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Global Stability Analysis of Primal Internet Congestion Control Schemes with Heterogeneous Delays

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Abstract: The aim of network congestion control for the Internet is to allocate the available bandwidth to the users in a scalable, efficient, fair and distributed fashion. This paper is concerned with the stability properties of a class of such schemes in the presence of heterogeneous delays. In particular, we present a time-domain methodology for scalable, global stability analysis of *primal* network congestion control schemes for the Internet. The conditions we obtain are delay-dependent and are similar to the ones that were obtained previously based on the analysis of the linearized system. The structure of the dynamics for the sources and links allows the construction of appropriate Lyapunov certificates in a scalable way.

Keywords: Internet Congestion Control, Heterogeneous Delays, stability, nonlinear.

1. INTRODUCTION

Congestion collapse is a result of serious packet losses that occur when the transmission rates on the links exceed link capacities. The aim of congestion control for the Internet is to avoid this phenomenon by allocating bandwidth to the users in a distributed, scalable way. Such congestion control schemes were proposed in the late 90's by Jacobson [1988] and several improvements have been introduced since then. The basic feature of these schemes is that each user adapts his transmission rate depending on the congestion that earlier packets experienced. Congestion signals are usually in the form of packet loss or delay. See Srikant [2003] for more details.

In a parallel effort, it was shown in Kelly et al. [1998] and Low and Lapsley [1999] that the full, centralized resource allocation problem that models bandwidth allocation for the Internet can be decomposed into a primal and a dual problem by introducing duality-based price signals. Algorithms for solving these problems in a distributed fashion have also been proposed in these articles. It is interesting to note that the proposed designs relate closely to the algorithms already implemented. In particular, the *dual variables* play the role of congestion signals, and are generated by the Active Queue Management (AQM) part of the algorithm at the routers (links); the congestion measure is usually based on either delay or packet loss. On the other hand, the source transmission rates play the role of *primal variables* — each source decides on its transmission rate based on the aggregate price signal.

In the past years, new TCP and AQM algorithms have been designed in order to make the earlier designs scalably stable for arbitrary network topologies and steer the system dynamics towards the global optimum of the resource allocation problem, irrespective of channel capacities. The dynamics that are chosen in Kelly et al. [1998] have this property, as they are based on a gradient algorithm to guarantee convergence. This inevitably leads to a *weighted potential system*, with a positive definite potential function. This potential function can be used as a Lyapunov function to verify the global attractivity of the optimal point, which is also the equilibrium point by construction.

It is important to appreciate that the simplest adequate model for network congestion control is in the form of nonlinear deterministic delay-differential equations — see Mathis et al. [1997]: delays cannot be ignored as in general their presence causes transmission rates to oscillate which could result in a reduction in the link utilization. Moreover short-lived small packets on which congestion control is difficult get dropped, and predictability of the behaviour of the system is lost.

The introduction of increasing bandwidth-delay product links in the network requires the design of scalably stable congestion control schemes with respect not only to the network size and link capacities, but also with respect to the Round Trip Time (delay). Such designs have already appeared (see Paganini et al. [2001], Vinnicombe [2002]), and were formulated at the linearization level, using, e.g., the frequency domain results developed in Vinnicombe [2000] and Paganini et al. [2001]. The control laws are then embedded into nonlinear equations including delays, but it is difficult to obtain conditions that guarantee the stability of the equilibrium of the nonlinear delay differential equations; this is because delay differential equations, and in particular nonlinear ones, are far more difficult to analyze — see, e.g., Jack K. Hale [1993].

Several research areas, such as population dynamics, have motivated the development of hand-crafted time-domain methodologies for specific classes of nonlinear time delay systems. Such constructions have also been used in Internet Congestion Control – see, e.g., Wang and Paganini [2002], Deb and Srikant [2003], Mazenc and Niculescu [2003]. New tools have also been introduced, such as passivity theory formulations (Wen and Arcak [2003]) and IQC formulations and absolute stability theory (Peet and Lall [2007]) but these approaches usually treat only simple network topologies. The only results reported for the full nonlinear system are delay-independent conditions in Ying et al. [2006] and Ranjan et al. [2006]; the novelty in this paper is the development of delay-dependent conditions for nonlinear stability for arbitrary delays and arbitrary topologies for primal congestion control algorithms: for the dual case, see Papachristodoulou [2004a]. The main difficulty in obtaining such conditions lies in the construction of scalable Lyapunov certificates and the conservativeness of the results is usually due to their type: Lyapunov-Razumikhin functions or Lyapunov-Krasovskii functionals. For more details see Gu et al. [2003].

In this paper we take advantage of the structure of the system and the particular choice of dynamics to derive scalable conditions for stability of the system described by nonlinear delay differential equations for the case of what are known as 'primal' congestion control algorithms using Lyapunov-Krasovskii functionals. The conditions we obtain are delay-dependent and are similar to the ones that the linearization gives. This will allow us to generalize our methodology to the full nonlinear system with delays, thus obtaining a stability condition in this case too.

The paper is organized as follows. In Section 2 we will present the problem we wish to tackle. In Section 3 we present the stability analysis of the linearization including the effect of delays, based on Lyapunov arguments. In Section 4 we present the stability analysis of the nonlinear system with delays. We conclude the paper in Section 5.

2. PROBLEM FORMULATION

Consider a network of L communication links shared by S sources. The routing matrix R is:

$$R_{li} = \begin{cases} 1 \text{ if source } i \text{ uses link } l \\ 0 \text{ otherwise.} \end{cases} \tag{1}$$

Associated to each source i is a transmission rate x_i. All sources whose flow passes through resource l contribute to the *aggregate rate*, y_l, the rates being added with some forward time delay $\tau_{i,l}^f$:

$$y_l(t) = \sum_i R_{li} x_i(t - \tau_{i,l}^f) \triangleq r_f(x_i, \tau_{i,l}^f). \tag{2}$$

The resources l react to the aggregate rate y_l by setting a *price* p_l at the Active Queue Management (AQM) part of the algorithm. The prices of all the links that source i uses are added to form q_i, the *aggregate price* for source i, again through a delay $\tau_{i,l}^b$:

$$q_i(t) = \sum_l R_{li} p_l(t - \tau_{i,l}^b) \triangleq r_b(p_l, \tau_{i,l}^b). \tag{3}$$

The prices q_i can then be used to set the rate, x_i, of source i. This is the Transmission Control Protocol (TCP) part of the algorithm, which completes the picture shown in Figure 1. The capacity of link l is denoted by c_l. The forward and backward delays can be combined to yield the Round Trip Time (RTT) for source i, τ_i:

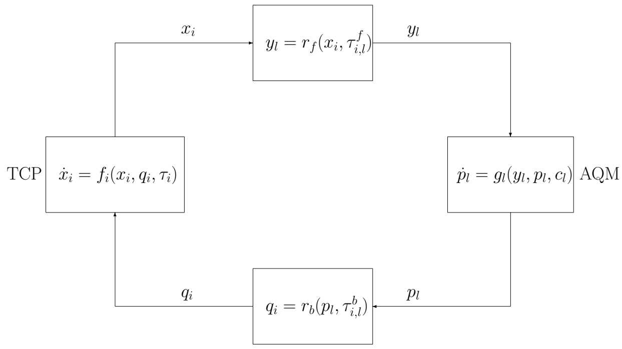

Fig. 1. The Internet as an interconnection of sources and links through delays.

$$\tau_i = \tau_{i,l}^f + \tau_{i,l}^b. \tag{4}$$

This setting is *universal*: what needs to be specified are two control laws that describe how the ith source reacts to the price signal q_i that it sees

$$\dot{x}_i(t) = f_i(x_i(t), q_i(t), \tau_i), \tag{5}$$

and how the lth router reacts to the signal y_l

$$\dot{p}_l(t) = g_l(y_l(t), p_l(t), c_l). \tag{6}$$

Here f_i models TCP algorithms (e.g. Reno or Vegas) and g_l models AQM algorithms (e.g. RED, REM).

The resource allocation problem can be cast as an optimization problem as discussed in Low [2003] and Kelly et al. [1998]. Each user i is identified with a continuously differentiable, strictly concave, non-decreasing *utility function*, $U_i(x_i)$, when allowed to have a transmission rate x_i — meaning that the sources' 'benefit' does not decrease when their rates are increased. The resource allocation problem is then:

$$\begin{aligned} \max_{x_i \geq 0} \quad & \sum_{i=1}^{S} U_i(x_i) \\ \text{s.t.} \quad & \sum_{i=1}^{S} R_{li} x_i \leq c_l, \ \forall\ l = 1, \dots, L, \end{aligned}$$

where the inequality constraint is the natural limitation that the sum of all transmission rates through link l has to be less than or equal to its capacity. The uniqueness of solution to the above problem is guaranteed since the U_i are strictly concave functions. Using a duality argument the above centralized problem can be decomposed into a primal problem that the sources are trying to solve, and a dual that the links are trying to solve. The source rates, x_i, are then primal variables and the prices set by the links, p_l, are the dual variables. Under specific assumptions the optimal point of the two sub-problems coincides with the optimal point of the original problem, which is unique. More details can be found in Low [2003].

The system defined by (5–6), with delays ignored, aims to drive the system close to or exactly at the optimal point (x^*, p^*), using gradient algorithms. A congestion control scheme with dynamics at the sources but a static link law is termed 'primal', which is the type of congestion control algorithm that we will be investigating in this paper. The gradient algorithm results in dynamics of a weighted *potential system*, i.e., there is a potential function V so that $\dot{x}_i = -\kappa_i \frac{\partial V}{\partial x_i}$ — see Srikant [2003]. Therefore scalable stability in the undelayed case can be obtained.

Here we will present a Lyapunov-based construction to obtain a delay-dependent result that holds for arbitrary network topologies when delays are introduced. The tools we will be using are Lyapunov-Krasovskii functionals (see Jack K. Hale [1993]), which are the relevant Lyapunov-based tools for functional differential equations as Lyapunov functions are for ordinary differential equations.

2.1 Nonlinear Delay Differential Equations and stability

For $h > 0$, let $C([-h,0],\mathbb{R}^S)$ be the Banach space of continuous functions mapping $[-h,0]$ into $\mathbb{R}^S$ with the topology of uniform convergence. Consider an autonomous functional differential equation of the form

$$\dot{x}(t) = f(x_t), \quad x(\theta) = \phi(\theta), \tag{7}$$

where $x_t = x(t+\theta)$, $\theta \in [-h,0]$ is the state of the system. We assume that solutions to this differential equation exist and are unique locally. We also assume without loss of generality that $x = 0$ is an equilibrium, i.e., $f(0) = 0$. Definitions of stability of the steady state can be found in Jack K. Hale [1993]. A Lyapunov-type theorem for assessing the stability properties of this differential equation is the following:

Theorem 1. **Lyapunov-Krasovskii** Assume $a(\cdot)$ and $b(\cdot)$ are nonnegative continuous, $a(0) = b(0) = 0$, $\lim_{s\to\infty} a(s) = +\infty$ and that $V : C \to \mathbb{R}$ is continuous and satisfies:

$$V(\phi) \geq a(|\phi(0)|), \quad -\dot{V}(\phi) \geq b(|\phi(0)|). \tag{8}$$

Then the solution $x = 0$ is stable, and every solution is bounded. If in addition $b(s) > 0$ for $s > 0$, then $x = 0$ is globally asymptotically stable; that is, every solution of (7) approaches $x = 0$ as $t \to \infty$.

2.2 The primal congestion control scheme

Let us assume that the routing matrix R is fixed and full row rank. This means that there are no algebraic constraints between link flows, i.e., they can vary independently by choice of source flows x_i. As a consequence, equilibrium prices are uniquely determined. The condition for optimality from the resource allocation problem formulation is given by:

$$U_i'(x_i^*) - q_i^* = 0.$$

For the primal case, we consider a gradient control algorithm to drive the source dynamics to achieve this optimality condition. We therefore have the following link and source laws — see Kelly et al. [1998] and Srikant [2003]:

$$p_l(t) = f_l(y_l(t))$$
$$\dot{x}_i(t) = \kappa_i(x_i(t))(U_i'(x_i(t)) - q_i),$$

where f_l is a strictly increasing function, such that $f_l > 0$, $f_l' > 0$. In particular, the following law has been proposed in Vinnicombe [2002]:

$$p_l(t) = f_l(y_l(t))$$
$$\dot{x}_i(t) = \kappa_i x_i(t-\tau_i)\left[1 - \frac{q_i}{U_i'(x_i(t))}\right]$$

We can combine the equations shown above with the network structure interconnection shown in Figure 1 to get the following closed loop dynamics for an arbitrary network with routing matrix R:

$$\dot{x}_i(t) = \kappa_i x_i(t-\tau_i) \times \left[1 - \frac{\sum_{l=1}^{L} R_{li} f_l\left(\sum_{j=1}^{S} R_{lj} x_j(t-\tau_{i,l}^b - \tau_{j,l}^f)\right)}{U_i'(x_i(t))}\right] \tag{9}$$

The initial conditions for (9) are non-negative functions defined on $C([-h,0],\mathbb{R}^S)$, where $h = \max h_i$ and

$$h_i = \max_{\{j,l:R_{li}=R_{lj}=1\}} \{\tau_{i,l}^b + \tau_{j,l}^f\}. \tag{10}$$

3. STABILITY OF THE LINEARIZATION

The presence of delays is often destabilizing and can affect the performance of a system. Stability analysis of linear time delay systems has been investigated actively in the past years, see e.g., Niculescu [2001] and Gu et al. [2003]. Just as in the stability analysis of system described by linear ODEs, there are in general two methodologies for investigating stability: using time-domain (Lyapunov) or frequency domain (eigenvalue) arguments. Frequency domain methodologies usually result in more accurate descriptions of the stability boundaries and are scalable for the special case of Internet Congestion Control, as they are developed in Vinnicombe [2000]. Lyapunov-based arguments often more conservative; they are however useful for the investigation of the stability of nonlinear systems (Jack K. Hale [1993]).

In this section we will use a Lyapunov argument for the stability analysis: we will be constructing a Lyapunov-Krasovskii functional, taking advantage of the structure of the system, to get conditions for stability similar to the ones that frequency domain methodologies produce. We accept the fact that choosing a Lyapunov functional structure will introduce some conservativeness in the condition for which stability is going to be retained. More complicated, 'richer' structures may be less conservative. In this section we consider the linearized system only. Section 4 will consider analysis of the full nonlinear system.

The linearization of (9) about the equilibrium x_i^* is:

$$\dot{x}_i = \frac{\kappa_i x_i^*}{q_i^*} U_i''(x_i^*) x_i - \frac{\kappa_i x_i^*}{q_i^*} \sum_{l=1}^{L} R_{li} p_l'^* \sum_{j=1}^{S} R_{lj} x_j(t-\tau_{i,l}^b - \tau_{j,l}^f) \tag{11}$$

where $q_i^* = U_i'^{-1}(x_i^*)$, $y_l^* = \sum_{l=1}^{L} R_{li} x_i^*$ and $p_l'^* = f_l'(y_l^*)$.

We have the following result:

Theorem 2. System (11) is asymptotically stable if R is full row rank and

$$\frac{\kappa_i}{q_i^*} \sum_{l=1}^{L} \sum_{j=1}^{S} R_{lj} R_{li} x_j^* p_l'^* \frac{(\tau_i + \tau_j)}{2} < 1, \quad \forall\, i. \tag{12}$$

Proof. Consider the following function:

$$V_1(t) = -\frac{1}{2}\sum_{i=1}^{S} U_i''(x_i^*) x_i(t)^2 + \frac{1}{2}\sum_{l=1}^{L} p_l'^* \left(\sum_{j=1}^{S} R_{lj} x_j(t)\right)^2$$

This is positive definite, as R is full row rank and the U_i are strictly concave. Define the undelayed version of (11) by

$$\dot{x}_{i,u}(t) = \kappa_i \frac{x_i^*}{q_i^*}\left[U_i''(x_i^*) x_i(t) - \sum_{l=1}^{L} R_{li} p_l'^* \sum_{j=1}^{S} R_{lj} x_j(t)\right]$$

We have:

$$\dot{V}_1(t) = -\sum_{i=1}^{S} \frac{q_i^*}{\kappa_i x_i^*}\dot{x}_i(t)\dot{x}_{i,u}(t)$$
$$= -\sum_{i=1}^{S} \frac{q_i^*}{\kappa_i x_i^*}\dot{x}_i(t)^2 - \sum_{i=1}^{S} \frac{q_i^*}{\kappa_i x_i^*}\dot{x}_i(t)(\dot{x}_{i,u}(t) - \dot{x}_i(t))$$

We now manipulate the second term to get:

$$-\sum_{i=1}^{S} \frac{q_i^*}{\kappa_i x_i^*}\dot{x}_i(t)(\dot{x}_{i,u}(t) - \dot{x}_i(t))$$
$$= \sum_{i=1}^{S} \dot{x}_i(t) \sum_{l=1}^{L} R_{li} p_l^{'*} \sum_{j=1}^{S} R_{lj}\left(x_j(t) - x_j(t - \tau_{i,l}^b - \tau_{j,l}^f)\right)$$
$$= \sum_{i=1}^{S}\sum_{l=1}^{L} R_{li} p_l^{'*} \sum_{j=1}^{S} R_{lj} \int_{-\tau_{i,l}^b - \tau_{j,l}^f}^{0} \dot{x}_i(t)\dot{x}_j(t+\theta)d\theta$$

where the Leibniz rule was used to distribute the delay over an interval. We combine this with the inequality $ab \leq \frac{k}{2}a^2 + \frac{1}{2k}b^2$ for any $k > 0$ to get:

$$-\sum_{i=1}^{S} \frac{q_i^*}{\kappa_i x_i^*}\dot{x}_i(t)(\dot{x}_{i,u}(t) - \dot{x}_i(t))$$
$$\leq \sum_{i=1}^{S}\sum_{l=1}^{L}\sum_{j=1}^{S} \frac{R_{lj}R_{li}}{2} p_l^{'*}(\tau_{i,l}^b + \tau_{j,l}^f)k_{ij}\dot{x}_i(t)^2$$
$$+ \sum_{i=1}^{S}\sum_{l=1}^{L}\sum_{j=1}^{S} \frac{R_{lj}R_{li}}{2k_{ij}} p_l^{'*} \int_{-\tau_{i,l}^b - \tau_{j,l}^f}^{0} \dot{x}_j(t+\theta)^2 d\theta$$

where the $k_{ij} > 0$ are arbitrary constants. Introduce now the following functional:

$$V_2(t) = \sum_{i=1}^{S}\sum_{l=1}^{L}\sum_{j=1}^{S} \frac{R_{lj}R_{li}}{2k_{ij}} p_l^{'*} \int_{-\tau_{i,l}^b - \tau_{j,l}^f}^{0} \int_{t+\theta}^{t} \dot{x}_j^2(\zeta)d\zeta d\theta$$

Then this satisfies:

$$\dot{V}_2(t) = \sum_{i=1}^{S}\sum_{l=1}^{L}\sum_{j=1}^{S} \frac{R_{lj}R_{li}(\tau_{i,l}^b + \tau_{j,l}^f)}{2k_{ij}} p_l^{'*}\dot{x}_j(t)^2$$
$$- \sum_{i=1}^{S}\sum_{l=1}^{L}\sum_{j=1}^{S} \frac{R_{lj}R_{li}}{2k_{ij}} p_l^{'*} \int_{-\tau_{i,l}^b - \tau_{j,l}^f}^{0} \dot{x}_j(t+\theta)^2 d\theta$$

Now let $V = V_1 + V_2$. Then we have

$$\dot{V}(t) \leq -\sum_{i=1}^{S} \frac{q_i^*}{\kappa_i x_i^*}\dot{x}_i(t)^2$$
$$+ \sum_{i=1}^{S}\sum_{l=1}^{L}\sum_{j=1}^{S} \frac{R_{lj}R_{li}(\tau_{i,l}^b + \tau_{j,l}^f)}{2k_{ij}} p_l^{'*}\dot{x}_j(t)^2$$
$$+ \sum_{i=1}^{S}\sum_{l=1}^{L}\sum_{j=1}^{S} \frac{R_{lj}R_{li}}{2} p_l^{'*}(\tau_{i,l}^b + \tau_{j,l}^f)k_{ij}\dot{x}_i(t)^2.$$

We now switch the index i and j in the last summation to get:

$$\dot{V}(t) \leq -\sum_{i=1}^{S} \frac{q_i^*}{\kappa_i x_i^*}\dot{x}_i(t)^2$$
$$+ \sum_{i=1}^{S}\sum_{l=1}^{L}\sum_{j=1}^{S} \frac{R_{lj}R_{li}}{2} p_l^{'*}(\tau_i + \tau_j)\frac{x_j^*}{x_i^*}\dot{x}_i(t)^2$$

where we have used $k_{ij} = \frac{x_j^*}{x_i^*}$. Stability is guaranteed if $\dot{V} \leq 0$, i.e., if for each i we have:

$$\frac{\kappa_i}{q_i^*}\sum_{l=1}^{L}\sum_{j=1}^{S} R_{lj}R_{li}x_j^* p_l^{'*}\frac{(\tau_i + \tau_j)}{2} < 1$$

In order to prove asymptotic stability, consider the following argument. The set $S = \{\phi \in C([-h, 0], \mathbb{R}^n) : \dot{V}(\phi) = 0\}$ is the set

$$S = \left\{ \phi : \begin{array}{c} \sum_{l=1}^{L} R_{li}p_l^{'*} \sum_{j=1}^{n} R_{lj}\phi_j(-\tau_{i,l}^b - \tau_{j,l}^f) = U_i^{''}(x_i^*)\phi_i(0), \\ i = 1, \ldots, n \end{array} \right\}.$$

The largest set in S that is invariant with respect to the system satisfies $\dot{x}_i = 0\ \forall i = 1, \ldots, n$, i.e., $x_i = K$, a constant. But the only constant ϕ that is in S is the zero equilibrium. Therefore the equilibrium of the system is asymptotically stable by an extension of LaSalle's theorem (Theorem 5.3.1 in Jack K. Hale [1993]) provided (12) holds and R is full rank.

The above theorem gives a conservative bound, but has shown how a time-domain argument can prove delay-dependent stability — this argument will be used in the next section to obtain stability conditions for the nonlinear delayed system.

4. NONLINEAR STABILITY ANALYSIS

The nonlinear system is given by Equation (9), and the existence and uniqueness of solutions is assumed. To ensure stability for arbitrary topologies in the nonlinear case with delays we have to use a Lyapunov based argument, similar to the one we used in the linearization. Before we go into that, let us recall the argument about global stability of the undelayed nonlinear system without delays.

4.1 Nonlinear Undelayed Model

The undelayed closed loop system is

$$\dot{x}_i = \kappa_i x_i \left[1 - \frac{\sum_{l=1}^{L} R_{li} f_l\left(\sum_{j=1}^{S} R_{lj}x_j\right)}{U_i'(x_i)}\right] \tag{13}$$

Theorem 3. For fixed full rank R, the (unique) equilibrium of (13) is asymptotically stable for all non-negative initial conditions.

The proof is omitted, it can be found in Kelly et al. [1998]. The Lyapunov function that is used is

$$V(x) = -\sum_{i=1}^{S}[U_i(x_i) - U_i(x_i^*)]$$
$$+ \sum_{l=1}^{L}\int_{y_l^*}^{\sum_{j=1}^{S} R_{lj}x_j} f_l(Y)dY \tag{14}$$

It is important to note that the derivative of this Lyapunov function satisfies $\dot{V}(x) = -\sum_{i=1}^{S} k_i\dot{x}_i^2$ for $k_i > 0$ which is key to the scalability of the result. In particular, we have:

$$
\begin{aligned}
\dot{V}(x) =& -\sum_{i=1}^{S} U_i'(x_i)\dot{x}_{i,u} \\
&+ \sum_{l=1}^{L} f_l\left(\sum_{j=1}^{S} R_{lj}x_j\right)\left(\sum_{i=1}^{S} R_{li}\dot{x}_{i,u}\right) \\
=& -\sum_{i=1}^{S}\left[U_i'(x_i) - \sum_{l=1}^{L} R_{li} f_l\left(\sum_{j=1}^{S} R_{lj}x_j\right)\right]\dot{x}_{i,u} \\
=& -\sum_{i=1}^{S} \frac{U_i'(x_i)}{\kappa_i x_i}\dot{x}_{i,u}^2 \le 0
\end{aligned}
$$

where $\dot{x}_{i,u}$ refers to Equation (13). A simple LaSalle argument ensures the asymptotic stability of the equilibrium – see Kelly et al. [1998] for more details.

We now turn to the nonlinear delayed system.

4.2 Nonlinear Delayed Model

In this section we will consider the system given by Equation (9). We will use the following definition for the undelayed system, instead of the one introduced earlier.

$$
\begin{aligned}
\dot{x}_{i,u}(t) =& \kappa_i x_i(t-\tau_i) \\
&- \kappa_i x_i(t-\tau_i)\frac{\sum_{l=1}^{L} R_{li} f_l\left(\sum_{j=1}^{S} R_{lj}x_j(t)\right)}{U_i(x_i(t))}
\end{aligned}
\qquad (15)
$$

Before we proceed, recall that the U_i are strictly concave, non-decreasing functions and $f_l(y_l)$ are strictly increasing functions, so that $f_l(x) > 0$ and $f_l'(x) > 0$ for all $x \ge 0$. This inevitably means that x_i are upper bounded – a feature that was used in Wang and Paganini [2002] – as shown in the following proposition:

Proposition 4. Let $x_i(t)$ be a solution of (9). Then there is a $T > 0$ such that, for $t \ge T$, $x_i(t) < U_i'^{-1}\left(\sum_{l=1}^{L} R_{li} f_l(0)\right) := \overline{x}_i$.

Proof. First, note that

$$
\begin{aligned}
U_i'^{-1}\left(\sum_{l=1}^{L} R_{li} f_l(0)\right) >& U_i'^{-1}\left(\sum_{l=1}^{L} R_{li} f_l(y_l^*)\right) \\
=& U_i'^{-1}(q_i^*) = x_i^*.
\end{aligned}
$$

If $x_i(t) \ge x_i^*$ $\forall i$ for all large t, say $t \ge T$, then $\dot{x}_i(t) \le 0$ for $t \ge T$ and hence $\lim_{t\to\infty} x_i(t) = x_i^*$. Thus we may assume $x_i(t)$ are oscillatory about x_i^*. Let $\bar{t} > 2h_i$, where h_i is defined by (10) be such that $x_i(\bar{t}) > x_i^*$ and $\dot{x}_i(\bar{t}) = 0$, i.e. a maximum of the trajectory. Then we have that:

$$
U_i'(x_i(\bar{t})) = \sum_{l=1}^{L} R_{li} f_l\left(\sum_{j=1}^{S} R_{lj}x_j(\bar{t} - \tau_{i,l}^b - \tau_{j,l}^f)\right)
$$

Therefore

$$
\begin{aligned}
x(\bar{t}) =& U_i'^{-1}\left(\sum_{l=1}^{L} R_{li} f_l\left(\sum_{j=1}^{S} R_{lj}x_j(\bar{t} - \tau_{i,l}^b - \tau_{j,l}^f)\right)\right) \\
<& U_i'^{-1}\left(\sum_{l=1}^{L} R_{li} f_l(0)\right)
\end{aligned}
$$

This completes the proof.

Let $x_i(t) \le \overline{x}_i$ where $\overline{x}_i$ is defined in the above proposition. Similarly define $\overline{y}_l = \sum_{i=1}^{S} R_{li}\overline{x}_i$ so that $y_l(t) \le \overline{y}_l$. Then we have:

Theorem 5. The non-zero equilibrium of (9) is asymptotically stable provided R is full rank and

$$
\frac{\kappa_i}{U_i'(\overline{x}_i)}\sum_{l=1}^{L}\sum_{j=1}^{S} R_{lj}R_{li}\overline{x}_j f_l'(\overline{y}_l)\frac{(\tau_i+\tau_j)}{2} < 1.
$$

Proof. Consider a function of the form:

$$
\begin{aligned}
V_1 =& -\sum_{i=1}^{S}[U_i(x_i) - U_i(x_i^*)] \\
&+ \sum_{l=1}^{L}\int_{y_l^*}^{\sum_{j=1}^{S} R_{lj}x_j} f_l(Y)dY
\end{aligned}
$$

This function is positive definite and radially unbounded. Then we have:

$$
\begin{aligned}
\dot{V}_1 =& -\sum_{i=1}^{S}\left[U_i'(x_i) - \sum_{l=1}^{L} R_{li} f_l\left(\sum_{i=1}^{S} R_{lj}x_j\right)\right]\dot{x}_i \\
=& -\sum_{i=1}^{S}\frac{U_i'(x_i)}{\kappa_i x_i(t-\tau)}\dot{x}_{i,u}\dot{x}_i \\
=& -\sum_{i=1}^{S}\frac{U_i'(x_i)}{\kappa_i x_i(t-\tau)}\left(\dot{x}_i^2 + \dot{x}_i(\dot{x}_{i,u} - \dot{x}_i)\right)
\end{aligned}
$$

Now

$$
\begin{aligned}
&-\sum_{i=1}^{S}\frac{U_i'(x_i)}{\kappa_i x_i(t-\tau)}\dot{x}_i(\dot{x}_{i,u} - \dot{x}_i) \\
=& \sum_{i=1}^{S}\sum_{l=1}^{L} R_{li} f_l\left(\sum_{j=1}^{S} R_{lj}x_j(t)\right)\dot{x}_i \\
&- \sum_{i=1}^{S}\sum_{l=1}^{L} R_{li} f_l\left(\sum_{j=1}^{S} R_{lj}x_j(t-\tau_{i,l}^b-\tau_{j,l}^f)\right)\dot{x}_i \\
\le& \sum_{i=1}^{S}\sum_{l=1}^{L}\sum_{j=1}^{S} R_{li}R_{lj}f_l'(\overline{y}_l)\int_{-\tau_{i,l}^b-\tau_{j,l}^f}^{0}|\dot{x}_i||\dot{x}_j(t+\theta)|d\theta
\end{aligned}
$$

as f_l is globally Lipschitz continuous and strictly increasing. The rest of the proof is the same as in the linear case. Since $x_i < \overline{x}_i$, we have:

$$
U_i'(x_i) > U_i'(\overline{x}_i)
$$

from the strict concavity of U_i and therefore $\dot{V}$ can be written as follows:

$$
\begin{aligned}
\dot{V} \le& -\sum_{i=1}^{S}\frac{U_i'(x_i)}{\kappa_i x_i(t-\tau)}\dot{x}_i^2 \\
&+ \sum_{i=1}^{S}\sum_{l=1}^{L}\sum_{j=1}^{S}\frac{R_{lj}R_{li}}{2}f_l'(\overline{y}_l)(\tau_i+\tau_j)\frac{\overline{x}_j}{\overline{x}_i}\dot{x}_i^2 \\
\le& -\sum_{i=1}^{S}\frac{U_i'(\overline{x}_i)}{\kappa_i\overline{x}_i}\dot{x}_i^2 \\
&+ \sum_{l=1}^{L}\sum_{j=1}^{S}\frac{R_{lj}R_{li}}{2}f_l'(\overline{y}_l)(\tau_i+\tau_j)\frac{\overline{x}_j}{\overline{x}_i}\dot{x}_i^2
\end{aligned}
$$

Therefore stability is retained if

$$\frac{\kappa_i}{U_i'(\overline{x}_i)}\sum_{l=1}^{L}\sum_{j=1}^{S}R_{lj}R_{li}\overline{x}_j f_l'(\overline{y}_l)\frac{(\tau_i+\tau_j)}{2}<1$$

Asymptotic stability follows from LaSalle's argument, in a similar way as in the linear case.

It is important to remark that the proof technique in the nonlinear case is very similar to the one for the linearization.

5. CONCLUSIONS

In this paper we have constructed a Lyapunov-Krasovskii functional for arbitrary network topologies that use primal congestion control schemes in Kelly's framework. The conditions are the first to take into account the size of the delays, i.e., they are the first for delay-dependent stability for such congestion control schemes.

As we have seen, the nonlinear results are more conservative than the linearizations, but several improvements may be possible. First, the estimated bounds in Proposition 4 can be strengthened if information on the initial condition is taken into account, as was done in Ying et al. [2006]. Furthermore, a more complicated functional structure can prove stability with a less conservative condition; this has been observed when constructing these functionals algorithmically using LMIs in the linear case, but also in the nonlinear case using SOSTOOLS (see Prajna et al. [2002] and Papachristodoulou [2004b]) for simple network topologies. Also, another functional structure may help obtain a condition for stability that can be implemented in a decentralized way.

Future research will concentrate on stability analysis of primal-dual congestion control schemes.

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