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Off-line Robustification of Model Predictive Control for Uncertain Multivariable Systems

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Abstract: This paper proposes an off-line state-space control methodology for enhancing the robustness of multivariable Model Predictive Control (MPC) through the convex optimization of the Youla parameter. The Youla parameter-based optimization strategy allows convex specifications in closed-loop representation, focusing on the robustification of an initial controller using LMIs (Linear Matrix Inequalities) techniques. It is well established that such kind of robustification improves among others robustness towards unstructured uncertainties, however modifying the robustness of the initial controller towards system polytopic uncertainties. On the other hand, these polytopic uncertainties are not straightforward to deal with, imposing non-convex specifications in the Youla parameter. To overcome these difficulties, a novel structure is presented, including an additional convex condition on the Youla parameter to preserve robustness of the initial controller towards system polytopic uncertainties while managing the compromise with robust stability under unstructured uncertainties for the nominal controlled system. The potential of the developed robustified multivariable MPC controller is further illustrated in simulation on a stirred tank reactor.

1. INTRODUCTION

Studies on the robustness of Model Predictive Control (MPC) laws have developed significantly in recent years (Kothare *et al.*, 1996; Goulart and Kerrigan, 2007; Camacho and Bordons, 2004). However, many of them are on-line strategies, for which the computational load may become a limitation factor, sometimes inducing a loss of performance. In order to extend the applicability of robust MPC to large scale systems and fast processes, methods have been elaborated which compute off-line a set of controllers, while leaving on-line only the selection of current controller (Wan and Kothare, 2003; Lee and Kouvaritakis, 2006). Other off-line methods can be found in (Rossiter, 2003; Rodriguez and Dumur, 2005), the last one dealing with the transfer function formalism applied to SISO systems, which generalization to the multivariable case appears to be complicated.

This paper proposes a unified off-line methodology elaborating a unique robustified controller which guarantees robust stability of uncertain systems. As a starting point, an initial stabilizing multivariable controller with specified robustness properties towards polytopic uncertainties is considered. It is further robustified towards unstructured uncertainties through the convex optimization of the Youla parameter, leading to the minimisation of a H_∞ norm solved with Linear Matrix Inequality (LMI) tools (Stoica *et al.*, 2007). However this robustification modifies the robustness of the initial controller towards system polytopic uncertainties. The purpose of the proposed new structure is thus to preserve robustness under polytopic uncertainties by adding supplementary stability conditions to the previous LMI, in order to guarantee the robust stability on the entire domain. Since the considered uncertainty domain is chosen as a convex polytope, this involves checking stability of the controlled system only for the vertices of the polytope (Kothare *et al.*, 1996). The developed strategy overcomes the fact that polytopic uncertainties impose non-convex specifications in the Youla parameter, and results in managing the compromise between robustness under unstructured uncertainties and robustness towards polytopic uncertainties. One of the advantages of this robustification technique, completely formulated in the state-space framework, comes from the fact that it can handle very easily both SISO and multivariable systems.

This paper is organized as follows. Section 2 reminds the main steps leading to the design of an initial MIMO MPC in state-space formalism. The theoretical background required to formulate the robustification strategy via the Youla parametrization is presented in Section 3. Section 4 provides the main result, the elaboration through a state-space description of a robustified controller under both unstructured and polytopic uncertainties. This control strategy is applied in Section 5 to the control of a stirred tank reactor. Finally, some concluding remarks are presented in Section 6.

2. MODEL PREDICTIVE CONTROL DESIGN

This section focuses on the state-space procedure leading to the elaboration of the multivariable MPC law. Consider the following discrete time LTI (Linear Time Invariant) system (1) with m inputs and p outputs, with the system matrices $\mathbf{A} \in \mathbf{R}^{n\times n}$, $\mathbf{B} \in \mathbf{R}^{n\times m}$, $\mathbf{C} \in \mathbf{R}^{p\times n}$, the system states $\mathbf{x}$, the control input vector $\mathbf{u}$ and the system output vector $\mathbf{y}$.

$$\begin{cases} \mathbf{x}(k+1) = \mathbf{A}\,\mathbf{x}(k) + \mathbf{B}\mathbf{u}(k) \\ \mathbf{y}(k) = \mathbf{C}\,\mathbf{x}(k) \end{cases} \tag{1}$$

The elimination of steady-state errors is further achieved by adding an integral action on the control signal:

$$\mathbf{u}(k) = \mathbf{u}(k-1) + \Delta\mathbf{u}(k) \tag{2}$$

leading to the extended state-space formulation:

$$\begin{cases}\mathbf{x}_e(k+1) = \mathbf{A}_e\,\mathbf{x}_e(k) + \mathbf{B}_e\,\Delta\mathbf{u}(k)\\ \mathbf{y}(k) = \mathbf{C}_e\,\mathbf{x}_e(k)\end{cases} \tag{3}$$

with $\mathbf{x}_e(k) = \begin{bmatrix}\mathbf{x}(k)\\ \mathbf{u}(k-1)\end{bmatrix}, \mathbf{A}_e = \begin{bmatrix}\mathbf{A} & \mathbf{B}\\ \mathbf{0}_{m,n} & \mathbf{I}_m\end{bmatrix}, \mathbf{B}_e = \begin{bmatrix}\mathbf{B}\\ \mathbf{I}_m\end{bmatrix}, \mathbf{C}_e^{\mathrm{T}} = \begin{bmatrix}\mathbf{C}^{\mathrm{T}}\\ \mathbf{0}_{p,m}^{\mathrm{T}}\end{bmatrix}.$

The control signals $\Delta\mathbf{u}(k)$ result from the minimization of the following quadratic objective function with the weighting matrices $\widetilde{\mathbf{Q}}_J$ and $\widetilde{\mathbf{R}}_J$ and the setpoint $\mathbf{y}_r$:

$$J = \sum_{i=N_1}^{N_2} \left\|\hat{\mathbf{y}}(k+i) - \mathbf{y}_r(k+i)\right\|^2_{\widetilde{\mathbf{Q}}_J(i)} + \sum_{i=0}^{N_u-1} \left\|\Delta\mathbf{u}(k+i)\right\|^2_{\widetilde{\mathbf{R}}_J(i)} \tag{4}$$

assuming that the future control increments $\Delta\mathbf{u}(k+i)$ are zero for $i \geq N_u$ and using the same output prediction horizons N_1, N_2 and control horizons N_u for all input/output transfers. The predicted output vector can be written as:

$$\hat{\mathbf{y}}(k+i) = \mathbf{C}\,\mathbf{A}^i\,\hat{\mathbf{x}}(k) + \sum_{j=0}^{i-1} \mathbf{C}\,\mathbf{A}^{i-j-1}\mathbf{B}\underbrace{[\mathbf{u}(k-1) + \sum_{m=0}^{j}\Delta\mathbf{u}(k+m)]}_{\mathbf{u}(k+j)} \tag{5}$$

where the state estimate $\hat{\mathbf{x}}(k)$ is derived from the observer:

$$\hat{\mathbf{x}}_e(k+1) = \mathbf{A}_e\,\hat{\mathbf{x}}_e(k) + \mathbf{B}_e\,\Delta\mathbf{u}(k) + \mathbf{K}[\mathbf{y}(k) - \mathbf{C}_e\,\hat{\mathbf{x}}_e(k)] \tag{6}$$

The observer gain $\mathbf{K}$ is designed through a classical method of eigenvectors, arbitrarily placing the eigenvalues of $\mathbf{A}_e - \mathbf{K}\mathbf{C}_e$ in a stable region, as detailed in (Magni, 2002).

In order to obtain the control signals, the matrix formalism described in (Maciejowski, 2001) is used leading to (7), with the same control gain matrix $\mathbf{L}$ and the same setpoint pre-filter $\mathbf{F}_r$ (Fig. 5) as in (Stoica *et al.*, 2007).

$$\Delta\mathbf{u}(k) = \mathbf{F}_r\mathbf{y}_r(k+N_2) - \mathbf{L}\,\hat{\mathbf{x}}_e(k) \tag{7}$$

3. YOULA PARAMETRIZATION BACKGROUND

This section examines a technique that improves the robustness of the previous MIMO MPC law in terms of the Youla-Kucera parameter, also called $\mathbf{Q}$ parameter. It is proved in the literature (Boyd and Barratt, 1991; Maciejowski, 1989) that any stabilizing controller can be represented by a particular state-space feedback controller coupled with an observer and a Youla parameter. This part focuses on the main steps leading to the multivariable $\mathbf{Q}$ parameter that robustifies the MPC law described in Section 2.

3.1 Stabilizing control law

Starting from an initial controller, the class of all stabilizing controllers can be obtained via the Youla parametrization. The first step considers an additional input vector $\mathbf{u}'$ and output vector $\mathbf{y}'$ with a zero transfer between them (Fig. 1).

Next step is the addition of the Youla parameter between $\mathbf{y}'$ and $\mathbf{u}'$ without restricting closed-loop stability. In this case, the transfer from $\mathbf{u}$ to $\mathbf{y}$ remains unchanged. As a result, the closed-loop function between $\mathbf{w}$ and $\mathbf{z}$ is linearly parametrized by the $\mathbf{Q}$ parameter, allowing convex specifications (Boyd and Barratt, 1991):

$$\mathbf{T_{zw}} = \mathbf{T}_{11\,\mathbf{zw}} + \mathbf{T}_{12\,\mathbf{zw}}\,\mathbf{Q}\,\mathbf{T}_{21\,\mathbf{zw}} \tag{8}$$

with $\mathbf{T}_{11_{\mathbf{zw}}}, \mathbf{T}_{12\mathbf{zw}}, \mathbf{T}_{21\mathbf{zw}}$ depending on the considered input/output ($\mathbf{w}/\mathbf{z}$).

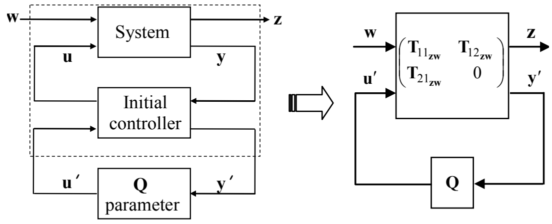

Fig. 1. Family of stabilizing controllers

3.2 Robust stability under unstructured uncertainties

Practical applications always deal with neglected dynamics and potential disturbances, so that robustness towards unstructured uncertainties Δ_u must be addressed (Fig. 2).

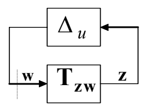

Fig. 2. Unstructured uncertainty

According to the small gain theorem (Maciejowski, 1989; Zhou *et al.*, 1996), robustness under unstructured uncertainties is maximized formulating a H_∞ norm minimization:

$$\min_{\mathbf{Q}\in\Re H_\infty} \left\|\mathbf{T_{zw}}\right\|_\infty \tag{9}$$

where the transfer $\mathbf{T_{zw}}$ also contains the weighting terms included to accomplish the desired robustness requirements.

The following theorem (see (Boyd *et al.*, 1994) for proof) formulates the previous H_∞ norm minimization.

Theorem 1 (Clement and Duc, 2000; Boyd *et al.*, 1994). A discrete time system $(\mathbf{A}_{cl}, \mathbf{B}_{cl}, \mathbf{C}_{cl}, \mathbf{D}_{cl})$ in the state-space formalism is stable and admits a H_∞ norm lower than γ iff:

$$\exists \mathbf{X}_1 = \mathbf{X}_1^{\mathrm{T}} \succ 0 / \begin{bmatrix} -\mathbf{X}_1^{-1} & \mathbf{A}_{cl} & \mathbf{B}_{cl} & \mathbf{0}\\ \mathbf{A}_{cl}^{\mathrm{T}} & -\mathbf{X}_1 & \mathbf{0} & \mathbf{C}_{cl}^{\mathrm{T}}\\ \mathbf{B}_{cl}^{\mathrm{T}} & \mathbf{0} & -\gamma\mathbf{I} & \mathbf{D}_{cl}^{\mathrm{T}}\\ \mathbf{0} & \mathbf{C}_{cl} & \mathbf{D}_{cl} & -\gamma\mathbf{I}\end{bmatrix} \prec 0 \tag{10}$$

where the notation '$\succ 0$'/'$\prec 0$' denotes respectively a strictly positive/negative definite matrix. The expression (10) can be transformed into a LMI, which decision variables are $\mathbf{X}_1$, γ and the $\mathbf{Q}$ parameter included in the closed-loop matrices, as shown in (Clement and Duc, 2000; Scherer, 2000). As a result, the optimization problem is formulated as the minimization of γ under this LMI constraint.

3.3 *Robust stability under polytopic uncertainties*

The aim of this part is to guarantee robust stability for systems under polytopic uncertainties. Let us consider the following time-varying system, as a generalization of the polytopic system described in (Kothare *et al.*, 1996):

$$\begin{cases} \mathbf{x}(k+1) = \mathbf{A}(k)\mathbf{x}(k) + \mathbf{B}(k)\mathbf{u}(k) \\ \mathbf{y}(k) = \mathbf{C}(k)\mathbf{x}(k) \end{cases} \quad (11)$$

where $[\mathbf{A}(k)\ \mathbf{B}(k)\ \mathbf{C}(k)] \in \mathbf{\Omega}$ (Fig.3), denoted by:

$$\mathbf{\Omega} = \mathrm{Co}\{[\mathbf{A}_1\ \mathbf{B}_1\ \mathbf{C}_1], [\mathbf{A}_2\ \mathbf{B}_2\ \mathbf{C}_2], \cdots, [\mathbf{A}_l\ \mathbf{B}_l\ \mathbf{C}_l]\} \quad (12)$$

with Co the notation for the convex hull defined by vertices $[\mathbf{A}_i\ \mathbf{B}_i\ \mathbf{C}_i]$. This means that if $[\mathbf{A}\ \mathbf{B}\ \mathbf{C}] \in \mathbf{\Omega}$, then $[\mathbf{A}\ \ \mathbf{B}\ \ \mathbf{C}] = \sum_{i=1}^{l} \lambda_i [\mathbf{A}_i\ \ \mathbf{B}_i\ \ \mathbf{C}_i]$ for $\lambda_i \geq 0$, $i = \overline{1, l}$, $\sum_{i=1}^{l} \lambda_i = 1$.

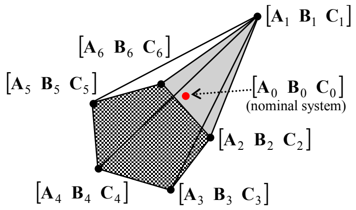

Fig. 3. Polytopic uncertainty representation with $l = 6$

As $\mathbf{\Omega}$ is a polytope, thus a convex set, guaranteeing the stability of (11) on the entire space $\mathbf{\Omega}$ is equivalent to guarantee the stability for all the vertices of the polytope (Kothare *et al.*, 1996).

In this multi-model case, the transfer from $\mathbf{w}$ to $\mathbf{z}$ has the expression (13). The difficulty appears due to the presence of the non-zero transfer $\mathbf{T}_{22_{\mathbf{zw}}}$. It can be noticed easily that if $\mathbf{T}_{22_{\mathbf{zw}}} = 0$ (Fig. 1), then the expression (13) is reduced to (8). Theorem 2 contains a significant related result.

$$\mathbf{T}_{\mathbf{zw}} = \mathbf{T}_{11_{\mathbf{zw}}} + \mathbf{T}_{12_{\mathbf{zw}}} (\mathbf{I} - \mathbf{Q}\mathbf{T}_{22_{\mathbf{zw}}})^{-1} \mathbf{Q}\mathbf{T}_{21_{\mathbf{zw}}} \quad (13)$$

Theorem 2. The transfer $\mathbf{T}_{\mathbf{zw}}$ given in (13) exists and is stable if the transfer $\widetilde{\mathbf{T}} = (\mathbf{I} - \mathbf{Q}\mathbf{T}_{22_{\mathbf{zw}}})^{-1}$ exists and is stable.

Sketch of proof: Looking at the state-space form of the transfers $\mathbf{T}_{\mathbf{zw}} = \mathbf{T}_{11_{\mathbf{zw}}} + \mathbf{T}_{12_{\mathbf{zw}}} \widetilde{\mathbf{T}} \mathbf{Q} \mathbf{T}_{21_{\mathbf{zw}}}$ and $\widetilde{\mathbf{T}}$, it appears that they have the same evolution matrix and implicitly the same eigenvalues. Thus $\widetilde{\mathbf{T}}$ stable implies that $\mathbf{T}_{\mathbf{zw}}$ is stable. ■

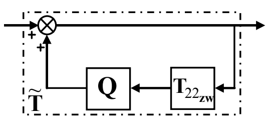

Fig. 4. Block diagram of $\widetilde{\mathbf{T}}$

Checking stability of $\widetilde{\mathbf{T}}$ for each vertex of the polytope leads to a non-convex problem. However if $\mathbf{T}_{22_{\mathbf{zw}}}$ is stable, the small gain theorem (Fig.4) can be applied to $\widetilde{\mathbf{T}}$, leading to a conservative solution:

$$\text{If } \left\| \mathbf{Q}\mathbf{T}_{22_{\mathbf{zw}}} \right\|_\infty < 1 \text{ then } \widetilde{\mathbf{T}} \text{ is stable} \quad (14)$$

The submultiplicative property of the H_∞ norm provides $\left\| \mathbf{Q}\mathbf{T}_{22_{\mathbf{zw}}} \right\|_\infty \leq \left\| \mathbf{Q} \right\|_\infty \left\| \mathbf{T}_{22_{\mathbf{zw}}} \right\|_\infty$, which implies that $\left\| \mathbf{Q}\mathbf{T}_{22_{\mathbf{zw}}} \right\|_\infty < 1$ is satisfied if:

$$\left\| \mathbf{Q} \right\|_\infty < \frac{1}{\left\| \mathbf{T}_{22_{\mathbf{zw}}} \right\|_\infty} \quad (15)$$

Condition (15) can be transformed into a LMI which is further added to the H_∞ norm minimization of Section 3.2 in order to guarantee the robust stability under polytopic uncertainties on the entire uncertain domain.

Hence, the expressions (9) and (15) guarantee robust stability under both unstructured and polytopic uncertainties.

4. OFF-LINE ROBUSTIFICATION PROCEDURE

The robustification strategy based on the $\mathbf{Q}$ parameter developed in Section 3 is now applied to an initial stabilizing MIMO MPC from Section 2. To shorten the presentation, it will be considered in the following only maximization of robustness under additive unstructured uncertainties. This maximization is equivalent to the minimization of $\left\| \mathbf{T}_{\mathbf{zb}} \right\|_\infty$ (Fig. 5), which will be solved using LMIs techniques. On the other hand, when dealing with polytopic uncertainties, the following situation can appear: the initial stabilizing MIMO controller can loose its stability property in some regions of the polytopic domain after robustification towards unstructured uncertainties (as the result of the optimization compromise); this aspect is the key motivation of the paper. Therefore supplementary stability conditions have to be added in order to guarantee robust stability on the entire validity domain, even if the consequence can be a decrease of robustness under unstructured uncertainties for the nominal controlled system. As mentioned in Section 3, guaranteeing robust stability under polytopic uncertainties will add other LMIs to the previous optimization problem. The result will be a necessary compromise between both robustness aspects.

4.1 *Stabilizing control law*

Consider the LTI discrete time MIMO system (3) (Fig. 5). Using additional inputs $\mathbf{u}'$ and outputs $\mathbf{y}'$ (Boyd and Barratt, 1991), the control law (7) applied to this system is:

$$\Delta\mathbf{u}(k) = \mathbf{F}_r \mathbf{y}_r (k + N_2) - \mathbf{L}\,\hat{\mathbf{x}}_e(k) - \mathbf{u}'(k) \quad (16)$$

with the following observer:

$$\hat{\mathbf{x}}_e(k+1) = \mathbf{A}_e\, \hat{\mathbf{x}}_e(k) + \mathbf{B}_e \Delta\mathbf{u}(k) + \mathbf{K}[\mathbf{y}(k) - \mathbf{C}_e \hat{\mathbf{x}}_e(k) + \mathbf{b}(k)] \quad (17)$$

To calculate the closed-loop transfer, the initial state-space form (3) is then extended by adding the prediction error:

$$\boldsymbol{\varepsilon}(k) = \mathbf{x}_e(k) - \hat{\mathbf{x}}_e(k) \quad (18)$$

Only the terms related to $\mathbf{b}(k)$ are considered in the following, as they are part of the minimization process:

$$\begin{bmatrix}\mathbf{x}_e(k+1)\\ \boldsymbol{\varepsilon}(k+1)\end{bmatrix}=\begin{bmatrix}\mathbf{A}_1 & \mathbf{A}_3\\ \mathbf{0} & \mathbf{A}_2\end{bmatrix}\begin{bmatrix}\mathbf{x}_e(k)\\ \boldsymbol{\varepsilon}(k)\end{bmatrix}+\begin{bmatrix}\mathbf{0} & -\mathbf{B}_e\\ -\mathbf{K} & \mathbf{0}\end{bmatrix}\begin{bmatrix}\mathbf{b}(k)\\ \mathbf{u}'(k)\end{bmatrix} \tag{19}$$

$$\mathbf{y}'(k)=\begin{bmatrix}\mathbf{0} & \mathbf{C}_e\end{bmatrix}\begin{bmatrix}\mathbf{x}_e(k)\\ \boldsymbol{\varepsilon}(k)\end{bmatrix}+\begin{bmatrix}\mathbf{I} & \mathbf{0}\end{bmatrix}\begin{bmatrix}\mathbf{b}(k)\\ \mathbf{u}'(k)\end{bmatrix} \tag{20}$$

with $\mathbf{A}_1=\mathbf{A}_e-\mathbf{B}_e\mathbf{L}$, $\mathbf{A}_2=\mathbf{A}_e-\mathbf{K}\mathbf{C}_e$, $\mathbf{A}_3=\mathbf{B}_e\mathbf{L}$.

According to the theory given in 3.1, the Youla parameter can be added to robustify the initial controller, since the transfer between $\mathbf{y}'(k)$ and $\mathbf{u}'(k)$ is zero (without measurement noise, the output $\mathbf{y}'$ depends only on $\boldsymbol{\varepsilon}(k)$, which is independent from $\mathbf{x}_e(k)$ and $\mathbf{u}'(k)$). All these state-space equations lead to the diagram of Fig. 5.

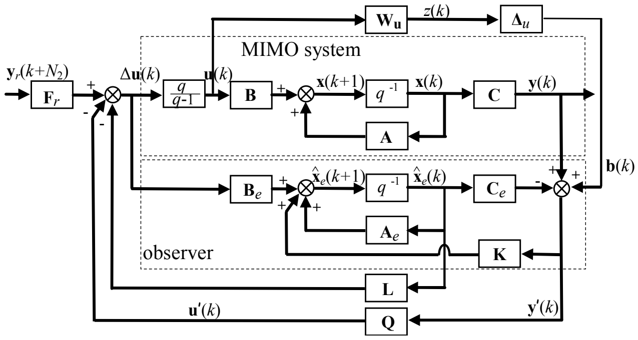

Fig. 5. Stabilizing MPC using Youla parametrization

4.2 Robust stability under unstructured uncertainties

Next step is the definition of the weighting as a high-pass filter with the following state-space representation:

$$\begin{cases}\mathbf{x}_w(k+1)=\mathbf{A}_w\mathbf{x}_w(k)+\mathbf{B}_w\mathbf{u}(k)\\ \mathbf{z}(k)=\mathbf{C}_w\mathbf{x}_w(k)+\mathbf{D}_w\mathbf{u}(k)\end{cases} \tag{21}$$

with $\mathbf{A}\in\mathbf{R}^{n_w\times n_w}$, $\mathbf{B}\in\mathbf{R}^{n_w\times m}$, $\mathbf{C}\in\mathbf{R}^{p\times n_w}$ and $\mathbf{D}\in\mathbf{R}^{p\times m}$. Including the $\mathbf{W_u}$ weighting, a new extended state-space description can be emphasized:

$$\begin{bmatrix}\overline{\mathbf{x}}_1(k+1)\\ \boldsymbol{\varepsilon}(k+1)\end{bmatrix}=\begin{bmatrix}\overline{\mathbf{A}}_1 & \overline{\mathbf{A}}_3\\ \mathbf{0} & \mathbf{A}_2\end{bmatrix}\begin{bmatrix}\overline{\mathbf{x}}_1(k)\\ \boldsymbol{\varepsilon}(k)\end{bmatrix}+\begin{bmatrix}\mathbf{0} & -\overline{\mathbf{B}}_{u_1'}\\ -\mathbf{K} & \mathbf{0}\end{bmatrix}\begin{bmatrix}\mathbf{b}(k)\\ \mathbf{u}'(k)\end{bmatrix} \tag{22}$$

$$\begin{bmatrix}\mathbf{z}(k)\\ \mathbf{y}'(k)\end{bmatrix}=\begin{bmatrix}\overline{\mathbf{C}}_1 & \overline{\mathbf{C}}_2\\ \mathbf{0} & \mathbf{C}_e\end{bmatrix}\begin{bmatrix}\overline{\mathbf{x}}_1(k)\\ \boldsymbol{\varepsilon}(k)\end{bmatrix}+\begin{bmatrix}\mathbf{0} & -\mathbf{D}_w\\ \mathbf{I} & \mathbf{0}\end{bmatrix}\begin{bmatrix}\mathbf{b}(k)\\ \mathbf{u}'(k)\end{bmatrix} \tag{23}$$

where $\overline{\mathbf{x}}_1(k)=[\mathbf{x}^{\mathrm{T}}(k)\ \ \mathbf{u}^{\mathrm{T}}(k-1)\ \ \mathbf{x}_w^{\mathrm{T}}(k)]^{\mathrm{T}}$,

$$\overline{\mathbf{A}}_1=\begin{bmatrix}\mathbf{A}-\mathbf{B}\mathbf{L}_1 & \mathbf{B}-\mathbf{B}\mathbf{L}_2 & \mathbf{0}\\ -\mathbf{L}_1 & \mathbf{I}-\mathbf{L}_2 & \mathbf{0}\\ -\mathbf{B}_w\mathbf{L}_1 & \mathbf{B}_w(\mathbf{I}-\mathbf{L}_2) & \mathbf{A}_w\end{bmatrix},\ \overline{\mathbf{A}}_3=\begin{bmatrix}\mathbf{B}\mathbf{L}\\ \mathbf{L}\\ \mathbf{B}_w\mathbf{L}\end{bmatrix},\ \overline{\mathbf{B}}_{u_1'}=\begin{bmatrix}\mathbf{B}\\ \mathbf{I}\\ \mathbf{B}_w\end{bmatrix},$$

$$\overline{\mathbf{C}}_1=[-\mathbf{D}_w\mathbf{L}_1\ \ \mathbf{D}_w(\mathbf{I}-\mathbf{L}_2)\ \ \mathbf{C}_w],\ \overline{\mathbf{C}}_2=\mathbf{D}_w\mathbf{L}.$$

As described in Section 3.2, a multivariable Youla parameter is added for robustification purposes. Since $\mathbf{Q}\in\Re H_\infty$, a sub-optimal solution is to consider for each input/output transfer a finite-dimensional subspace generated by an orthonormal base of discrete stable transfer functions such as polynomial or Finite Impulse Response (FIR) filters:

$$Q^{ij}=\sum_{t=0}^{n_Q}q_t^{ij}q^{-t}\text{, with } 1\le i\le m,\ 1\le j\le p \tag{24}$$

A state-space representation of this Youla parameter can be obtained using a fixed pair $(\mathbf{A}_Q,\mathbf{B}_Q)$, thus only the variable pair $(\mathbf{C}_Q,\mathbf{D}_Q)$ must be designed (Clement and Duc, 2000):

$$\begin{cases}\mathbf{x}_Q(k+1)=\mathbf{A}_Q\mathbf{x}_Q(k)+\mathbf{B}_Q\mathbf{y}'(k)\\ \mathbf{u}'(k)=\mathbf{C}_Q\mathbf{x}_Q(k)+\mathbf{D}_Q\mathbf{y}'(k)\end{cases} \tag{25}$$

with $\mathbf{A}_Q=diag(\mathbf{a}_Q,\cdots,\mathbf{a}_Q)$, $\mathbf{B}_Q=diag(\mathbf{b}_Q,\cdots,\mathbf{b}_Q)$,

$$\mathbf{C}_Q=\begin{bmatrix}\mathbf{c}_Q^{11} & \cdots & \mathbf{c}_Q^{1p}\\ \vdots & \ddots & \vdots\\ \mathbf{c}_Q^{m1} & \cdots & \mathbf{c}_Q^{mp}\end{bmatrix},\ \mathbf{D}_Q=\begin{bmatrix}d_Q^{11} & \cdots & d_Q^{1p}\\ \vdots & \ddots & \vdots\\ d_Q^{m1} & \cdots & d_Q^{mp}\end{bmatrix}\text{ and denoting}$$

$$\mathbf{a}_Q=\begin{bmatrix}\mathbf{0}_{1,n_Q-1} & 0\\ \mathbf{I}_{n_Q-1} & \mathbf{0}_{n_Q-1,1}\end{bmatrix},\ \mathbf{b}_Q=\begin{bmatrix}1\\ \mathbf{0}_{n_Q-1,1}\end{bmatrix},\ \mathbf{c}_Q^{ij}=\begin{bmatrix}q_1^{ij}\\ \vdots\\ q_{n_Q}^{ij}\end{bmatrix}^{\mathrm{T}},\ d_Q^{ij}=q_0^{ij}.$$

Adding the $\mathbf{Q}$ parameter leads to the state-space formulation:

$$\begin{cases}\mathbf{x}_{cl}(k+1)=\mathbf{A}_{cl}\mathbf{x}_{cl}(k)+\mathbf{B}_{cl}\mathbf{b}(k)\\ \mathbf{z}(k)=\mathbf{C}_{cl}\mathbf{x}_{cl}(k)+\mathbf{D}_{cl}\mathbf{b}(k)\end{cases} \tag{26}$$

where $\mathbf{x}_{cl}=\begin{bmatrix}\overline{\mathbf{x}}_1\\ \boldsymbol{\varepsilon}\\ \mathbf{x}_Q\end{bmatrix}$, $\mathbf{A}_{cl}=\begin{bmatrix}\overline{\mathbf{A}}_1 & \overline{\mathbf{A}}_3-\overline{\mathbf{B}}_{u_1'}\mathbf{D}_Q\mathbf{C}_e & -\overline{\mathbf{B}}_{u_1'}\mathbf{C}_Q\\ \mathbf{0} & \mathbf{A}_2 & \mathbf{0}\\ \mathbf{0} & \mathbf{B}_Q\mathbf{C}_e & \mathbf{A}_Q\end{bmatrix}$,

$\mathbf{B}_{cl}=\begin{bmatrix}-\mathbf{D}_Q\overline{\mathbf{B}}_{u_1'}^{\mathrm{T}} & -\mathbf{K}^{\mathrm{T}} & \mathbf{B}_Q^{\mathrm{T}}\end{bmatrix}^{\mathrm{T}}$, $\mathbf{C}_{cl}=\begin{bmatrix}\overline{\mathbf{C}}_1 & \overline{\mathbf{C}}_2-\mathbf{D}_w\mathbf{D}_Q\mathbf{C}_e & -\mathbf{D}_w\mathbf{C}_Q\end{bmatrix}$,

$\mathbf{D}_{cl}=-\mathbf{D}_w\mathbf{D}_Q$.

This closed-loop state-space formulation $(\mathbf{A}_{cl},\mathbf{B}_{cl},\mathbf{C}_{cl},\mathbf{D}_{cl})$ is the crucial point of the robustification method. Using the result of Theorem 1, the expression (10) can be transformed into a first LMI. Finally, the robustification under additive unstructured uncertainties can be transformed into (27), where $LMI1$ is the inequality (26) from (Stoica *et al.*, 2007):

$$\min_{LMI1}\gamma \tag{27}$$

4.3 Robust stability under polytopic uncertainties

This subsection describes the robustification methodology in the case of polytopic uncertain systems (11). The procedure is the following: firstly an initial stabilizing controller is developed for the nominal plant, and is further robustified under unstructured uncertainties based on the $\mathbf{Q}$ parameter. To overcome the case for which the polytopic system controlled by this robustified controller may become unstable, the procedure described in Section 3.3 is applied to guarantee the stability for every vertex of the polytopic domain.

Using Theorem 1, the expression (15) can be rewritten as:

$$\begin{bmatrix} -\mathbf{X}_2 & \mathbf{X}_2\mathbf{A}_Q & \mathbf{X}_2\,\mathbf{B}_Q & \mathbf{0} \\ \mathbf{A}_Q^{\mathrm{T}}\,\mathbf{X}_2 & -\mathbf{X}_2 & \mathbf{0} & \mathbf{C}_Q^{\mathrm{T}} \\ \mathbf{B}_Q^{\mathrm{T}}\,\mathbf{X}_2 & \mathbf{0} & -\varepsilon\,\mathbf{I} & \mathbf{D}_Q^{\mathrm{T}} \\ \mathbf{0} & \mathbf{C}_Q & \mathbf{D}_Q & -\varepsilon\,\mathbf{I} \end{bmatrix} \prec 0 \tag{28}$$

with $\varepsilon = \max(\varepsilon_i)$ and $\varepsilon_i = 1/\left\|\mathbf{T}_{22\,i_{\mathbf{zb}}}\right\|_\infty$, where $\mathbf{T}_{22\,i_{\mathbf{zb}}}$ is the transfer $\mathbf{T}_{22_{\mathbf{zb}}}$ of the i^{th} input/output channel. The expression (28) is a LMI having $\mathbf{X}_2$, $\mathbf{C}_Q$, and $\mathbf{D}_Q$ as decision variables. In order to guarantee the robust stability under both additive unstructured and polytopic uncertainties, a sub-optimal solution is to verify (29), where $LMI2$ is the inequality (28).

$$\min_{LMI1,LMI2} \gamma \tag{29}$$

5. EXAMPLE

This section focuses on the results obtained while applying the previous robustification methodology to an academic example consisting into a simplified multivariable model of a stirred tank reactor. Let us consider as nominal system, the transfer function formulation (Camacho and Bordons, 2004):

$$\begin{bmatrix} Y_1(s) \\ Y_2(s) \end{bmatrix} = \begin{bmatrix} 1/(1+0.7s) & 5/(1+0.3s) \\ 1/(1+0.5s) & 2/(1+0.4s) \end{bmatrix} \begin{bmatrix} U_1(s) \\ U_2(s) \end{bmatrix} \tag{30}$$

with the effluent concentration Y_1, the reactor temperature Y_2, the feed flow rate U_1 and the coolant flow U_2. An equivalent discretized (with a sampling period $T_e = 0.03$ minute) state-space representation $(\mathbf{A}_0, \mathbf{B}_0, \mathbf{C}_0)$ of the nominal system (1) is obtained with:

$$\mathbf{A}_0 = diag\big(0.9580,\ 0.9418,\ 0.9048,\ 0.9277\big),$$

$$\mathbf{B}_0^T = \begin{bmatrix} 0.25 & 0.25 & 0 & 0 \\ 0 & 0 & 0.25 & 0.25 \end{bmatrix}, \mathbf{C}_0 = \begin{bmatrix} 0.1678 & 0 & 0.9516 & 0 \\ 0 & 0.2329 & 0 & 0.2890 \end{bmatrix}$$

To find an initial stabilizing MPC (called MPC0) for the nominal system, an integral action is first added, leading to an extended model belonging to $\mathbf{R}^6$. The same tuning parameters $N_1 = 1$, $N_2 = 3$, $N_u = 2$ are chosen for all outputs and control signals, and the same weights $\widetilde{\mathbf{Q}}_J = \mathbf{I}_{N_2-N_1+1}$ and $\widetilde{\mathbf{R}}_J = 0.05\mathbf{I}_{N_u}$, as in (Camacho and Bordons, 2004).

It is now considered that the nominal system is affected by high frequency neglected dynamics that can be represented as additive unstructured uncertainties. The previous initial controller is thus robustified, providing a controller (called MPC1) which will guarantee the stability robustness under additive unstructured uncertainties. Following the procedure described in Section 4.2, the state-space representation of the following weighting $\mathbf{W_u}$ is considered:

$$\mathbf{W_u}(q^{-1}) = diag\Big((1-0.7q^{-1})/0.3, \quad (1-0.7q^{-1})/0.3\Big) \tag{31}$$

Solving the optimization problem (27) provides a multivariable Youla parameter of a chosen order $n_Q = 40$. Analyzing the singular values before robustification (MPC0) and after robustification under additive unstructured uncertainties (MPC1) from Fig. 6a, we can remark that the H_∞ norm (which is the greatest value of the maximal singular values) has been reduced using MPC1. Therefore the robust stability for the nominal system with respect to additive unstructured uncertainties is improved.

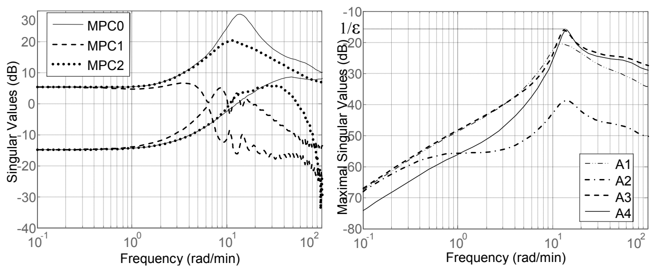

Fig 6. a. Singular values of $\mathbf{T_{zb}}$ (with the nominal system) before and after robustification (left); b. Maximal singular values of $\mathbf{T}_{22_{\mathbf{zb}}}$ transfer (right)

The next part refers to the robustification under polytopic uncertainties. Let consider that the polytopic system (11) where the matrix $\mathbf{A} \in \mathbf{\Omega} = \text{Co}\{\mathbf{A}_1,\ \mathbf{A}_2,\ \mathbf{A}_3,\ \mathbf{A}_4\}$ has the following explicit form:

$$\mathbf{A}(k) = diag\big(0.9580,\ 0.9418,\ 0.9048(1+\alpha_2),\ 0.9277(1+\alpha_1)\big) \tag{32}$$

with $\alpha_1 \in [\underline{\alpha}_1; \overline{\alpha}_1]$, $\alpha_2 \in [\underline{\alpha}_2; \overline{\alpha}_2]$ and $\underline{\alpha}_1 = -0.3$, $\underline{\alpha}_2 = 0$, $\overline{\alpha}_1 = 0.03$; $\overline{\alpha}_2 = 0.06$. This leads to the next vertices of $\mathbf{\Omega}$:

$\mathbf{A}_1 = diag\big(0.9580,\ 0.9418,\ 0.9048(1+\overline{\alpha}_2),\ 0.9277(1+\overline{\alpha}_1)\big)$,

$\mathbf{A}_2 = diag\big(0.9580,\ 0.9418,\ 0.9048(1+\underline{\alpha}_2),\ 0.9277(1+\overline{\alpha}_1)\big)$,

$\mathbf{A}_3 = diag\big(0.9580, 0.9418, 0.9048(1+\overline{\alpha}_2), 0.9277(1+\underline{\alpha}_1)\big)$,

$\mathbf{A}_4 = diag\big(0.9580, 0.9418, 0.9048(1+\underline{\alpha}_2), 0.9277(1+\underline{\alpha}_1)\big)$.

The nominal system corresponds to $\alpha_1 = \alpha_2 = 0$.

It can be proved that the closed-loop of the polytopic uncertain system with the initial controller MPC0 is stable for all vertices, but that MPC1 destabilizes the polytopic system (32) for $\mathbf{A}_3$, since there exists an eigenvalue with a magnitude of 1.0083. This difficulty justifies the design of the robustified controller MPC2 that will guarantee the robustness towards polytopic uncertainties. In order to find MPC2, it can be noticed in Fig. 6b that the maximum of the transfer $\left\|\mathbf{T}_{22_{\mathbf{zb}}}\right\|_\infty$ is -16 dB that corresponds to $\varepsilon = 6.3$ from (28).

With the controller MPC2, a stable closed-loop with the polytopic system (32) is obtained, even if the robust stability for the nominal system is decreased in comparison with the result obtained with MPC1 for the nominal system (Fig. 6a). Indeed, a compromise is achieved: with MPC2 the robustness under unstructured uncertainties is less improved compared to MPC1 (still remaining in acceptable limits), but the robustness under polytopic uncertainties is satisfied. Fig. 7a shows the maximal singular values of the $\mathbf{Q}$ parameter obtained with MPC1 and MPC2. Notice that using MPC2 the H_∞ norm of the Youla parameter is decreased in order to satisfy $\left\|\mathbf{Q}\mathbf{T}_{22_{\mathbf{zb}}}\right\|_\infty < 1$, guaranteeing the robust stability under the considered polytopic uncertainties.

The size of the optimization problem (27) is determined by $n_{1,\text{var}}$ scalar decision variables:

$$n_{1,\text{var}} = 0.5p^2 n_Q^{\ 2} + pn_Q(2n + 3m + n_w + 0.5) + \\ + (n+m)(2n + 2m + 2n_w + 1) + 0.5n_w(n_w + 1) + pm + 1 \quad (33)$$

The addition of LMI (28) increases this number $n_{1,\text{var}}$ by $0.5pn_Q(pn_Q+1)$ which corresponds to $\mathbf{X}_2$, leading to the following number of scalar decision variables:

$$n_{2,\text{var}} = p^2 n_Q^{\ 2} + pn_Q(2n + 3m + n_w + 1) + \\ + (n+m)(2n + 2m + 2n_w + 1) + 0.5n_w(n_w + 1) + pm + 1 \quad (34)$$

It should be noticed that the number of scalar decision variables (34) can be reduced to (33) using the relaxation condition: $\mathbf{X}_2$ equals to the part of $\mathbf{X}_1$ in (10) which multiplies the **Q** parameter.

Finally, a high-frequency neglected dynamics is considered acting on the u_1 actuator, so that the y_1/u_1 transfer corresponds to $1/[(1+0.7s)(1+0.07s)]$. As shown in Fig. 7b, this neglected dynamics destabilizes the initial controller MPC0, but it is also illustrated that the robustified controllers MPC1 and MPC2 remain stable. Figure 7b also shows the coupling influence of the considered neglected dynamics on y_2. In Fig. 8 the same neglected dynamics is acting on $\mathbf{A}_3$ and here only MPC2 remains stable.

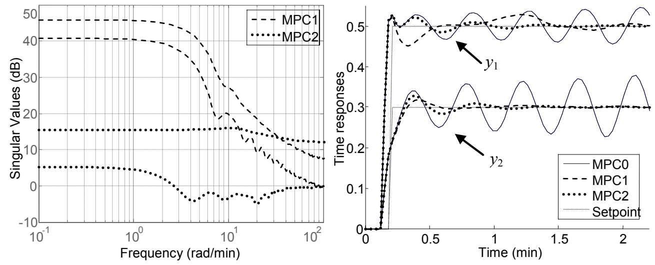

Fig. 7. a Singular values of **Q** parameter(left), b. Influence of a neglected dynamics on y_1/u_1 transfer (right)

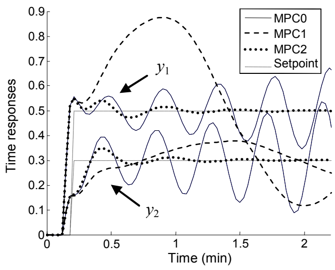

Fig. 8. Influence in $\mathbf{A}_3$ of a neglected dynamics on A_3

As mentioned in previous sections, the proposed methodology guarantees the robustness towards neglected dynamics acting only on the nominal system. The problem of guaranteeing explicitly the robustness towards neglected dynamics for the entire polytopic domain is a non-convex problem for which solutions have to be found.

6. CONCLUSIONS

This paper has presented a complete methodology which enables robustifying an initial multivariable MPC controller in state-space formalism using the Youla parameter framework, formulated as a convex optimization problem solved with LMIs tools. The major advantage consists in managing the compromise between robust stability under unstructured uncertainties for a nominal system and the robust stability under polytopic uncertainties for an entire variation domain, considered as a polytopic domain enclosing the nominal system. Therefore this method can ensure with only one robustified controller robustness under polytopic uncertainties, with a trade-off that may decrease (within acceptable limits) the robust stability under unstructured uncertainties for the nominal system.

Perspectives mainly focus on the possibility of finding a solution for the non-convex problem of the stability robustness for all the vertices of the polytopic uncertain domain.

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