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Reachability and Robust Control of PWA Systems with Parameter Variations and Bounded Disturbance

J. Thomas*, D. Dumur**, S. Olaru**, and J. Buisson***

* Faculty of Industrial, Beni Sueif Egypt (Tel: 20 (0)12 52 63 476; e-mail: jhh_thomas@yahoo.com)
** Supélec F 91192 Gif sur Yvette cedex, France (e-mail:{didier.dumur, sorin.olaru}@supelec.fr)
*** Supélec F 35 511 Cesson-Sévigné cedex, France, (e-mail: jean.buisson@supelec.fr)

Abstract: This paper considers discrete-time, uncertain PWA (piecewise affine) systems affected by parameter variations and bounded disturbances, where reachability technique based on polyhedral approach is developed and the robust control problems are investigated. Checking attainability and calculating the state space regions for which a robust control is assured despite the possible disturbance and the parameter variations is performed using a geometrical approach. A model predictive control law derived from a quadratic cost function minimization is further examined as an alternative sub-optimal approach for decreasing the computational load. The proposed technique is applied in simulation to a two-tank benchmark.

1. INTRODUCTION

Hybrid systems, including both continuous and discrete variables, are now of common use in many industrial applications, e.g. in control of mechanical systems, process control, automotive industry, power systems, aircraft and traffic control, since this formalism opens promising perspectives for optimisation of more and more complex systems. However this requires the use of sufficiently simple and tractable models, thus taking into account uncertainties related to model simplification and system knowledge, parameters variations depending on the operating modes and environment.

Control robustness becomes mandatory, so that performances of the controlled system are preserved in spite of these different causes of uncertainty. Indeed, the parameters uncertainties or disturbances influences may cause severe practical problems and safety, attainability and robust control become interesting questions for researchers. In this direction, this paper examines a class of discrete-time uncertain Piecewise Affine (PWA) systems, where the uncertainties are coming from parameter variations and bounded disturbances. For this class of systems, some solutions to the above mentioned problems are already proposed in the literature. For example, in (Lin and Antsaklis, 2003), an attainability checking that employs the predecessor operator, and a controller technique using finite automata and linear programming is presented. In (Necoara *et al.*, 2004, Bemporad *et al.*, 2003), a control technique based on minimizing the worst-case cost function (min-max problem) is proposed to solve the control problem.

In (Thomas *et al.*, 2006) an attainability checking based on polyhedral approach for PWA systems affected by bounded disturbances were presented, in the same direction this paper is based on a polyhedral approach enabling the elaboration of the state space regions for which a robust control exists which drives the plant to a desired behavior in despite of the parameter variations and the possible disturbances. The safety, reachability and attainability questions are examined through this framework and a robust Model Predictive Control (MPC) with quadratic cost function is presented as a fast suboptimal robust control.

The paper is organized as follows. A brief description of PWA systems and the related class is given in Section 2. Section 3 develops the polyhedral approach which will elaborate the state space regions where reachability, safety and attainability questions can be assured. A fast and suboptimal robust control is then developed in Section 4 for the considered class. An application of the proposed technique to a two-tank benchmark is presented in Section 5. Finally the conclusions and some remarks are given in Section 6.

2. UNCERTAIN PIECEWISE AFFINE SYSTEMS

Piecewise affine systems are powerful tools for describing or approximating both nonlinear and hybrid systems, and represent a straightforward extension from linear to hybrid systems (Sontag, 1981). This paper focuses on the particular class of uncertain discrete-time piecewise affine systems subject to both parameter variation and bounded disturbances, defined as:

$$S^i : \left\{ \begin{array}{l} \mathbf{x}_{k+1} = \mathbf{A}^i(w_k)\mathbf{x}_k + \mathbf{B}^i(w_k)\mathbf{u}_k + \mathbf{f}^i(w_k) + \mathbf{C}^i\mathbf{d}_k \\ \text{for } \begin{bmatrix} \mathbf{x}_k \\ \mathbf{u}_k \end{bmatrix} \in \chi_i \end{array} \right\} \quad (1)$$

$\mathbf{x}_k \in \mathbf{X}, \mathbf{u}_k \in \mathbf{U}, w_k \in \mathbf{W}, \mathbf{d}_k \in \mathbf{D}$ denote the system state, the discretized control input, the uncertainty and the disturbance vector respectively at instant k (for the i th model) with $\mathbf{X}, \mathbf{U}, \mathbf{W}, \mathbf{D}$ assigned polytopes, and $\mathbf{D}$ contains the origin.

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$\{\chi_i\}_{i=1}^{s}$ is the polyhedral coverage of the state and input spaces $\mathbf{X}\times\mathbf{U}$, s being the number of subsystems. Each χ_i is given by:

$$\chi_i = \left\{ \begin{bmatrix} \mathbf{x}_k \\ \mathbf{u}_k \end{bmatrix} \mid \mathbf{Q}^i \begin{bmatrix} \mathbf{x}_k \\ \mathbf{u}_k \end{bmatrix} \le \mathbf{q}^i \right\} \tag{2}$$

Exact state measurement $\mathbf{x}$ is supposed to be available.

In this formalism, the existence of logical decision variables is taken into account by developing an affine model (1) for each possible combination in $\{1,\cdots,s\}$. However the selection of $i\in\{1,\cdots,s\}$ is valid only if the system of linear inequality constraints (2) are satisfied.

Each subsystem S^i defined by the 6-uple $\left(\mathbf{A}^i,\mathbf{B}^i,\mathbf{C}^i,\mathbf{f}^i,\mathbf{Q}^i,\mathbf{q}^i\right)$, $i\in I=\{1,\cdots,s\}$ is a component of the global hybrid system with I the collection of all subsystems. $\mathbf{A}^i\in\Re^{n\times n}, \mathbf{B}^i\in\Re^{n\times m}, \mathbf{C}^i\in\Re^{n\times r}, \mathbf{f}^i\in\Re^n$, $\mathbf{Q}^i\in\Re^{p_i x(n+m)}$ and $\mathbf{q}^i\in\Re^{p_i}$, where n, m, r are respectively the dimension of state, input and disturbance vectors, and p_i is the number of hyperplanes defining the χ_i polyhedral.

Taking into account uncertainty as it appears in (1), the following considers polytopic uncertainty in $\mathbf{A}^i(w),\mathbf{B}^i(w)$ and $\mathbf{f}^i(w)$ for every mode $i\in I$. In general a polyhedral set can be represented either by a set of linear inequalities, or by its dual representation in terms of a convex hull of the vertices. In what it may concern the polytopic uncertainty the structure is defined as follows:

$$\mathbf{A}^i(w)=\sum_j^{v_i} w^j\mathbf{A}^{ij},\quad \mathbf{B}^i(w)=\sum_j^{v_i} w^j\mathbf{B}^{ij},\quad \mathbf{f}^i(w)=\sum_j^{v_i} w^j\mathbf{f}^{ij} \tag{3}$$

where $w^j\ge 0$ and $\sum_{j=1}^{v_i} w^j=1$. $\left(\mathbf{A}^{ij},\mathbf{B}^{ij},\mathbf{f}^{ij}\right)$ is the j-th vertex of the i-th model, v_i being the number of vertices. The matrices $\left(\mathbf{A}^i(w),\mathbf{B}^i(w),\mathbf{f}^i(w)\right)$ represents the model subject to uncertainty, described by the polytopic set $ConvexHull\left\{(\mathbf{A}^{ij},\mathbf{B}^{ij},\mathbf{f}^{ij}), j=1,\cdots,v_i\right\}$ for each mode $i\in I$. The coefficients w^j are unknown and possibly time varying. In the following v_i will be assumed to be the same for each partition, noting $v=\max_{i=1\cdots s} v_i$. The next results can be extended to the case with different v_i.

3. DIRECT REACHABILITY: A POLYHEDRAL APPROACH

Let consider the region $\mathbf{R}_k, k>1$, as a target region in the global state space $\mathbf{X}$. This section considers the robust one-step control region $\mathbf{R}_{k-1}$ as the region in the state space for which there exist a feasible mode (1) and an admissible control signal able to drive the states from $\mathbf{R}_{k-1}$ into $\mathbf{R}_k$ in one-step despite all allowable disturbances and parameter variations, i.e.:

$$\mathbf{R}_{k-1}=\left\{\begin{array}{l} \mathbf{x}_{k-1}\in\mathbf{X} \left| \exists i\wedge\mathbf{u}_{k-1}\in\mathbf{U}, \begin{bmatrix}\mathbf{x}_{k-1}\\ \mathbf{u}_{k-1}\end{bmatrix}\in\chi_i \right. \\ s.t.: \mathbf{A}^i(w_{k-1})\mathbf{x}_{k-1}+\mathbf{B}^i(w_{k-1})\mathbf{u}_{k-1}+ \\ \qquad +\mathbf{f}^i(w_{k-1})+\mathbf{C}^i\mathbf{d}_{k-1}\in\mathbf{R}_k, \\ \qquad \forall w_{k-1}\in\mathbf{W},\ \forall\mathbf{d}_{k-1}\in\mathbf{D} \end{array}\right\} \tag{4}$$

In the following, the computation of this region $\mathbf{R}_{k-1}$ is achieved through a polyhedral approach.

Consider the global state space defined by the following constraints:

$$\mathbf{X}:=\left\{\mathbf{F}_s\mathbf{x}\le\mathbf{g}_s,\ \mathbf{F}_s\in\Re^{p\times n},\ \mathbf{g}_s\in\Re^p\right\} \tag{5}$$

The control input is supposed to be bounded:

$$\mathbf{U}:=\left\{\mathbf{m}\,\mathbf{u}\le\mathbf{n},\ \mathbf{m}\in\Re^{p_u\times m},\mathbf{n}\in\Re^{p_u}\right\} \tag{6}$$

With disturbance given inside an assigned polytope $\mathbf{d}_{k-1}\in\mathbf{D}$, with the target region $\mathbf{R}_k\subset\mathbf{X}$, defined by:

$$\mathbf{R}_k:=\left\{\mathbf{F}\mathbf{x}_k\le\mathbf{g}\right\} \tag{7}$$

In the first step, the effect of presence of disturbances is considered, for each valid model i where $i\in\{1,\cdots,s\}$, this leading to the computation of the set:

$$\hat{\mathbf{R}}_k^i=\mathbf{R}_k-\mathbf{C}^i\mathbf{D} \tag{8}$$

where the subtraction is computed in the Minkowsky sense (exact geometric operation, based on the double representation of polyhedral domains). The set $\mathbf{C}^i\mathbf{D}$ is the image of $\mathbf{D}$ by the linear mapping:

$$f:\mathbf{D}\to\Re^n, f(\mathbf{d})=\mathbf{C}^i\mathbf{d}$$

The new polyhedral set $\hat{\mathbf{R}}_k^i$ can be represented by a set of linear inequalities:

$$\hat{\mathbf{R}}_k^i=\left\{\mathbf{x}\left|\hat{\mathbf{F}}^i\mathbf{x}\le\hat{\mathbf{g}}^i\right.\right\}, \tag{9}$$

In the second step, the effect of parameter variation is considered. For the system in the mode i where $i\in\{1,\cdots,s\}$ and the j-th vertex of the polytopic model ($1\le j\le v$), and using the system evaluation (1), equation (9) can be rewritten as follows (where the disturbance effect is the one considered before):

$$\mathbf{R}_{k-1}^{ij}=\left\{\begin{array}{l} \mathbf{x}_{k-1}\in\mathbf{X}\left|\exists\mathbf{u}_{k-1}\in\mathbf{U}, \begin{bmatrix}\mathbf{x}_{k-1}\\ \mathbf{u}_{k-1}\end{bmatrix}\in\chi_i\right. \\ \qquad s.t.\quad \hat{\mathbf{F}}^i(\mathbf{A}^{ij}\mathbf{x}_{k-1}+\mathbf{B}^{ij}\mathbf{u}_{k-1}+\mathbf{f}^{ij})\le\hat{\mathbf{g}}^i \end{array}\right\} \tag{10}$$

Taking in account the i th model constraints and the global state space constraints, the following set can be introduced:

$$\mathbf{T}_{k-1}^{ij}=\left\{\begin{bmatrix}\hat{\mathbf{F}}^i\mathbf{A}^{ij} & \hat{\mathbf{F}}^i\mathbf{B}^{ij}\\ \mathbf{F}_s & 0\\ & \mathbf{Q}^i\end{bmatrix}\begin{bmatrix}\mathbf{x}_{k-1}\\ \mathbf{u}_{k-1}\end{bmatrix}\le\begin{bmatrix}\hat{\mathbf{g}}^i-\hat{\mathbf{F}}^i\mathbf{f}^{ij}\\ \mathbf{g}_s\\ \mathbf{q}^i\end{bmatrix}\right\} \quad (11)$$

The maximal admissible region in the state space for mode i and j vertex is deduced from $\mathbf{T}_{k-1}^{ij}$:

$$\widetilde{\mathbf{R}}_{k-1}^{ij}=\mathrm{Pr}_{\mathbf{X}}\,\mathbf{T}_{k-1}^{ij} \quad (12)$$

Remark 1: The projection of polyhedral sets can be efficiently handled in a double representation (generators/ constraints) and related tools can be found as for example - POLYLIB (Wilde, 1994).

For polytopic uncertain piecewise affine systems, the state space region $\mathbf{R}_{k-1}$ under the i-th mode can be determined by (Lin and Antsaklis, 2002):

$$\mathbf{R}_{k-1}^{i}=\bigcap_{j=1}^{v}\widetilde{\mathbf{R}}_{k-1}^{ij} \quad (13)$$

With these sets constructed for each linear sub-model, the global one-step robust controllable region of the state space being able to drive the states into the region $\mathbf{R}_k$ in one-step despite all possible parameter variations and possible disturbances is thus given by:

$$\mathbf{R}_{k-1}=\bigcup_{i=1}^{s}\mathbf{R}_{k-1}^{i} \quad (14)$$

The procedure presented above can be repeated in a recursive way to find the domain for any limited N steps horizon. Precautions have to be taken for the case where (14) is not leading to a convex set. In this case, the use of a suitable polytopic region included in the union (14) has to be found. Using a dynamic programming approach, after defining the target region $\mathbf{R}_{k+N}$, the state space domain $\mathbf{R}_k$ can be recursively calculated, that includes all the states having a feasible control policy that can in N steps derive the states to $\mathbf{R}_{k+N}$ despite the parameter variation and the possible disturbances.

Remark 2: Number of regions is less than s^N depending on the infeasibility of the set of constraints (11) which may lead to empty regions. For PWA systems with many sub-models s and for long horizon N, this may imply the exploration of a large number of regions (exponential complexity, Figure 1), even if those calculations are made off-line.

Safety, a well-known geometric condition for a set to be safe (control invariant) is the following (Lin and Antsaklis, 2002):

the set $\mathbf{R}_{k+1}$ is safe if and only if $\mathbf{R}_{k+1}\subseteq\mathbf{R}_k$

Attainability, given a finite number of regions $(\mathbf{R}_k,\mathbf{R}_{k+1},\cdots,\mathbf{R}_{k+N})\in I\times\chi$, the attainability for this sequence of regions is equivalent to the following two different properties:

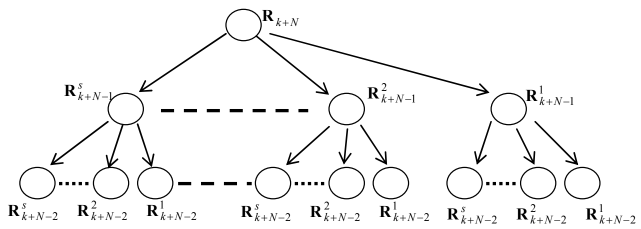

Fig. 1. Complete regions exploration.

1. *the direct reachability from region* $\mathbf{R}_{k+j}$ *to* $\mathbf{R}_{k+j+1}$ *for* $0\le j\le N-1$,
2. *the safety (or control invariance) for region* $\mathbf{R}_{k+N}$.

4. ROBUST MODEL PREDICTIVE CONTROL

The min-max control technique is proposed in the literature as a robust control for such problems, which minimizes the maximum cost, to try to counteract the worst disturbance. This paper focuses on the model predictive control for PWA systems with quadratic cost function as a fast suboptimal robust solution. Model predictive control (MPC) has proved to efficiently control a wide range of applications in industry for non-hybrid and hybrid systems as well (Camacho and Bordons, 1999), (Bemporad and Morari, 1999a) (Schutter and Boom, 2004), and also used as robust control (Bemporad and Morari, 1999b).

The control object for the closed-loop system is to exhibit certain desired behaviour despite the uncertainties. Specifically, given finite number of regions $\{\mathbf{R}_0,\mathbf{R}_1,\cdots,\mathbf{R}_N\}$ in the state space, the goal for the closed-loop system trajectories is that starting from the given initial region $\mathbf{R}_0$ goes through the sequence of finite number of regions $\mathbf{R}_1,\mathbf{R}_2,\cdots,\mathbf{R}_N$ in the desired order and finally reach the final region $\mathbf{R}_N$.

A compromise between the computation load and the system performance can be considered; as the object is to deliver the system states to the target region $\mathbf{R}_N$. The computation load can be decreased considering no switch between sub-models over the N steps horizon (exploration according to Figure 2). This technique will not grant the optimal system response but it insures the arriving to the desired region and leads to a lower complexity mechanism, while it may imply more conservatism. However this suboptimal construction appears to be convenient for many applications.

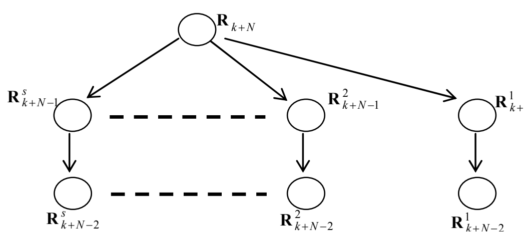

Fig. 2. Exploration with no switch over the N steps.

The model predictive control proposed here requires solving at each sampling time the following problem:

$$\min_{\mathbf{u}_k^{k+N-1}} J(\mathbf{u}_k^{k+N-1}, \mathbf{x}_k) = \sum_{j=1}^{N} \left\| \mathbf{x}_{k+j} - \mathbf{x}_e \right\|_{\mathbf{\Lambda}}^2 + \sum_{j=0}^{N-1} \left\| \mathbf{u}_{k+j} \right\|_{\mathbf{\Gamma}}^2$$
$$\text{s.t.:} \begin{cases} \mathbf{x}_{k+1} = \mathbf{A}^i(w_k^i)\mathbf{x}_k + \mathbf{B}^i(w_k^i)\mathbf{u}_k + \mathbf{f}^i(w_k^i), \\ \mathbf{Q}^i \begin{bmatrix} \mathbf{x}_{k+j} \\ \mathbf{u}_{k+j} \end{bmatrix} \le \mathbf{q}^i \quad , \quad \mathbf{u}_{k+j-1} \in \mathbf{U}, \\ \mathbf{x}_{k+j} \in \hat{\mathbf{R}}_{k+j}^i, \quad \text{for} \quad j = 1, 2, \cdots, N \end{cases} \tag{15}$$

where $\mathbf{x_e}$ is the states reference, Λ, Γ are the weighting diagonal matrices in the sense $\|x\|_{\Lambda}^2 = x^T \Lambda x$.

Equation (15) is solved according to the following steps:

- solve this quadratic problem for each possible dynamic sequence among the s sub-models, Fig. 2, excluding the non feasible sequences,
- compute all the resulting costs,
- retain the model with the lowest cost and the associated control sequence,
- apply only the first value of this sequence and restart the procedure at the next sampling time.

Remark 3: If the initial state $\mathbf{x}_k$ is included in the union of regions $\mathbf{R}_{k-N}^i$ of different modes (i), the MPC technique can select a suboptimal solution among all feasible modes. The feasibility at instant k implies feasibility at any instant $k+1$ to $k+N$. The longest the prediction, the largest the feasible domain will be.

5. APPLICATION

Let consider as application of the previous theory the following benchmark consisting of two tanks (Figure 3), filled by pump acting on tank 1, continuously manipulated from 0 up to a maximum flow Q_1.

One switching valve V_{12} controls the flow between the tanks, this valve is assumed to be either completely opened or closed ($V_{12} = 1$ or 0 respectively). The V_{N2} manual valve controls the nominal outflow of the second tank. It is assumed in further simulations that the manual valves, V_{N1} is always closed and V_{N2} is open.

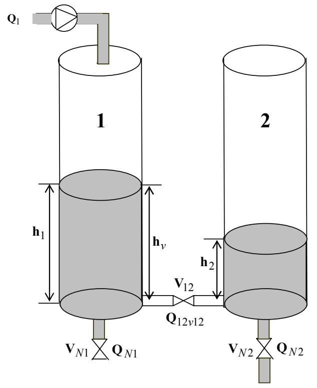

Fig. 3. Two-tank benchmark.

The liquid levels to be controlled are denoted h_1 and h_2 for each tank respectively. The conservation of mass in the tanks provides the following differential equations:

$$\dot{h}_1 = \frac{1}{A}(Q_1 - Q_{12V12})$$
$$\dot{h}_2 = \frac{1}{A}(Q_{12V12} - Q_{N2}) \tag{16}$$

where the Qs denote the flows and A is the cross-sectional area of each of the tanks. The Toricelli law defines the flows in the valves by following expressions:

$$Q_{12V12} = V_{12} a S_{12} sign(h_1 - h_2)\sqrt{|2g(h_1 - h_2)|}$$
$$Q_{N2} = V_{N2} a S_{N2} \sqrt{2gh_2} \tag{17}$$

where S_i represents the area of valves V_i and a is a constant depending on the liquid. From this, a simplified linear model can be obtained under the form:

$$Q_{12V12} \approx k_{12} V_{12} (h_1 - h_2)$$
$$Q_{N2} \approx k_{N2} V_{N2} h_2 \tag{18}$$
$$\text{where: } k_{12} = a S_{12}\sqrt{\frac{2g}{h_{\max}}}, \; k_{N2} = a S_{N2}\sqrt{\frac{2g}{h_{\max}}}$$

The Euler discretisation technique is used to further derive the discrete form :

$$h_1(k+1) = h_1(k) + \frac{T_s}{A}(Q_1(k) - k_{12}V_{12}(h_1(k) - h_2(k))$$
$$h_2(k+1) = h_2(k) + \tag{19}$$
$$\frac{T_s}{A}(k_{12}V_{12}(h_1(k) - h_2(k)) - k_{N2}V_{N2}h_2(k))$$

where T_s is the sampling time, equal to 10 s.

This benchmark can be considered as a piecewise affine system of form (1), with two subsystems (two modes), described as follows. For mode one which corresponds to valve $V_{12} = 1$ (open), two vertices for the uncertainty description are considered:

$$\mathbf{A}^{11} = \begin{bmatrix} 0.9188 & 0.0812 \\ 0.0812 & 0.8377 \end{bmatrix}, \quad \mathbf{B}^{11} = \begin{bmatrix} 721.5007 & 0 \\ 0 & 0 \end{bmatrix}$$
$$\mathbf{A}^{12} = \begin{bmatrix} 0.9336 & 0.0664 \\ 0.0664 & 0.8672 \end{bmatrix}, \quad \mathbf{B}^{12} = \begin{bmatrix} 590.3188 & 0 \\ 0 & 0 \end{bmatrix}$$
$$\mathbf{Q}^1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \quad \mathbf{q}^1 = \begin{bmatrix} 0.62 \\ 0.62 \\ 0 \\ 0 \\ 0.0001 \\ 0 \\ 1 \\ -1 \end{bmatrix}$$

For mode two which corresponds to valve $V_{12} = 0$ (closed), two vertices for the uncertainty description are also considered:

$$\mathbf{A}^{21} = \begin{bmatrix} 1 & 0 \\ 0 & 0.9188 \end{bmatrix}, \quad \mathbf{B}^{21} = \begin{bmatrix} 721.5007 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\mathbf{A}^{22} = \begin{bmatrix} 1 & 0 \\ 0 & 0.9336 \end{bmatrix}, \quad \mathbf{B}^{22} = \begin{bmatrix} 590.3188 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\mathbf{Q}^{2} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \quad \mathbf{q}^{2} = \begin{bmatrix} 0.62 \\ 0.62 \\ 0 \\ 0 \\ 0.0001 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The previous constraints have integrated limitations on the global state space:

$$\mathbf{X} := \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix}}_{\mathbf{F}_s} \mathbf{x} \le \underbrace{\begin{bmatrix} 0.62 \\ 0.62 \\ 0 \\ 0 \end{bmatrix}}_{\mathbf{g}_s} \qquad (20)$$

and limitations on the control signal as well where $\mathbf{u} = [Q_1 \quad V_{12}]'$. The target region, to which system states will be derived to, is defined by the following constraints:

$$\mathbf{R}_{k+N} := \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix}}_{\mathbf{F}} \mathbf{x} \le \underbrace{\begin{bmatrix} 0.55 \\ 0.25 \\ -0.45 \\ -0.15 \end{bmatrix}}_{\mathbf{g}} \qquad (21)$$

A polytope for bounded disturbance is finally considered with:

$$\begin{bmatrix} -0.01 \\ -0.01 \end{bmatrix} \le \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} \le \begin{bmatrix} 0.01 \\ 0.01 \end{bmatrix} \qquad (22)$$

The approach presented above is first applied to elaborate the region $\mathbf{R}_k$ in the state space which includes the states that can be derived in finite N steps to $\mathbf{R}_{k+N}$ despite both of disturbance and parameter variations.

Figure 4 presents the feasible regions for $N = 4$ where 16 regions are found according to the technique presented above (Fig. 1), and Figure 5 shows 72 feasible region for $N = 7$ where there is 56 empty regions.

The Multi-Parametric (MPT) toolbox (Kvasnica *et al.*, 2004) was used to deal with the polyhedral operations; to find the intersection, deleting the redundant constraints, Minkowsky subtraction, projections and also plotting the polyhedral regions.

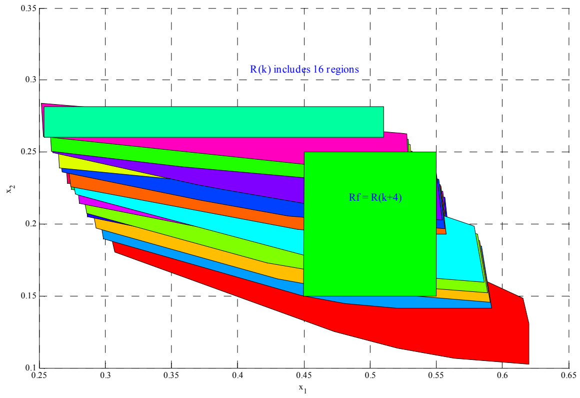

Fig. 4. 16 Regions for $N = 4$.

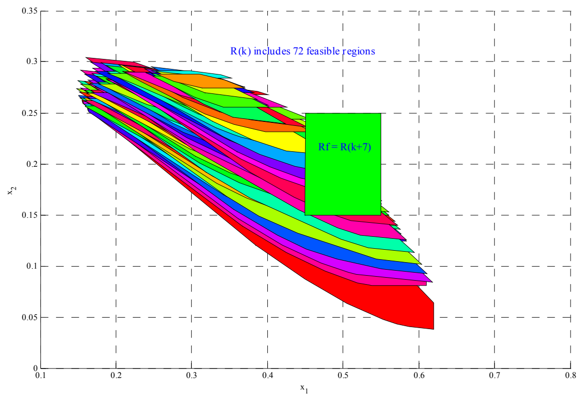

Fig. 5. 72 Regions for $N = 7$.

For $N = 10$ the regions evaluation is shown in Figure 6, where vertical axis corresponds to the sampling time (from 0 to N). Number of regions for $N = 10$ is 252 regions.

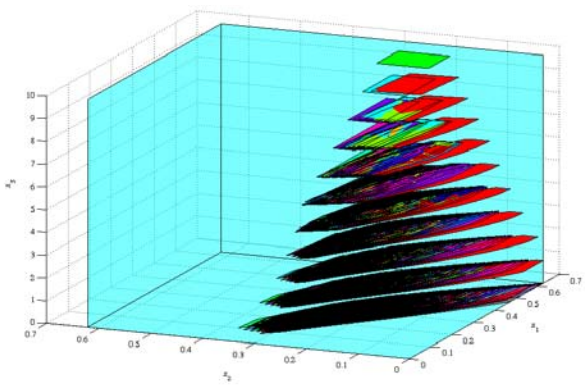

Fig. 6. Regions evaluations for $N = 10$.

The robust model predictive control presented above (15) is applied where the model of state evaluation is chosen to be the epicenter of the state matrix $\frac{1}{2}\left(\mathbf{A}^{i1} + \mathbf{A}^{i2}\right)$ for each mode, in order to remain in the middle of the two extreme vertices, since the current real matrix values are unknown (considering that the coefficients w are unknown and possibly time varying). The robust model predictive control is applied so many times, each with different initial states inside the region $\mathbf{R}_k$, and in each simulation a random uncertainty w is applied, as well as a random disturbance is added to the system.

The weighting diagonal terms in the cost function are chosen such that $\Lambda = 1000 * I_2$ and $\Gamma = 1$, and the states reference is $(0.5, 0.2)$.

Figure 7 shows some results of robust MPC with $N = 3$ for extreme initial states inside $\mathbf{R}_k$ with both of random uncertainty and random disturbance, and as Figure 7 shows, all the states in $\mathbf{R}_k$ are derived in three steps ($N = 3$) to the desired region $\mathbf{R}_{k+3}$ despite the parameter variations and the possible disturbance.

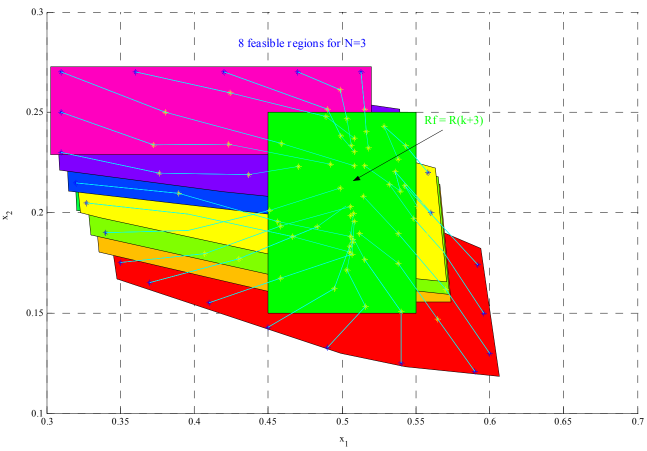

Fig. 7. Robust MPC for different initial states, with $N = 3$.

When talking about complexity, one has to mention that the convex regions computed here, $\mathbf{R}^i_{k-1}$ as in (13), are obtained in a dual representation (extreme points/constraints), which does not represent a computational challenge as long as the number of vertices does not increase (there are polytopic regions with either 4 or 5 vertices). This fact is strongly related to the particular shape of the target region. In this case, neither the projections nor the difference of polyhedral regions should require an important computational effort, keeping in mind that those calculations are made off-line.

Finally, in Figure 7 one can remark several state trajectories generated based on random uncertainty and random disturbance realizations validating any physical extreme combination of states.

6. CONCLUSION

This paper has examined a class of uncertain discrete-time piecewise affine systems with both of bounded disturbance and parameter variations, for which a polyhedral technique has been proposed to find the regions in the state space where a feasible mode and a robust control is assured to derive the system states to the desired region despite the parameter variation and the possible disturbance. Model predictive control technique has been proposed as a fast and suboptimal robust control for the considered problem. A validation for the proposed technique through a two tanks benchmark is presented.

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