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Timing and deadlock-freeness in Continuous Petri nets ⋆

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Abstract: Timing an unforced (discrete or continuous) net model preserves deadlock-freeness, but not the stronger liveness property, in general. The converse is not true, and if the autonomous net model has deadlocks, the timing may transform it into deadlock-free. Under infinite servers semantics, here we investigate the conditions on the firing rates of continuous timed models that makes deadlock-free a given timed system.

1. INTRODUCTION

It is a well known fact that the addition of timing constraints to the firing of transitions does not preserve liveness or non-liveness in (*discrete*) Petri net systems. It is also in the folklore of the field that, for stochastic discrete models, these properties are preserved when the support of the stochastic functions associated to the firing of transitions is infinite. Let us study this a little more deeply by means of two simple examples. The net system in Fig. 1, seen as autonomous (i.e., with no timing), is obviously live. Nevertheless, if we associate deterministic timings θ_1 and θ_3 to transitions t_1 and t_3, respectively, with θ_3 smaller (thus faster) than θ_1, t_2 will never be enabled, thus cannot be fired, and non liveness follows. Considering now the net system in Fig. 2(a), it is also immediate to asses that it is non-live as autonomous; nevertheless, if $\theta_1 = \theta_2$, and both transitions are deterministically timed, the system becomes live. Therefore, liveness of the discrete autonomous model is nor necessary, neither sufficient for that of the (at least partially) deterministically-timed interpreted model. Regarding deadlock-freeness, things are a bit simpler, if a system is deadlock-free as autonomous it will be deadlock-free as timed.

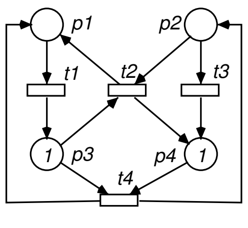

Fig. 1. Live as autonomous discrete net system but non-live under certain deterministic timing: $\theta_1 > \theta_2$

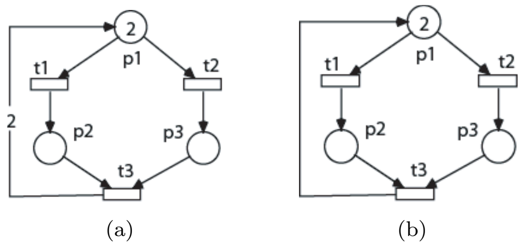

Fig. 2. (a) Non live as autonomous, but live as timed if $\lambda_1 = \lambda_2$, (b) is neither live as autonomous, nor for any lambda

If we consider now a classical markovian timed interpretation (i.e., all transitions have associated an exponential probability distribution function, pdf), then the Markov chain and the reachability graph are isomorphous (see Molloy [1982]). Thus any autonomous discrete net system, and the result of arbitrarily timing it, are both simultaneously live (deadlock-free) or both equally non-live (non-deadlock-free). Therefore, even if the net system in Fig. 1 is non live for the above mentioned deterministic timing, it is live for any positive exponential timing; moreover, the net system in Fig. 2(a) is live for the defined deterministic timing, but non live for any markovian case, even if timing rates are equal: $\lambda_1 = \lambda_2$.

In principle, infinite server semantics in *continuous* nets can be interpreted as a limit case for the markovian interpretation of the *discrete* net model (Recalde and Silva [2001]), and we might expect that the system in Fig. 2(a) deadlocks, even for $\lambda_1 = \lambda_2$. Nevertheless, under $\lambda_1 = \lambda_2$ the timed continuous Petri net ($TCPN$) model is deadlock-free, what "can be understood" as the mean time to deadlock being extremely long (in fact, for the discrete *model*, if $\mathbf{m_0}(p_1) = 2k$, it would take $2k^2$ firings of t_3 (see Silva and Recalde [2002]); but continuization "is like k going to be extremely large", thus, even if with probability 1 the deadlock will be reached, it would take an "enormous" amount of time).

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The kind of question we address here is: which conditions should the firing rate $\boldsymbol{\lambda}$ verify so that it can be guaranteed that the *timed continuous* Petri net is deadlock-free? The results in this context are of two types: If the continuous Petri net (CPN) is already deadlock-free, it will remain deadlock-free for any infinite servers semantics interpretation (already advanced for a particular case in Júlvez et al. [2006]), and the new contribution: if the autonomous CPN deadlocks, it can eventually be transformed into deadlock-free (somehow, very long time to deadlock) for particular numerical timing of the continuous model. The results hold even for deadlock-free non-monotonic systems (a system that being deadlock-free, has a deadlock if the initial marking is increased). Intuitively speaking, deadlocks, being associated to net *siphons* that are emptied, are avoided by creating some token conservations laws around siphons, thus avoiding them to become empty.

The material in this paper is structured as follows: After recalling elementary concepts and notations, basics of deadlock-freeness in CPNs is considered, both in autonomous and timed systems. Later, using stability results in linear systems theory and concepts and structures from Petri net theory, the timing enforcing of token conservations laws is addressed.

2. NOTATION

We assume that the reader is familiar with Petri nets (PNs) (for notation we use the standard one, see for instance Silva [1993]).

The structure $\mathcal{N} = \langle P, T, \mathbf{Pre}, \mathbf{Post}\rangle$ of *continuous Petri nets* (CPN) is the same as the structure of the usual (discrete) PNs. That is, P is a finite set of places, T is a finite set of transitions with $P \cap T = \emptyset$, $\mathbf{Pre}$ and $\mathbf{Post}$ are $|P| \times |T|$ sized, natural valued, *pre- and post-incidence matrices*. We assume that $\mathcal{N}$ is connected and that every place has a successor, i.e. $|p^\bullet| \geq 1$. The usual PN system, $\langle \mathcal{N}, \mathbf{m_0}\rangle$, will be said to be *discrete* so as to distinguish it from a *continuous* PN system, in which $\mathbf{m_0} \in (\mathbb{R}_{\geq 0})^{|P|}$. Here, we always consider net systems whose $\mathbf{m_0}$ marks all P-semiflows. The main difference between both formalisms is in the evolution rule, since in continuous PNs firing is not restricted to be done in integer amounts (Alla and David [1998], Silva and Recalde [2002]). As a consequence the marking is not forced to be integer. More precisely, a transition t is enabled at $\mathbf{m}$ iff for every $p \in {}^\bullet t$, $\mathbf{m}[p] > 0$, and its *enabling degree* is $\mathrm{enab}(t, \mathbf{m}) = \min_{p \in {}^\bullet t}\{\mathbf{m}[p]/\mathbf{Pre}[p, t]\}$. The firing of t in a certain amount $\alpha \leq \mathrm{enab}(t, \mathbf{m})$ leads to a new marking $\mathbf{m}' = \mathbf{m} + \alpha \cdot \mathbf{C}[P, t]$, where $\mathbf{C} = \mathbf{Post} - \mathbf{Pre}$ is the token-flow matrix.

As in discrete systems, right and left integer annullers of the token flow matrix are called *T-* and *P-flows*, respectively. When they are non-negative, they are called *T-* and *P-semiflows*. If there exists $\mathbf{y} > \mathbf{0}$ s.t. $\mathbf{y} \cdot \mathbf{C} = \mathbf{0}$, the net is said to be *conservative*, and if there exists $\mathbf{x} > \mathbf{0}$ s.t. $\mathbf{C} \cdot \mathbf{x} = \mathbf{0}$ the net is said to be *consistent*. A set of places Σ is a *siphon* iff ${}^\bullet\Sigma \subseteq \Sigma^\bullet$ (the set of input transitions is smaller or equal to the corresponding output one), and it is *minimal* if it does not contain another siphon. For example, in the net in Fig. 3(a), $\{p_4, p_5, p_6\}$ defines a minimal siphon, while $\{p_3, p_4, p_5, p_6\}$ is also a siphon, but it is non minimal. Transitions whose firing increases (decreases) the marking of Σ are called generator (consumer) transitions. Transition t_1 is a consumer and t_3 is a generator for the first siphon; for the second one the generator is t_2 not t_3.

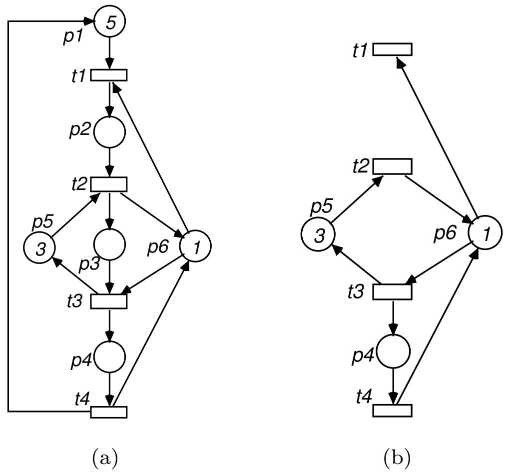

Fig. 3. (a) Deadlocks as autonomous, but is deadlock-free as timed if $\lambda_1 > \lambda_3$. (b) Minimal siphon of the net of 3(a).

For reachability, as in Júlvez et al. [2006], the limit concept is used, and a marking reached in the limit of an infinitely long sequence is considered reachable. $\langle \mathcal{N}, \mathbf{m_0}\rangle$ is deadlock-free iff for every reachable marking there exists $t \in T$ such that $\mathrm{enab}(t, \mathbf{m}) > 0$. If a marking $\mathbf{m}_D$ is a deadlock, no transition is enabled and the set $\Sigma_D = \{p \in P \mid \mathbf{m}_D(p) = 0\}$ is a (usually non minimal) siphon whose outputs cover all transitions (i.e. ${\Sigma_D}^\bullet = T$). For example, for the system in Fig. 3(a), $[1, 1, 3, 0, 0, 0]$ is the only deadlock marking, and defines the minimal siphon $\{p_4, p_5, p_6\}$ as seen before.

A simple and interesting way to introduce time in *discrete* PNs is to assume that all the transitions are timed with exponential probability distribution functions. For the timing interpretation of *continuous* PNs we will use a first order (or deterministic) approximation of the discrete case, assuming that the delays associated to the firing of transitions can be approximated by their mean values. Hence, a *Timed Continuous Petri Net (TCPN)* is a continuous PN together with a vector $\boldsymbol{\lambda} \in \mathbb{R}^{|T|}_{>0}$.

Different semantics have been defined for continuous timed transitions, the two most important being *infinite server* or *variable speed*, and *finite server* or *constant speed* (see Alla and David [1998], Silva and Recalde [2002]). Here infinite server semantics will be considered. Like in purely markovian discrete net models, under infinite server semantics, the flow through a timed transition t is the product of the speed, $\boldsymbol{\lambda}[t]$, and $\mathrm{enab}(t, \mathbf{m})$, the instantaneous enabling of the transition, i.e., $\mathbf{f}(\mathbf{m})[t] = \boldsymbol{\lambda}[t] \cdot \mathrm{enab}(t, \mathbf{m}) = \boldsymbol{\lambda}[t] \cdot \min_{p \in {}^\bullet t}\{\mathbf{m}[p]/\mathbf{Pre}[p, t]\}$. For the flow to be well defined, every transition must have at least one input place, hence in the following we will assume $\forall t \in T, |{}^\bullet t| \geq 1$. The "min" in the definition leads to the concept of *configurations*: a configuration assigns to each transition one place that for some markings will control its firing rate. An upper bound for the number of configurations is $\prod_{t \in T} |{}^{\bullet}t|$. The reachability space can be divided into *marking regions*, denoted by $\Re$, according to the configurations. These regions are polyhedrons, and are disjoint, except on the borders.

The flow through the transitions can be written in a vectorial form as $\mathbf{f}(\mathbf{m}) = \mathbf{\Lambda\Pi}(\mathbf{m})\mathbf{m}$, where $\mathbf{\Lambda}$ is a diagonal matrix whose elements are those of $\boldsymbol{\lambda}$, and $\mathbf{\Pi}(\mathbf{m})$ is the configuration operator matrix at $\mathbf{m}$, which is defined s.t. the i-th entry of the vector $\mathbf{\Pi}(\mathbf{m})\mathbf{m}$ is equal to the enabling degree of transition t_i (see Mahulea et al. [2008]). For example, in the net of Fig. 2(b) the marking $\mathbf{m} = [\,1\ 1\ 0\,]^T$ defines a configuration at which p_3 is controlling the firing rate of t_3, so, the transitions flow vector is

$$\mathbf{f}(\mathbf{m}) = \mathbf{\Lambda\Pi}(\mathbf{m})\mathbf{m} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ 0 \end{bmatrix}$$

The dynamical behavior of a $TCPN$ system is described by its state equation: $\dot{\mathbf{m}} = \mathbf{C\Lambda\Pi}(\mathbf{m})\mathbf{m}$. Inside each region, the state equation is linear ($\mathbf{\Pi}(\mathbf{m})$ is constant).

3. DEADLOCK-FREENESS IN CPN SYSTEMS

Regarding autonomous (i.e. unforced) continuous net systems, it has been proved in Recalde et al. [2007] that deadlock-freeness is decidable. However, timing can change the deadlock-freeness properties of these continuous net systems, although just in one sense.

If a system reaches a deadlock as timed, it also deadlocks as untimed. This result was stated in Júlvez et al. [2006] for a subclass of nets, but it is clearly a general one, because the evolution of the timed system just gives a particular trajectory, i.e., a firing sequence, that can be fired also in the untimed system (it is immediate that this property holds also for discrete net systems).

Proposition 1. If the CPN system $\langle \mathcal{N}, \mathbf{m_0} \rangle$ is deadlock-free, then for any $\boldsymbol{\lambda} > \mathbf{0}$, $\langle \mathcal{N}, \boldsymbol{\lambda}, \mathbf{m_0} \rangle$ is timed-deadlock-free.

The reverse is not true, as already shown (Fig. 2(a)). It may seem that the set of rates for which this kind of things occur has to be of null measure (i.e., a smaller dimension manifold), but it is not so. For example, the CPN system in Fig. 3(a) deadlocks as autonomous, but is deadlock-free as timed if $\lambda_3 > \lambda_1$. Let us prove it by showing that under that timing the siphon defined by the deadlock, $\{p_4, p_5, p_6\}$, will never empty. The deadlock belongs to a configuration in which $m_6 \leq m_3$. However, inside this configuration, the marking of the siphon is always increasing, since $m_4 + m_5 + m_6 = m_{0,4} + m_{0,5} + m_{0,6} + \int(f_3 - f_1)d\tau$, and $\int(f_3 - f_1)d\tau = \int(\lambda_3 - \lambda_1) \cdot m_6 \cdot d\tau > 0$. Clearly, the siphon never empties either if $\lambda_3 = \lambda_1$, and so the system does not deadlock, unless the initial marking was a deadlock already. If $\lambda_3 < \lambda_1$ and the system enters in this configuration, the siphon will empty and a deadlock will be reached. Moreover, the deadlock does not occur if the initial marking of p_1 is 3 instead of 5. That is, deadlock freeness is non monotonic with respect to marking: increasing the number of resources ($m_{0,1} > 3$) can kill the system!

But there may exist cases also for which no set of rates makes the system deadlock-free. For example, the net system in Fig. 2(b). This net system has two minimal siphons $\{p_1, p_2\}$ and $\{p_1, p_3\}$. Their markings are non-increasing for any set of rates ($\dot{m}_1 + \dot{m}_2 = -\lambda_2 \cdot m_1$, and $\dot{m}_1 + \dot{m}_3 = -\lambda_1 \cdot m_1$). For any λ_1 and λ_2, in infinite time one of the siphons will empty.

4. DEADLOCKS AND EQUILIBRIUM MARKINGS

Steady states in $TCPN$ systems are equilibrium markings, i.e. solutions of $\dot{\mathbf{m}} = \mathbf{C\Lambda\Pi}(\mathbf{m})\mathbf{m} = 0$. Clearly, deadlock markings are equilibrium markings in which the transitions flow is null. In this section, the existence of non-deadlock equilibrium markings, related to a deadlock configuration (a configuration associated to a deadlock marking), is studied.

Deadlock markings can appear in different regions. In general, when more than one deadlock appear, they can be isolated or connected in the reachability space, in last case there exist infinite deadlocks. However, inside a region (in which the system is linear), deadlocks are always connected, i.e. there exists a unique deadlock or infinite and connected deadlocks.

In Mahulea et al. [2008], it was pointed out that, if for a given configuration operator matrix $\mathbf{\Pi_i}$ there exists η s.t.

$$\begin{bmatrix} \mathbf{\Pi_i} \\ \mathbf{B_y}^T \end{bmatrix} \eta = 0 \tag{1}$$

and its associated region $\Re_i$ includes an equilibrium marking (in particular a deadlock), then $\Re_i$ has infinite equilibrium markings with the same flow (infinite deadlocks). Notice that such vector η is an eigenvector of $\mathbf{C\Lambda\Pi_i}$ related to a zero-valued eigenvalue.

In general, eigenvectors η related to a zero eigenvalue of $\mathbf{C\Lambda\Pi_i}$ can be interpreted as the difference between a pair of equilibrium markings, even when both markings have different flows. Also, notice that, by definition, $\mathbf{\Lambda\Pi_i}\eta$ is either $\mathbf{0}$ or a T-flow of $\mathbf{C}$.

In the sequel, we call *fixed eigenvalues* of $\mathbf{C\Lambda\Pi_i}$ those that do not depend on $\boldsymbol{\lambda}$, i.e. they are timing independent (constant). In the same way, other eigenvalues are called *variable*.

Proposition 2. Let $\langle \mathcal{N}, \boldsymbol{\lambda}, \mathbf{m_0} \rangle$ be a $TCPN$ system. Let $\mathbf{\Pi_D}$ and $\Re_D$ be a deadlock configuration operator matrix and its associated region, respectively.

1) If all the eigenvalues of $\mathbf{C\Lambda\Pi_D}$, non associated to P-flows, are not null, then $\mathbf{C\Lambda\Pi_D}\eta = \mathbf{0}$ iff $\mathbf{\Pi_D}\eta = 0$. As a consequence, all equilibrium markings in $\Re_D$ are deadlocks.

2) If there exists an eigenvector η, associated to a variable zero valued eigenvalue, s.t. $\Lambda\mathbf{\Pi_D}\eta$ is a T-semiflow, $dim(\Re_D) = rank(\mathbf{C})$, and there exists a deadlock $\mathbf{m_D} \in \Re_D$ with only one associated configuration, then there exist infinite non deadlock equilibrium markings in $\Re_D$.

Proof. Since $\mathbf{\Pi_D}$ is related to a deadlock configuration, then there exists a similarity transformation $\left[\, \mathbf{Z^T}\ \mathbf{B_y} \,\right]^T$, where $\mathbf{Z}$ is a matrix built with all the elementary rows related to the places that are constraining some transition at $\mathbf{\Pi_D}$ ($\mathbf{Z}$ exists because in a deadlock configuration there always exists a place in the support of each P-flow that is not constraining any transition). Denoting by $[\,\mathbf{A}\ \mathbf{B}\,]$ the inverse transformation, the transformed system is described by

$$\begin{bmatrix} \mathbf{Z} \\ \mathbf{B_y^T} \end{bmatrix} \mathbf{C}\Lambda\mathbf{\Pi_D}\,[\,\mathbf{A}\ \mathbf{B}\,] = \begin{bmatrix} \mathbf{Z}\mathbf{C}\Lambda\mathbf{\Pi_D}\mathbf{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$$

since $\mathbf{B_y^T}\mathbf{C} = \mathbf{0}$ and $\mathbf{\Pi_D}\mathbf{B} = \mathbf{0}$ (because $\mathbf{ZB} = \mathbf{0}$ and, by definition of $\mathbf{Z}$, the rows of $\mathbf{\Pi_D}$ are scaled rows of $\mathbf{Z}$). Since this matrix is diagonal (by blocks) and a similarity transformation does not change the eigenvalues, then the eigenvalues of $\mathbf{Z}\mathbf{C}\Lambda\mathbf{\Pi_D}\mathbf{A}$ are those of $\mathbf{C}\Lambda\mathbf{\Pi_D}$ non associated to P-flows.

Statement 1). By hypothesis, all the eigenvalues, non associated to the P-flows, are not null. Then, all the eigenvalues of $\mathbf{Z}\mathbf{C}\Lambda\mathbf{\Pi_D}\mathbf{A}$ are not null, which implies that it has full rank. So, every vector in the right annuler of $\mathbf{C}\Lambda\mathbf{\Pi_D}$ is in the form $\eta = \mathbf{Ba}$, for some vector $\mathbf{a}$. However, since $\mathbf{\Pi_D}\mathbf{B} = \mathbf{0}$ then $\mathbf{\Pi_D}\eta = \mathbf{0}$ (i.e. $\forall \eta$ s.t. $\mathbf{C}\Lambda\mathbf{\Pi_D}\eta = \mathbf{0}$, $\Lambda\mathbf{\Pi_D}\eta = \mathbf{0}$). Finally, since every equilibrium marking $\mathbf{m}_1 \in \Re_D$ must satisfy $\mathbf{C}\mathbf{\Lambda}\mathbf{\Pi_D}\mathbf{m}_1 = \mathbf{0}$, then $\mathbf{m}_1$ is in the right annuler of $\mathbf{\Lambda}\mathbf{\Pi_D}$, i.e. $\mathbf{\Lambda}\mathbf{\Pi_D}\mathbf{m}_1 = \mathbf{0}$, so, every equilibrium marking is a deadlock.

Statement 2). By hypothesis, η fulfills that $\mathbf{C}\Lambda\mathbf{\Pi_D}\eta = \mathbf{0}$ since $\Lambda\mathbf{\Pi_D}\eta$ is a T-semiflow. Now, let us define $\eta' = \mathbf{AZ}\eta$ ($\eta' \neq \mathbf{0}$ since η is associated to a variable eigenvalue). Notice that $\Lambda\mathbf{\Pi_D}\eta = \Lambda\mathbf{\Pi_D}\,[\,\mathbf{A}\ \mathbf{B}\,]\,[\,\mathbf{Z^T}\ \mathbf{B_y}\,]^T\,\eta$, but $\mathbf{\Pi_D}\mathbf{B} = \mathbf{0}$, so $\Lambda\mathbf{\Pi_D}\eta = \Lambda\mathbf{\Pi_D}\,[\,\mathbf{A}\ \mathbf{0}\,]\,[\,\mathbf{Z^T}\ \mathbf{B_y}\,]^T\,\eta = \Lambda\mathbf{\Pi_D}\mathbf{AZ}\eta = \Lambda\mathbf{\Pi_D}\eta'$, so, $\Lambda\mathbf{\Pi_D}\eta'$ is also a T-semiflow. Besides, $\mathbf{C}\Lambda\mathbf{\Pi_D}\eta = \mathbf{C}\Lambda\mathbf{\Pi_D}\eta' = \mathbf{0}$ and, since $\mathbf{B_y^T}\mathbf{A} = \mathbf{0}$ (by definition of $\mathbf{A}$), then $\mathbf{B_y^T}\eta' = \mathbf{B_y^T}\mathbf{AZ}\eta = \mathbf{0}$.

Now, consider a vector $\mathbf{m}_1 = \mathbf{m_D} + \eta'\alpha$, notice that it is nonnegative for a small enough $\alpha \geq 0$ ($\forall p_j$ s.t. $\mathbf{m_D}(p_j) = 0$ it fulfills that $\eta'_j \geq 0$, because p_j is constraining a transition and $\Lambda\mathbf{\Pi_D}\eta'$ is a T-semiflow). Besides, since $\mathbf{B_y}^T\mathbf{m}_1 = \mathbf{B_y}^T\mathbf{m_D}$, $dim(\Re_D) = rank(\mathbf{C})$ and $\mathbf{m_D}$ is related to only one configuration, there always exists a small enough $\alpha \geq 0$ s.t. $\mathbf{m}_1 \in \Re_D$. Moreover, $\mathbf{C}\mathbf{\Lambda}\mathbf{\Pi_D}\mathbf{m}_1 = \mathbf{0}$ and $\mathbf{\Lambda}\mathbf{\Pi_D}\eta' \neq 0$ (which implies that $\mathbf{\Lambda}\mathbf{\Pi_D}\mathbf{m}_1 \neq \mathbf{\Lambda}\mathbf{\Pi_D}\mathbf{m_D} = \mathbf{0}$), i.e. $\mathbf{m}_1$ is a non deadlock equilibrium marking. Finally, by linearity, every marking in the convex described by $\mathbf{m}_1$ and $\mathbf{m_D}$ is also a non deadlock equilibrium marking.

5. REACHABILITY AND STABILITY OF DEADLOCK MARKINGS

In this section, deadlock-freeness of $TCPN$ systems is algebraically analyzed. In particular, the stability of a deadlock marking, seen as an equilibrium marking, is studied through the knowledge of the values of the poles of the linear subsystems at which it belongs. For a detailed introduction to stability concepts see Khalil [2002].

It is known that P-flows are related to zero valued poles that do not depend on the timing $\boldsymbol{\lambda}$ (Mahulea et al. [2008]). From an algebraic perspective, it means that the matrix $\mathbf{C}\mathbf{\Lambda}\mathbf{\Pi_i}$ has a *fixed* zero valued eigenvalue and its corresponding row and column eigenvectors (i.e. $\exists$ $\mathbf{y}$ and η s.t $\mathbf{y}\mathbf{C}\mathbf{\Lambda}\mathbf{\Pi_D} = \mathbf{0}$ and $\mathbf{C}\mathbf{\Lambda}\mathbf{\Pi_D}\eta = \mathbf{0}$). In this way, P-flows are row eigenvectors associated to the zero eigenvalue (but not all eigenvectors are P-flows) , and every related column eigenvector η fulfills that $\mathbf{\Lambda}\mathbf{\Pi_D}\eta$ is a T-flow.

As for eigenvalues, let us distinguish between *fixed* (λ-timing independent) and *variable* poles.

Not all fixed poles are related to P-flows. According to the Sylvester's inequality (Chen [1984]), for any $\mathbf{\Pi_i}$ and $\mathbf{\Lambda}$ $rank(\mathbf{C}\mathbf{\Lambda}\mathbf{\Pi_i}) \leq min(rank(\mathbf{C}), rank(\mathbf{\Pi_i}))$. So, whenever $rank(\mathbf{\Pi_i}) < rank(\mathbf{C})$ the dimension of the right annuller of $\mathbf{C}\mathbf{\Lambda}\mathbf{\Pi_i}$ is bigger than the dimension of P-flows. Thus there exist others zero valued poles, and since they are independent of $\mathbf{\Lambda}$, then they are fixed. Notice that these zero valued poles appear for a particular configuration, but not for all, as in case of those associated to P-flows. Moreover, whenever this new kind of pole appears, it is possible to find an eigenvector η that fulfills equation 1.

Some nets allow to reduce the rank of $\mathbf{C}\mathbf{\Lambda}\mathbf{\Pi_i}$ by choosing appropriately $\mathbf{\Lambda}$. It means that it is possible to add other zero valued poles and other marking conservation laws, which are not P-flows (i.e. $\exists\mathbf{y}$ s.t. $\mathbf{y}\dot{\mathbf{m}} = \mathbf{y}\mathbf{C}\mathbf{\Lambda}\mathbf{\Pi_D} = \mathbf{0}$ but $\mathbf{yC} \neq \mathbf{0}$). So, if this occurs for a given deadlock configuration matrix $\mathbf{\Pi_D}$, and the corresponding marking conservation law ensures that the places that are constraining the transitions at $\mathbf{\Pi_D}$ never become empty, then the deadlock markings, related to $\mathbf{\Pi_D}$, can be avoided. The sense of this idea is introduced in the following proposition.

Proposition 3. Let $\langle \mathcal{N}, \boldsymbol{\lambda}, \mathbf{m_0} \rangle$ be a $TCPN$ system. Consider that $\mathbf{m_0} > 0$ belongs to a deadlock region $\Re_D$, and let $\mathbf{\Pi_D}$ be its associated configuration operator matrix.

If there exists an eigenvector η, associated to a variable zero valued eigenvalue of $\mathbf{C}\mathbf{\Lambda}\mathbf{\Pi_D}$, s.t. $\Lambda\mathbf{\Pi_D}\eta$ is a T-semiflow, then deadlock markings in $\Re_D$ are not reachable through a trajectory in $\Re_D$.

Proof. Without loss of generality, suppose that η also fulfills that $\mathbf{B_y^T}\eta = \mathbf{0}$ (in the proof of Prop. 2 it is shown that, if the hypothesis of this proposition is fulfilled then there always exists an eigenvector η' s.t. $\mathbf{C}\Lambda\mathbf{\Pi_D}\eta' = \mathbf{0}$, $\mathbf{B_y^T}\eta' = \mathbf{0}$ and $\Lambda\mathbf{\Pi_D}\eta'$ is a T-semiflow).

Now, every marking $\mathbf{m}_1$ reachable from $\mathbf{m_0} > 0$, through a trajectory inside $\Re_D$, must fulfill the solution of the state equation (see Chen [1984]): $\mathbf{m}_1 = e^{\mathbf{C}\mathbf{\Lambda}\mathbf{\Pi_D}\tau}\mathbf{m_0}$, for some time τ. In this way, considering an initial marking $\mathbf{m_0}' = \mathbf{m_0} + \eta\alpha$, (for some suitable scalar α), the marking reachable at time τ is given by: $\mathbf{m}'_1 = e^{\mathbf{C}\mathbf{\Lambda}\mathbf{\Pi_D}\tau}\mathbf{m_0}'$. So, substituting $\mathbf{m_0}'$ and considering that $e^{\mathbf{C}\mathbf{\Lambda}\mathbf{\Pi_D}\tau}\eta = \eta$ (this equality is easy to see by expanding the exponential matrix in Taylor's series), it follows that $\mathbf{m}'_1 = \mathbf{m}_1 + \eta\alpha$.

Now, following a contradiction reasoning, suppose that for a given positive initial marking $\mathbf{m_0} \in \Re_D$ the system converges asymptotically towards a deadlock $\mathbf{m_D} \in \Re_D$. Then, according to previous equation, for a positive initial marking $\mathbf{m_0}' = \mathbf{m_0} + \eta\alpha$, in which $\alpha < 0$, the system converges asymptotically towards $\mathbf{m_D}' = \mathbf{m_D} + \eta\alpha$, but, since the entries of $\eta\alpha$ related to the places that constraint a transition in the support of the T-semiflow $\mathbf{\Lambda}\mathbf{\Pi_D}\eta$ are negative and the corresponding entries of $\mathbf{m_D}$ are zero (close enough to zero), then $\mathbf{m_D}'$ has negative entries, which is a contradiction. Therefore, $\mathbf{m_D}$ is not reachable, through a trajectory in $\Re_D$, from any positive marking $\mathbf{m_0} \in \Re_D$.

From a control theory perspective, deadlocks are equilibrium points, so, the knowledge of the value of the poles (in each deadlock configuration) is useful to decide if a deadlock will be reached or will not. This idea is captured in the following propositions:

Proposition 4. Let $\langle \mathcal{N}, \boldsymbol{\lambda}, \mathbf{m_0} \rangle$ be a $TCPN$ system. Given a deadlock $\mathbf{m_D}$ in only one associated region $\Re_D$, then

1) If the real part of the eigenvalues of $\mathbf{C\Lambda\Pi_D}$, non associated to the P-flows, are negative, then $\mathbf{m_D}$ is locally asymptotically stable, i.e. there exists a neighborhood of $\mathbf{m_D}$, named $N(\mathbf{m_D})$, s.t. if $\mathbf{m_0} \in N(\mathbf{m_D})$ then the system *inevitably* reach $\mathbf{m_D}$.

2) If $\mathbf{C\Lambda\Pi_D}$ has a zero valued eigenvalue, non associated to the P-flows, and the real part of the others are negative, then $\mathbf{m_D}$ is stable. ($\mathbf{m_D}$ could be reached or not.)

3) If there exists a variable zero eigenvalue of $\mathbf{C\Lambda\Pi_D}$, with an associated eigenvector η s.t. $\Lambda\mathbf{\Pi_D}\eta$ is a T-semiflow, then $\mathbf{m_D}$ is not reachable from a positive marking $\mathbf{m_0} \in \Re_D$, through a trajectory in $\Re_D$.

4) If there exist a positive eigenvalue of $\mathbf{C\Lambda\Pi_D}$ then $\mathbf{m_D}$ is unstable, so it is not reachable from another marking, through a trajectory in $\Re_D$.

Proof. The reduced order system $\dot{\mathbf{m}}' = (\mathbf{ZC\Lambda\Pi_D A})'\mathbf{m}'$, obtained by the similarity transformation introduced in the proof of Prop. 2, describes the dynamical behavior of the original one (in $\Re_D$). Besides, in this system, only the poles of $\mathbf{C\Lambda\Pi_D}$, non associated to P-flows, are present. Now, if the condition of statement 1) is fulfilled, then, according to Prop. 2, all equilibrium markings in $\Re_D$ are deadlocks. So, statements 1), 2) and 4) are immediate from the stability concepts of control theory (Khalil [2002]). Statement 3) is immediate from Prop. 3.

The stability analysis of a deadlock marking $\mathbf{m_D}$, which is related to different configurations, is more complex, since it is a stability problem of a piecewise linear system. However, it is possible to know what could happen for particular cases.

Proposition 5. Let $\langle \mathcal{N}, \boldsymbol{\lambda}, \mathbf{m_0} \rangle$ be a $TCPN$ system. Given a deadlock marking $\mathbf{m_D}$ that belongs to different regions $\Re_D^1, \ldots, \Re_D^k$, then

1) If for each region $\Re_D^i$ the eigenvalues of $\mathbf{C\Lambda\Pi_D}^i$ are real and negative, then $\mathbf{m_D}$ is locally asymptotically stable, i.e. there exists a neighborhood of $\mathbf{m_D}$, named $N(\mathbf{m_D})$, s.t. if $\mathbf{m_0} \in N(\mathbf{m_D})$ then the system inevitably reach $\mathbf{m_D}$.

2) If for all regions $\Re_D^i$, there exists an eigenvector η, associated to a variable zero eigenvalue of $\mathbf{C\Lambda\Pi_D}^i$, s.t. $\mathbf{\Lambda\Pi_D}^i\eta > 0$, then $\mathbf{m_D}$ is not reachable from $\mathbf{m_0} > 0$ through a trajectory in $\bigcup \Re_D^i$.

Proof. As in the proof of previous proposition, for every configuration $\mathcal{C}_D^i$ related to $\mathbf{m_D}$, there exists a reduced order system that describes the dynamical behavior of the system in $\Re_D^i$. If the condition of statement 1) is fulfilled, then every linear subsystem has real and negative poles. It is well known, in control theory (Khalil [2002]), that in such case the state is decreasing all time in such system. So, the marking $\mathbf{m}'$, and thus the flow, is decreasing while the system stay in $\Re_D^i$. Since that happens for every system

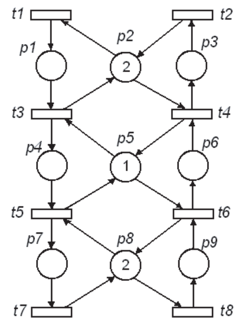

Fig. 4. $TCPN$ system with two independent T-semiflows.

associated to $\mathbf{m_D}$, then for a small enough neighborhood of $\mathbf{m_D}$, named $N(\mathbf{m_D})$, the flow is decreasing, so, the system inevitably will reach $\mathbf{m_D}$. Statement 2) derives from Prop. 3.

6. TOWARDS INTERPRETATION AT NET LEVEL

The introductory examples of section 3 show that, for some systems, deadlocks can be avoided by means of a suitable timing $\boldsymbol{\lambda}$. Proposition 3 establishes that, if $\mathbf{\Lambda}$ is s.t. a variable zero valued eigenvalue exists in $\mathbf{C\Lambda\Pi_D}$ with an associated eigenvector η s.t. $\Lambda\mathbf{\Pi_D}\eta$ is a T-semiflow, then no deadlock $\mathbf{m_D} \in \Re_D$ is reachable from any positive initial marking $\mathbf{m_0} \in \Re_D$, through a trajectory in $\Re_D$. On the other hand, if the real part of the eigenvalues of $\mathbf{C\Lambda\Pi_D}$ (non associated to P-flows) are negative for every $\mathbf{\Lambda}$, and $\mathbf{m_D}$ has only one associated configuration, then, according to Prop. 4, there exists a neighborhood $N(\mathbf{m_D})$ from which the system inevitably reach $\mathbf{m_D}$.

For instance, consider the $TCPN$ system of Fig. 2(b). This system has two possible configurations and infinite deadlocks related to each one (for both $\mathbf{\Pi_i}$, $\exists \eta \neq 0$ satisfying equation (1)). For each configuration, the linear subsystem, described by the transfer matrix $\mathbf{C\Lambda\Pi_i}$, has a fixed zero-valued pole. The real parts of the other poles, for both subsystems, are: $\text{Re}\{s_1\} = \text{Re}\{s_2\} = -\frac{1}{2}(\lambda_1 + \lambda_2 + \lambda_3)$, where λ_1, λ_2 and λ_3 are the transitions firing speeds. Since $\boldsymbol{\lambda}$ must be positive, then, there does not exist a timing that leads to a variable zero valued pole, and an asymptotically stable deadlock exists (the real part of the variable poles are always negative), i.e. this system is dead for any timing $\boldsymbol{\lambda}$. The same conclusion is easy to obtain, noting that both siphons of this system do not have generator transitions.

On the other hand, consider the $TCPN$ system of Fig. 3(a). In this case, there exists a unique deadlock $\mathbf{m_D} = [1\ 1\ 3\ 0\ 0\ 0]^T$, belonging to a unique configuration. This net has three P-semiflows, meaning three fixed zero valued poles. Here, we are interested in the possibility of finding $\boldsymbol{\lambda}$ that leads to a new zero valued pole. For that, it is sufficient to compute the lower order term of the characteristic polynomial of matrix $\mathbf{C\Lambda\Pi_D}$ in a parametric form.

Remark. Considering the characteristic polynomial of $\mathbf{C\Lambda\Pi_D}$, where $\mathbf{\Pi_D}$ is related to a deadlock configuration and $\mathbf{\Lambda}$ is in parametric form, the order of the lower order term is the number of fixed zero valued poles. Besides, a particular $\mathbf{\Lambda}$ that makes this lower order term become zero, leads to a variable zero valued poles.

For this example, the lower order term is $s^3(\lambda_4\lambda_1\lambda_2 - \lambda_4\lambda_2\lambda_3)$, it means that there exist 3 fixed zero valued poles, and that, a timing $\boldsymbol{\lambda}$ s.t $\lambda_2\lambda_4(\lambda_1 - \lambda_3) = 0$ creates an additional zero valued pole. It is easy to see that every timing $\boldsymbol{\lambda}$ in which $\lambda_1 = \lambda_3$ fulfills that condition, in which case, according to Prop. 4, $\mathbf{m_D}$ is not reachable from any $\mathbf{m_0} > 0$, thus the $TCPN$ system is deadlock-free (since this system is consistent and it has only one minimal T-semiflow, then the existence of a variable zero valued eigenvalue implies that $\exists\eta$ that fulfills the conditions of Prop. 4).

Furthermore, if $\lambda_3 > \lambda_1$, then the coefficient of this term becomes negative and it can be demonstrated, through the Routh-Hurwitz criterion (Khalil [2002]), that $\mathbf{m_D}$ is unstable (a pole becomes positive), then the system is deadlock-free (Prop. 4). Also, since this system is mono T-semiflow, deadlock-freeness implies liveness.

Now, consider the system of Fig. 4. This system has 16 configurations but only three with deadlocks:

$$\mathcal{C}_1 = \{(t_3, p_5), (t_4, p_6), (t_5, p_8), (t_6, p_5)\}$$
$$\mathcal{C}_2 = \{(t_3, p_5), (t_4, p_2), (t_5, p_8), (t_6, p_5)\}$$
$$\mathcal{C}_3 = \{(t_3, p_5), (t_4, p_2), (t_5, p_4), (t_6, p_5)\}$$

(the arcs that constraint transitions t_1, t_2, t_7 and t_8 are not written because they are the same for every configuration). Configuration $\mathcal{C}_2$ has infinite deadlocks ($\exists\eta \neq 0$ satisfying equation (1)), but all deadlocks in the system are connected. Computing the lower order terms of the characteristic polynomial for the three cases we obtain:

for $\mathcal{C}_1$ $\quad s^3\lambda_1\lambda_2\lambda_7\lambda_4(\lambda_3\lambda_8 - \lambda_5\lambda_6)$

for $\mathcal{C}_2$ $\quad s^4[\lambda_1\lambda_2\lambda_7(\lambda_3\lambda_8 - \lambda_5\lambda_6) + \lambda_2\lambda_7\lambda_8(\lambda_1\lambda_6 - \lambda_3\lambda_4)]$

for $\mathcal{C}_3$ $\quad s^3\lambda_2\lambda_7\lambda_8\lambda_5(\lambda_1\lambda_6 - \lambda_3\lambda_4)$

Notice that for any timing $\boldsymbol{\lambda}$ s.t. $\lambda_1\lambda_6 = \lambda_3\lambda_4$ and $\lambda_8\lambda_3 = \lambda_5\lambda_6$, a variable zero valued pole is added to every deadlock configuration. Even if this system has two different minimal T-semiflows, there does not exist a siphon whose output transitions correspond to the support of one of them. So, every eigenvector η associated to the new variable zero valued pole is s.t. $\mathbf{\Lambda\Pi_D}\eta$ is a linear combination of the minimal T-semiflows, i.e. a T-semiflow whose support covers all transitions. Then, according to Prop. 5, no deadlock is reachable from a positive initial marking.

Now, the system of Fig. 5 has two different minimal T-semiflows, whose supports are covered, independently, by siphons $\Sigma_1 = \{p_4, p_5, p_6\}$ and $\Sigma_2 = \{p_9, p_{10}, p_{11}\}$. If timing $\boldsymbol{\lambda}$ is s.t. $\lambda_1 = \lambda_3$, the siphon Σ_1 conserves its total marking (as in the system of Fig. 3(a)). But, if $\lambda_5 > \lambda_7$ then the siphon Σ_2 will empty, so, the system does not reach a deadlock, but it is non live. On the contrary, choosing $\boldsymbol{\lambda}$ s.t. $\lambda_1 = \lambda_3$ and $\lambda_5 = \lambda_7$, both siphons remain marked, for all time, i.e. the timed system is live.

Notice that, in such case, we are introducing two linearly independent right eigenvectors of $\mathbf{C\Lambda\Pi_D}$, i.e. we are adding two variable zero-valued poles.

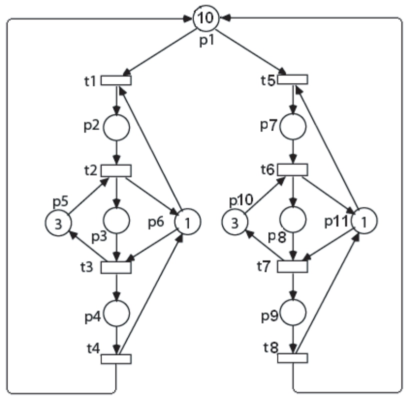

Fig. 5. $TCPN$ system with two independent siphons.

7. CONCLUSIONS

Starting with the idea that a deadlock leads to an empty siphon, we motivate the rest of the work by showing in an easy way that values for the firing rates may exist in order to make deadlock-free the timed continuous model under infinite servers semantics. After that, we algebraically study equilibrium markings (i.e. potentially steady states), because deadlocks are so. Looking for the stability of those equilibrium markings, we can prove conditions for transforming the system into deadlock-free one (checking if those markings are unstable). Finally, closing the loop, we came back on the net interpretations of the results partially proven in the framework of control theory.

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