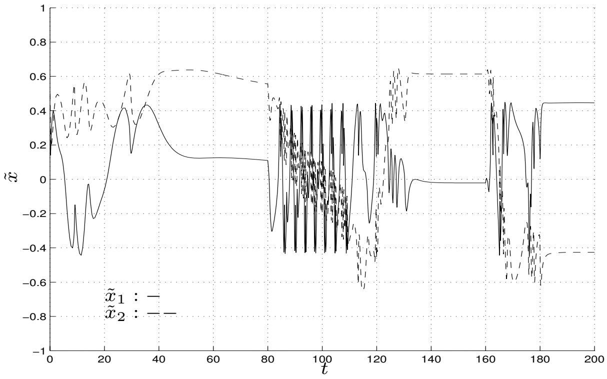

Fig. 2. Position tracking error $\tilde{x}$

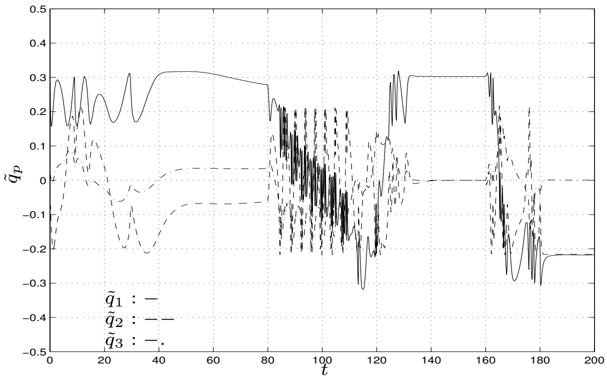

Fig. 3. Orientation tracking error $\tilde{q}_p$

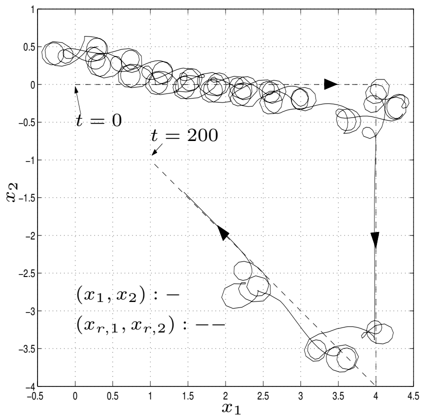

Fig. 4. Motion of the sphere's center and reference point in the plane

5. CONCLUSION

This paper has established that many controllable driftless systems have linearized equations along reference trajectories generated with constant inputs which are never controllable. The simplest physical system of this kind is the rolling sphere, the study of which is thus of particular interest. For these systems, most of the difficulties associated with the asymptotic stabilization of fixed points continue to hold when considering the stabilization of non-stationary admissible trajectories. As a matter of fact, they may even be amplified due to the drift term in the error system equations. Several related open questions have been formulated and a practical stabilizer, based on the Transverse Function approach, has been proposed for the rolling sphere.

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