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Some necessary and sufficient conditions of stability on the approximation of a distributed delay control law

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Abstract: Since the end of the seventies, several authors have proposed to use a distributed delay in the control law for poles-assignment of time delay systems, i. e. a finite integral over the past values of the state or the command of the system. However, since ten years publications have showed that the implementation of such control laws, by the means of an approximation, is not self-evident. Back to this topic, this paper is concerned with the conditions of stability of the controlled system after approximation of the distributed delay and, limited to the first-order SISO case, constitute the first step of a larger study. So, the proof of the necessity and the sufficiency of a set of conditions, some of them well known, is given. It is followed by a criterion of stabilizability of the primary system based on its only parameters.

Keywords: Time-delay, Distributed control, Stability analysis, Stabilizability, stability criterion.

1. INTRODUCTION

Time delays arise naturally in numerous control applications, and control of time delay systems is a challenging task, as it is shown by the extensive literature on the subject (see Niculescu (2001), Gu et al. (2003), Niculescu and Gu (2004) and the references therein). The presence of time delays imposes strict limitations on achievable performance and may considerably complicate controller design. In particular, for unstable time delay system, the use of distributed delays (i.e. finite integrals over the time, often called finite-impulse-response FIR blocks as well) is often a necessity as in the finite spectrum assignment problem (see Olbrot (1978), Manitius and Olbrot (1979),Watanabe (1986)).

Also, let us concern with the linear system with delay in control:

$$\begin{cases} \dot{x}(t) = ax(t) + bu(t-h) \ , \ t > 0 \\ y(t) = cx(t) \end{cases} \tag{1}$$

with $a \in \mathbb{R}^+$, $b \in \mathbb{R}$, $c \in \mathbb{R}$, and $u(t) \in \mathbb{R}$ defined for $t > -h$.

As suggested by Manitius and Olbrot (1979), the control law of the form:

$$u(t) = fx(t) + \int_0^h g(\tau)u(t-\tau)d\tau + v(t) \ , \tag{2}$$

where v is a command signal, and with the appropriate real f and real function $g(\tau)$ whose support is $[0, h]$, allows to stabilize the system and especially freely assign its pole

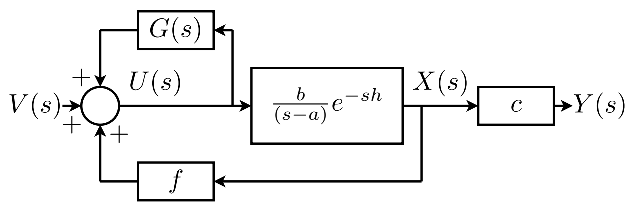

Fig. 1. Blocks diagrams of the closed-loop system

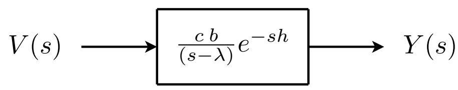

Fig. 2. Blocks diagrams of the equivalent system

to a value $\lambda < 0$: a prediction of the state variable over one delay interval is generated and then a feedback of the predicted state is applied, thereby compensating the effect of the time-delay (Fig. 1), (Fig. 2).

Of course, this type of control law cannot be implemented as it is, one has to approximate the integral term using some numerical quadrature method. And different publications, not so old, have shown that replacing the distributed delay by a finite sum of discrete delays is unsafe in terms of stability (see Van Assche et al. (1999), Santos and Mondié (2000), Mondié et al. (2001)).

Since, this topic has drawn a lot of attention from the delay community, see (Van Assche et al. (2001), Engelborghs et al. (2001), Mondié and Michiels (2003)). The analysis of the causes of such behavior was studied in Santos and Mondié (2000), Engelborghs et al. (2001), and in the survey paper (Richard (2003)) using a simple example. Another study (Mirkin (2004)) has pointed out an important property: the approximation problems reported in some recent research studies are caused by a combination of poor approximation accuracy in the high-frequency range and excessive sensitivity of the design method to high-frequency additive plant uncertainties. Some remedies have also been proposed such that the idea of adding low pass elements to overcome the problem of sensitivity (Mirkin (2004), Mondié and Michiels (2003)) or strategies of global approximation of the control law (Zhong (2004), Zhong (2005)).

Our attention here focuses on the result of the approximation of the distributed delay by a sum of commensurate delays in the scalar case, independently of the method used to get it. Thus, we study stability conditions of transfers of the type

$$\frac{c\,b\,e^{-sh}}{(s-a)\left[1-\sum_{k=0}^{N}\alpha_k e^{-ksh/N}\right]-bfe^{-sh}}, \qquad (3)$$

with $\alpha_k \in \mathbb{R}, \forall\ k$, substitute to the original feedback system transfer: $\dfrac{c.b.e^{-sh}}{(s-\lambda)}$.

Multiplying, for convenience, numerator and denominator of the transfer (3) by $\dfrac{e^{sh}}{1-\alpha_0}$ (if $\alpha_0 \neq 1$ or else by $\dfrac{e^{sh}}{\alpha_{k_0}}$ where α_{k_0} is the first non-zero coefficient of the sum), the object of our study appears clearly as being the class of the quasi-polynomials of the form:

$$D(s) = (s-a)P_N(e^{sh/N}) - \frac{bf}{1-\alpha_0}\ , \qquad (4)$$

where $P_N(e^{sh/N})$ is a monic polynomial of degree at most N with real coefficients in $e^{sh/N}$.

Also, in the first part, we are interested in the conditions which guarantee that the quasi-polynomial (4) has only stable zeros. Then, is derived from the conditions which are underlined an instability criterion based on the parameters of the initial system (1), independently of the approximation of the distributed delay as far as this is under the shape of a sum of commensurate delays.

2. STABILITY CONDITIONS

In this first part, the point is to bring forward a set of conditions, altogether necessary and sufficient, about stable zeros of the denominator:

$$Z_1(s) = (s-a)P_N(e^{sh/N}) - k_1\ , \qquad (5)$$

where a is a non-negative real, h a positive real, P_N a monic polynomial with real coefficients of degree at most N, and k_1 a real number.

For this, let us remind the well-known theorem due to L. S. Pontryagin (see Pontryagin (1955, 1958) or Hale and Verduyn Lunel (1993)) on the zeros of quasi-polynomials.

Theorem 1. (Pontryagin's Theorem). Let $Z(s) = P(s, e^s)$ and suppose $P(s,t)$ is a polynomial with principal term $a_{mn}s^m t^n$. All of the zeros of $Z(s)$ have negative real parts if and only if:

(1) The complex vector $Z(iw)$ rotates in the positive direction with a positive velocity for w ranging in $(-\infty, \infty)$.
(2) For $w \in [-2l\pi, 2l\pi]$, $l \geq 0$ an integer, there is an $\epsilon_l \to 0$ as $l \to \infty$ such that $Z(iw)$ subtends an angle $4l\pi n + \pi m + \epsilon_l$.

Based on this last, the first result can be apprehended as a transcription of it, taking into account the specificities of $Z_1(s)$.

So, let us introduce some notations: for any ω real, we denote by

- $\theta_0(\omega)$ the argument of $P_N(e^{i\omega h/N})$ and $\dot{\theta}_0(\omega)$ its derivative,
- $\theta_1(\omega)$ the argument of $Z_1(i\omega)$ and $\dot{\theta}_1(\omega)$ its derivative,
- $\delta_1(\omega)$ the argument of $(i\omega - a)$, and $\dot{\delta}_1(\omega)$ its derivative.

And note that, for any complex number s, $\overline{Z_1(s)} = Z_1(\overline{s})$.

Theorem 2. The complex function of the complex variable s:

$$Z_1(s) = (s-a)P_N(e^{sh/N}) - k_1\ ,$$

have all its roots in the left-half complex plane if and only if the three following conditions are fulfilled.

(1) All the roots of the polynomial P_N are inside the open unit disk ;
(2) The argument of $(i\omega - a)P_N(e^{i\omega h/N})$ has a negative velocity for $w = 0$:
$$\dot{\theta}_0(0) + \dot{\delta}_1(0) < 0 \quad ;$$
(3) The complex vector $Z_1(iw)$ rotates in the positive direction with a positive velocity for w ranging in $(-\infty, \infty)$:
$$\dot{\theta}_1(\omega) > 0\ , \quad \forall\ \omega \text{ real}.$$

2.1 Proof of the necessity of the conditions

First, one will agree that the necessity of the third condition, which has been copied out of the Pontryagin's theorem, follows directly from this last since the quasi-polynomial $Z_1(s)$ presents, by construction, a principal term.

Next, let us remind from Cauchy´s theory (see Rudin (1966)) that:

- the number of turns (positively weighed in the counterclockwise direction) that makes a closed curve $\mathcal{C}_1$ around a point z_0 of the complex plane less $\mathcal{C}_1$ can be expressed by the Index function $\mathrm{Ind}_{\mathcal{C}_1}$:
$$\mathrm{Ind}_{\mathcal{C}_1}(z_0) = \frac{1}{2i\pi}\int_{\mathcal{C}_1}\frac{dz}{z-z_0}\ ;$$
- for any holomorphic function $H(s)$ in the complex plane, and for any complex number k, the number of zeros of $H(s)-k$ enclosed by an one-loop-closed-curve $\mathcal{C}_2$ (not passing through the zeros) taken in the counterclockwise direction is equal to:

$$\frac{1}{2i\pi}\int_{\mathcal{C}_2}\frac{H'(s)}{H(s)-k}\,ds\ ,$$

what can be written:

$$\frac{1}{2i\pi}\int_{H(\mathcal{C}_2)}\frac{dz}{z-k}\ ,$$

and, therefore, corresponds to the number of turns of $H(\mathcal{C}_2)$ around the complex number k: $\mathrm{Ind}_{H(\mathcal{C}_2)}(k)$.

Now, suppose that all the zeros $\{\beta_i\}_i$ of the complex function $Z_1(s)$, which is quite obviously holomorphic in the complex plane, have negative real parts. We mean that

$$\sup_i\{\Re(\beta_i)\} < 0\ .$$

So there exists $\varepsilon \geq 0$ such that $\sup_i\{\Re(\beta_i)\} < -\varepsilon \leq 0$ and such that there is no root of P_N whose modulus is equal to $e^{-\varepsilon h/N}$. Therefore, if, for any $R > 0$, we denote by Γ the closed curve composed of the vertical segment Γ_i:

$$\Gamma_i = \{-\varepsilon + i\omega \mid -\varepsilon - R \leq \omega \leq \varepsilon + R\}$$

and of the bow of radius $\varepsilon + R$:

$$\Gamma_\theta = \{-\varepsilon + (\varepsilon + R)e^{i\theta} \mid \theta \in [-\pi/2, \pi/2]\}\ ,$$

we may state, since there is no zero of $Z_1(s)$ enclosed by Γ, that $\mathrm{Ind}_{Z_1(\Gamma)}(0) = 0$, whatever the positive value of R is and even at the limit when R tends toward the infinity.

To prove the first condition, suppose the opposite: the polynomial P_N has one or more roots outside the closed disk centered on zero and of radius $e^{\varepsilon h/N}$, roots we denote by r_j , $j = 1, \cdots, J \leq N$. Therefore, are zeros of $P_N(e^{sh/N})$ the values of s such that

$$e^{sh/N} = r_j = e^{\ln|r_j| + i(\arg(r_j) + 2k\pi)}\ ,$$

$\forall\ k \in \mathbb{Z}$, $\forall\ j = 1, \cdots, J$, that is whose expression is

$$s = \frac{N}{h}\ln|r_j| + i\frac{N}{h}(\arg(r_j) + 2k\pi)\ ,$$

$\forall\ k \in \mathbb{Z}$, $\forall\ j = 1, \cdots, J$. They have their real parts greater than $-\varepsilon$ since $|r_j| > e^{-\varepsilon h/N}$, $\forall\ j = 1, \cdots, J$, and are in infinite quantity.

Thus, one or more roots of P_N outside the closed disk centered on zero and of radius $e^{-\varepsilon h/N}$ means an infinite number of zeros of $(s - a)P_N(e^{sh/N}) = Z_1(s) + k_1$ at the right-hand side of Γ_i, or in others words, enclosed by Γ when R tends to the infinity.

However, this number, expressed by the formula

$$\frac{1}{2i\pi}\int_{\Gamma}\frac{Z_1'(s)}{Z_1(s)+k_1}\,ds = \frac{1}{2i\pi}\int_{Z_1(\Gamma)}\frac{dz}{z+k_1}\ ,$$

corresponds to the number of turns made by $Z_1(\Gamma)$ around $-k_1$: $\mathrm{Ind}_{Z_1(\Gamma)}(-k_1)$, if it is defined, that is if $Z_1(s)+k_1 \neq 0$, $\forall\ s \in \Gamma$.

So, let us concern with the curve $Z_1(\Gamma_\theta) + k_1$. For almost any infinitely large R, $|Z_1(\Gamma_\theta) + k_1|$ is infinitely large too. Indeed, writing $P_N(z) = \prod_{j=1}^{N}(z - r_j)$, the expression of the point of $Z_1(\Gamma_\theta) + k_1$ is:

$$\left[(\varepsilon + R)e^{i\theta} - a - \varepsilon\right] \times \prod_{j=1}^{N}\left(e^{(\varepsilon+R)\frac{h}{N}\cos\theta}\ e^{-\varepsilon\frac{h}{N}}\ e^{i(\varepsilon+R)\frac{h}{N}\sin\theta} - r_j\right)\ ,$$

with θ ranging from $-\pi/2$ to $\pi/2$. There, the term $e^{(\varepsilon + R)\frac{h}{N}\cos\theta}\ e^{-\varepsilon\frac{h}{N}}\ e^{i(\varepsilon + R)\frac{h}{N}\sin\theta}$ draws a pair of spirals which passes through at least one of the roots $\{r_j\}_{j=1}^{J}$ of the polynomial P_N if and only if:

$$\begin{cases} -\varepsilon\dfrac{h}{N} + (\varepsilon + R)\dfrac{h}{N}\cos\theta = \ln|r_j| \\ (\varepsilon + R)\dfrac{h}{N}\sin\theta = \arg(r_j) + 2k\pi \end{cases}$$

$$\Longleftrightarrow \begin{cases} (\varepsilon + R)\dfrac{h}{N}\cos\theta = \varepsilon\dfrac{h}{N} + \ln|r_j| \\ (\varepsilon + R)\dfrac{h}{N}\sin\theta = \arg(r_j) + 2k\pi \end{cases}$$

for some $j = 1, \cdots, J$, and some $k \in \mathbb{Z}$, and some $\theta \in [-\pi/2, \pi/2]$. So, this event only occurs if, for some $j = 1, \cdots, J$, and some $k \in \mathbb{Z}$,

$$(\varepsilon + R)^2 = (\varepsilon + \frac{N}{h}\ln|r_j|)^2 + \frac{N^2}{h^2}(\arg(r_j) + 2k\pi)^2\ ,$$

that is:

$$R = \sqrt{(\varepsilon + \frac{N}{h}\ln|r_j|)^2 + \frac{N^2}{h^2}(\arg(r_j) + 2k\pi)^2} - \varepsilon\ .$$

In consequence, for R infinitely large and out of the discrete set:

$$\left\{ \begin{array}{c} R_{j,k} = \sqrt{(\varepsilon + \frac{N}{h}\ln|r_j|)^2 + \frac{N^2}{h^2}(\arg(r_j) + 2k\pi)^2} - \varepsilon \\ j = 1, \cdots, J,\ k \in \mathbb{Z} \end{array} \right\},$$

$P_N(e^{sh/N})$, $\forall\ s \in \Gamma_\theta$, does not vanish, and the modulus of $Z_1(s)+k_1$, as the modulus of $(s-a_1)$, is infinitely large.

Now, let us concern with the second part of the curve $Z_1(\Gamma) + k_1$: $Z_1(\Gamma_i) + k_1$. From the expression of its points

$$Z_1(-\varepsilon + i\omega) + k_1 = (i\omega - (\varepsilon + a))\ P_N(e^{(-\varepsilon + i\omega)h/N})\ ,$$

with $\omega \in [-R, R]$, and since the polynomial P_N has no root on the circle centered on zero and of radius $e^{-\varepsilon h/N}$, it is clear that 0 does not belong to them.

Thus, for R infinitely large and out of the discrete set $\{R_{j,k},\ j = 1, \cdots, J,\ k \in \mathbb{Z}\}$, $Ind_{Z_1(\Gamma)}(-k_1)$ is well defined and supposed to be infinitely large. But from another point of view it is also equal to $\mathrm{Ind}_{Z_1(\Gamma)}(0)$, which is zero, plus the count of the intersections of the curve $Z_1(\Gamma)$ with the segment $]0, -k_1[$, intersection positively counted if the crossing is made with respect to 0 in the clockwise direction and negatively counted if not. Therefore, we have the following inequality:

$$|\mathrm{Ind}_{Z_1(\Gamma)}(-k_1)| \leq |\mathrm{Ind}_{Z_1(\Gamma)}(0)| + \mathrm{Card}\,\{s \in \Gamma \mid Z_1(s) \in\,]0, -k_1[\ \}\ ,$$

where $\mathrm{Card}(A)$ symbolizes the cardinal number of the set A.

Besides, it is clear that $Z_1(\Gamma_\theta)$, as $Z_1(\Gamma_\theta)+k_1$, presents an infinitely large modulus, and therefore is not involved in the intersections between $Z_1(\Gamma)$ and the segment $]0, -k_1[$.

On the other hand, according to the inequality built on the expression of $Z_1(s)$ with $s \in \Gamma_i$:

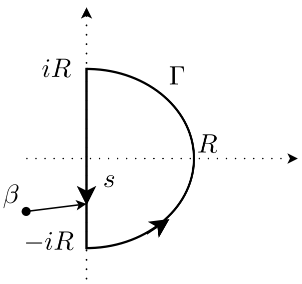

Fig. 3. plot of Γ

$$|Z_1(-\varepsilon+i\omega)| \geq |i\omega-(\varepsilon+a)|\,|P_N(e^{(-\varepsilon+i\omega)h/N})|-|k_1| \ ,$$

(and since $P_N(e^{(-\varepsilon+i\omega)h/N})$ never vanishes), we can see that, when $|\omega|$ tends to the infinity, the limit of the modulus $|Z_1(-\varepsilon+i\omega)|$ is infinite. That means there exists a real $\omega_1 > 0$ such that:

$$|Z_1(-\varepsilon+i\omega)| > |-k_1| \ , \ \forall\ \omega\ /\ |\omega| > |\omega_1| \ ,$$

hence we can deduce that $\{Z_1(-\varepsilon+i\omega)\ /\ |\omega| > |\omega_1|\}$ does not intersect the segment $]0,-k_1[$.

As a result, we can rewrite the set:

$$\{s \in \Gamma \mid Z_1(s) \in \,]0,-k_1[\ \}$$

as:

$$\{s \in [-\varepsilon-i\omega_1, -\varepsilon+i\omega_1] \mid Z_1(s) \in \,]0,-k_1[\} \ .$$

This last is part of the set:

$$\{s \in [-\varepsilon-i\omega_1, -\varepsilon+i\omega_1] \mid Z_1(s) \in \mathbb{R}\} \ ,$$

that is of the set:

$$\{s \in [-\varepsilon-i\omega_1, -\varepsilon+i\omega_1] \mid Z_1(s) - \overline{Z_1(s)} = 0\} \ ,$$

which is isomorphic to the zero set of the holomorphic function of the variable z:

$$z \longmapsto Z_1(-\varepsilon+z) - Z_1(-\varepsilon-z)$$

defined on the compact set $[-i\omega_1, i\omega_1]$:

$$\{z \in [-i\omega_1, i\omega_1] \mid Z_1(-\varepsilon+z) - Z_1(-\varepsilon-z) = 0\} \ ,$$

and whose cardinal number is obviously a finite number.

So, Card $\{s \in \Gamma \mid Z_1(s) \in \,]0,-k_1[\ \}$ is a finite number, and we may conclude that $|\mathrm{Ind}_{Z_1(\Gamma)}(-k_1)|$ is bounded from above, in contradiction with the direct consequence of the hypothesis: the polynomial P_N has one or more roots outside the closed disk centered on zero and of radius $e^{-\varepsilon h/N}$. Thus we have stated that the polynomial P_N has no root outside the open unit disk.

For the following, we can set $\varepsilon = 0$ such that Γ_i is on the imaginary axis (see Fig. 3).

To prove, now, the necessity of the second condition, let us note the symmetric character with respect to the real axis of the closed curve $Z_1(\Gamma)$ which presents so double crossings of the real axis except for $Z_1(0)$ and $Z_1(R)$. Since $\lim_{R\to\infty} Z_1(R) = +\infty$, the index of any real $k > Z_1(0)$ which does not belong to $Z_1(\Gamma)$ is odd and reciprocally (any real k whose index is odd belongs to $[Z_1(0), Z_1(R)]$). As the index of the origin is zero we deduce that $0 < Z_1(0)$ (see Fig. 4).

Now, since we proved that all the roots of P_N are inside the open unit disk, it follows that the mapping:

$$s \longmapsto (s-a)P_N(e^{sh/N}) \ ,$$

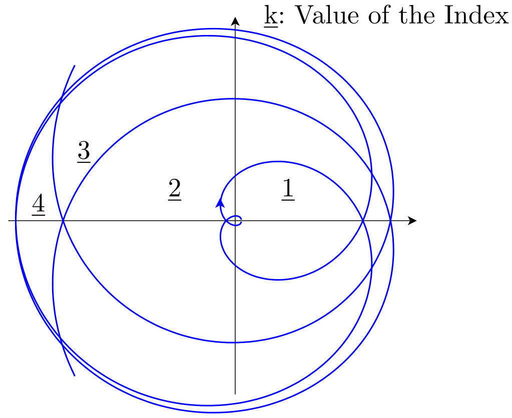

Fig. 4. Part of the Plot of $Z_1(\Gamma_i)$

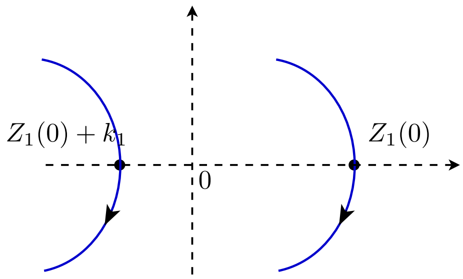

Fig. 5. Part of the Plot of $Z_1(\Gamma_i)$ and $Z_1(\Gamma_i)+k_1$

presents only one zero: a, enclosed by Γ what we can write through the formula:

$$\mathrm{Ind}_{Z_1(\Gamma)}(-k_1) = 1 \ .$$

From it we deduce that $Z_1(0) < -k_1$ or better $Z_1(0) + k_1 < 0$ (see Fig. 5).

Also, as the argument of $Z_1(i\omega)$ increases for any ω (third condition), so for $\omega = 0$, and $Z_1(0)$ is a positive real, it follows that, around $\omega = 0$, and since $Z_1(0) + k_1$ is a negative real, the argument of $Z_1(i\omega) + k_1$ decreases, what we can write:

$$\dot{\theta}_0(0) + \dot{\delta}_1(0) < 0 \ . \square$$

2.2 Proof of the sufficiency of the conditions

First, it must be clear from the expression of $Z_1(s)$ and under the first condition that the point of $Z_1(\Gamma_\theta)$:

$$Z_1(Re^{i\theta}) = (Re^{i\theta} - a) \prod_{j=1}^{N} \left(\exp(\frac{hRe^{i\theta}}{N}) - r_j \right) - k_1 \ ,$$

where $\theta \in [-\frac{\pi}{2}; \frac{\pi}{2}]$ and $\{r_j\}_j$ stand for the roots of the polynomial P_N, has an infinitely large modulus when R tends to the infinity.

So, from the third condition which means $Z_1(\Gamma_i)$ turns around the origin in the clockwise direction (when Γ_i is covered from the positive to the negative imaginary part), we can deduce that the Index function with respect to $Z_1(\Gamma)$, restricted to the real values:

$$\mathrm{Ind}_{Z_1(\Gamma)} \ : \quad \mathbb{R} - \{Z_1(\Gamma) \cap \mathbb{R}\} \quad \longrightarrow \quad \mathbb{N}$$

is a non-negative integer valued staircase function, non-increasing on $\mathbb{R}^- - \{Z_1(\Gamma) \cap \mathbb{R}^-\}$ and non-decreasing on $\mathbb{R}^+ - \{Z_1(\Gamma) \cap \mathbb{R}^+\}$.

Furthermore, since all the roots of the polynomial P_N are inside the open unit disk and generate only negative real part zeros of $(s-a_1)P_N(e^{sh/N}) = Z_1(s) + k_1$ which ergo presents a single zero in the right half-plane, we have:

$$\mathrm{Ind}_{Z_1(\Gamma)}(-k_1) = 1 \ ,$$

Also, it follows from the symmetric character with respect to the real axis of the closed curve $Z_1(\Gamma)$ and since $\lim_{R\to+\infty} Z_1(R) = +\infty$, that $Z_1(0) < -k_1$.

Moreover, from both condition 2 and condition 3, we know that $Z_1(0)+k_1$ (whose argument is $\theta_0(0)+\delta_1(0)$) and $Z_1(0)$ (whose argument is $\theta_1(0)$) are on both sides of the origin. As $Z_1(0) + k_1 < 0$, that means that $0 < Z_1(0)$.

Thus,we have

$$0 < Z_1(0) < -k_1 \ ,$$

and as an obvious consequence:

$$0 \le \mathrm{Ind}_{Z_1(\Gamma)}(0) < \mathrm{Ind}_{Z_1(\Gamma)}(-k_1) = 1 \ ,$$

such that we can but conclude that $\mathrm{Ind}_{Z_1(\Gamma)}(0) = 0$, which means that there is no zero of the complex function Z_1 enclosed by Γ, and so in the right half-plane. □

Remark 3. The third condition suggests without saying it that the complex vector $Z_1(i\omega)$, $\forall\ \omega \in (-\infty, \infty)$, never vanishes, notably for $\omega = 0$. That consequently means that, in the last part, $\mathrm{Ind}_{Z_1(\Gamma)}(0)$ is well defined and the value $k_1 = -aP(1)$ prohibited.

Remark 4. Furthermore, we will note that if the polynomial P_N possesses a main coefficient different to 1, the necessity and sufficiency of the three conditions still hold.

Remark 5. The necessity of the first condition is well known but nowhere we have found a direct and whole proof of it.

3. INSTABILITY CRITERION

Now, if we focus on the two first conditions of the *theorem* 2, we see emerging from a very simple criterion of instability:

Corollary 6. Let P_N be a real coefficients polynomial of degree N with all its roots inside the open unit disk, and let a and h be positive real numbers. The quasi-polynomial

$$(s-a)P_N(e^{sh/N}) - k \ , \qquad \forall\ k \in \mathbb{R} \ ,$$

possesses at least one root in the right half-plane if $ah \ge 2$.

In other words, the approximation of distributed delay control law (2) with a sum of commensurate delays can not stabilize the system (1) if the product of its parameters: ah is greater than 2.

To prove this last result we will lean on the following lemma:

Lemma 7. Let P_N be a real coefficients polynomial of degree N with all its roots inside the open unit disk, $\alpha(x) = \arg(P_N(e^{ix}))$ the argument function defined for any $x \in \mathbb{R}$, and $\dot\alpha(x)$ its derivative. The following inequality holds for any $x \in \mathbb{R}$:

$$\dot\alpha(x) > N/2 \ .$$

Proof of the lemma. First, let us consider the case $N = 1$, and denote by P_1 the polynomial of degree one whose root r may be supposed belonging to the interval $[0, 1[$ without any loss of generality. Denoting by $\alpha(x)$ the argument of $P_1(e^{ix})$, we have $\dot\alpha(x) = \dfrac{d}{dx}\arg(e^{ix} - r)$ whose expression can be obtained from the tangent derivative formula: $\dfrac{d}{dx}\tan(\alpha(x)) = (1 + \tan^2(\alpha(x)))\dot\alpha(x)$. Thus we get

$$\begin{aligned}\dot\alpha(x) &= \frac{\cos(x)(\cos(x) - r) + \sin^2(x)}{(\cos(x) - r)^2 + \sin^2(x)} \\ &= 1 + \frac{r(\cos(x) - r)}{(\cos(x) - r)^2 + \sin^2(x)} \ .\end{aligned}$$

where the denominator, which is $|e^{ix} - r|^2$, never vanishes. So, the optimums of the continuous and differentiable periodic function $\dot\alpha(x)$ are characterized by the equation

$$\ddot\alpha(x) = \frac{r(r^2 - 1)\sin(x)}{(1 - 2r\cos(x) + r^2)^2} = 0 \ ,$$

whose solutions are $x = 0 \pmod{\pi}$. Thus, the optimum values are

$$\dot\alpha(0) = \frac{1}{1-r} \ , \qquad \text{and} \ \ \dot\alpha(\pi) = \frac{1}{1+r} \ .$$

And it follows, taking into account the possible values of r, that $\dfrac{1}{2} < \dot\alpha(x) < \infty$.

At last, in the general case, considering P_N as the product of N polynomials of degree one, we have

$$\frac{d}{dx}\arg(P_N(e^{ix})) = \sum_{j=1}^{N} \frac{d}{dx}\arg(P_{1j}(e^{ix})) > N\frac{1}{2} \ . \square$$

Proof of the corollary. As a consequence of the previous lemma we know that the derivative of the argument function $\theta_0(\omega) = \arg(P_N(e^{i\omega h/N}))$ is bounded from above:

$$\dot\theta_0(\omega) = \dot\alpha(\omega h/N)\frac{h}{N} > \frac{N}{2}\frac{h}{N} = \frac{h}{2} \ .$$

In the same time we deduce the expression of the derivative of $\delta_1(\omega) = \arg(i\omega - a)$ from the tangent derivative formula:

$$\frac{d}{d\omega}\left(\frac{\omega}{-a}\right) = \frac{1}{1 + \frac{\omega^2}{a^2}}\dot\delta_1(\omega) \ .$$

We get $\dot\delta_1(\omega) = \dfrac{1}{-a}\dfrac{a^2 + \omega^2}{a^2}$. So, for $\omega = 0$, we have

$$\dot\delta_1(0) + \dot\theta_0(0) = \frac{1}{-a} + \dot\theta_0(0) > \frac{1}{-a} + \frac{h}{2} \ ,$$

which is greater than 0 as soon as $ah > 2$.

Thus, if $ah > 2$, $\dot\delta_1(0) + \dot\theta_0(0) > 0$, that is the second condition of the *theorem* 2 is not fulfilled, hence the conclusion. □

This result shows, if it is still necessary (it is not), that the approximation of the distributed delay (2) needs in general a more elaborated strategy, than a classical numerical quadrature, as the strategies proposed by Mondié and Michiels (2003) or Zhong (2004).

4. CONCLUSION

Completely dedicated to the scalar case, this work constitutes the first stage of a general study of the stability of the

Single Delayed Input - Single Output system after approximation by linear combination of commensurate delays of the distributed delay control law proposed by Manitius and Olbrot (1979). Indeed, the instability problems mentioned in the introduction do not concern every system in the same manner and depend on the parameters of the system as on the strategy of the approximation which provides the linear combination of commensurate delays. As *Theorem 2* and *Corollary 6*, tools can be brought out to develop some kind of classification and, in the same time, allow to build specific but simple strategies of approximation delivering satisfactory results when it is possible.

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