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Absolute Stability of Multivariable Lur'e-type Descriptor Systems ⋆

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Abstract: The purpose of this paper is to derive conditions for absolute stability as well as existence of solutions for multivariable Lur'e-type feedback systems whose linear part is expressed by a descriptor system. The nonlinearities are uncertain and satisfy multivariable sector conditions, or a part of the nonlinearities satisfies a norm bounded condition. Thus, the systems can be refered to as multivariable Lur'e-type descriptor systems. In the existing works on Lur'e-type descriptor systems, the nonlinearities were assumed to be smooth or given as a set of single-variable scalar functions, while in this paper, the smoothness assumption is relaxed and multivariable vector-valued nonlinearities are considered. The obtained stability conditions are described in terms of linear matrix inequalities, which are extensions of the authors' previous results on extended Popov criteria for multivariable Lur'e systems whose linear part is expressed by a state-space equation.

1. INTRODUCTION

Nonlinear dynamical systems with algebraic constraints are often appeared in control problems. As their model representations, nonlinear descriptor equations are useful and natural since they have much more flexibility in describing nonlinear systems than state equations, that is, they can simultaneously represent static constraints as well as dynamical parts of systems, and preserve physical parameters. Descriptor systems are referred to as semistate equations or differential/algebraic systems, and their theory were applicable to many physical systems (Newcomb [1981], Newcomb and Dziurla [1989], and Campbell et al. [1999], etc.).

In general, any autonomous nonlinear system with algebraic constraints can be represented by a feedback loop (Fig. 1) composed of a linear descriptor system block and a static nonlinear block if the descriptor variable includes the system variable, its derivatives, all of the inputs and outputs of all nonlinearities. When the static nonlinearity is uncertain and satisfies a multivariable sector condition, the feedback system represented by Fig. 1 can be considered as a multivariable Lur'e-type feedback system whose linear part is a descriptor system. Let us refer to it as a multivariable Lur'e-type descriptor system.

The absolute stability as well as the solvability of Lur'e-type descriptor systems was considered by Yang et al. [2007a,c,b], where the nonlinearities were assumed to be smooth enough and specific absolute stability called generalized (or strongly) absolute stability was considerd. In [Yang et al., 2007a,c], a quadratic Liapunov function was used, which is often appeared in stability problems for

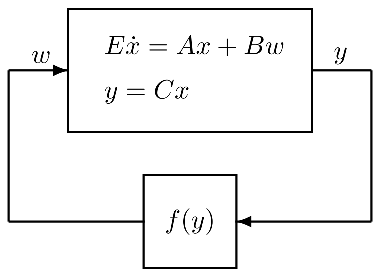

Fig. 1. A Multivariable Lur'e-type Descriptor System

linear descriptor systems. In [Yang et al., 2007b], a Lur'e-Postnikov type Liapunov function was introduced but the nonlinearity was given as a set of single-variable scalar functions. The smoothness assumption on the nonlinearities arose from the existence condition of solutions of general nonlinear descriptor systems [Hill and Mareels, 1990], [Müller, 1996, 1998], etc.

In the previous work by Wada et al. [2006], the authors dealt with general non-smooth nonlinear descriptor systems and developed stability conditions as well as an existence condition of solutions for any admissible initial values. Furthermore, in the former works by Wada and Ikeda [1993, 2004], the first two authors derived absolute stability criteria for multivariable Lur'e systems whose linear part is expressed by a state-space equation, where the differentiability assumption on the nonlinearity was relaxed and extensions of the Popov criterion was considered.

In this paper, based on the works by Wada et al. [2006], Wada and Ikeda [2004], we derive conditions for absolute stability as well as existence of solutions of multivariable

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Lur'e-type descriptor systems with a multivariable vector-valued nonlinearity. The nonlinearity satisfies multivariable sector conditions, or a part of the nonlinearities satisfies a norm bounded condition. The differentiability assumption on the nonlinearity is relaxed. The obtained stability conditions are described in terms of linear matrix inequalities (LMIs), which are extensions of the former results by Wada and Ikeda [2004] on extended Popov criteria for multivariable Lur'e systems.

Notations and Matrix Properties

For an $n \times n$ singular matrix E satisfying $\operatorname{rank} E = r < n$, E^+ denotes its pseudoinverse matrix. By using full column rank matrices $E_L, E_R \in \mathbf{R}^{n\times r}$ satisfying $E = E_L E_R^T$, the matrix E^+ can be written as $E^+ = E_R(E_R^T E_R)^{-1}(E_L^T E_L)^{-1}E_L^T$. Let U and V denote an $n \times (n-r)$ full column rank matrices whose column vectors compose a basis of $\operatorname{Null} E^T$ and $\operatorname{Null} E$, respectively. Thus, $E^T U = 0$, $EV = 0$, $E_L^T U = 0$, and $E_R^T V = 0$ are satisfied.

$\|\cdot\|$ denotes the Euclidean norm of a vector or its induced matrix norm. $\lambda_{\max}[\cdot]$ denotes the maximum eigenvalue of a symmetric matrix.

The function class $\mathcal{K}$ (or referred to as CIP) denotes the families of continuous increasing, positive definite functions from $\mathbf{R}^+ = [0, \infty)$ to $\mathbf{R}^+$.

2. MULTIVARIABLE LUR'E-TYPE DESCRIPTOR SYSTEMS

Let us consider a nonlinear descriptor system

$$E\dot{x} = Ax + Bf(Cx), \tag{1}$$

where $x \in \mathbf{R}^n$ is the descriptor variable, E and A are $n\times n$ constant matrices, $f : \mathbf{R}^m \to \mathbf{R}^m$ is a continuous function, B and C are appropriate dimensional constant matrices.

Another representation of (1) is a feedback system shown in Fig. 1 with its linear part

$$E\dot{x} = Ax + Bw \tag{1a}$$
$$y = Cx, \tag{1b}$$

and with its static nonlinear block

$$w = f(y). \tag{1c}$$

Let the matrix E be singular, that is, $\operatorname{rank} E \triangleq r < n$. In general, any autonomous nonlinear system with algebraic constraints can be described in this form without solving any algebraic equations nor calculating inverse functions if the inputs and outputs of all the nonlinearities are included in the elements of the descriptor variable.

We call this system a multivariable Lur'e-type descriptor system (MLDS). Let us assume $f(0) = 0$. Then, the MLDS (1) has the zero solution $x(\cdot) \equiv 0$ as an equilibrium, and its stability is discussed in this paper. If the equilibrium under consideration is not the origin, then an appropriate variable transformation can move the equilibrium to the origin.

2.1 Decomposition of the Nonlinearity

Let $\eta \triangleq Ex$. Then, the variable $\eta = Ex$ represents the dynamical behavior of the system (1) since its derivative $\dot{\eta}$ appears in (1). Thus, the variable $\eta = Ex$ plays a major role in analizing stability of the nonlinear system (1) by using a Liapunov function. For this reason, we decompose the nonlinearity $f(y)$ into the part depending only on Ex and the residual part.

The output y of the linear part can be rewritten as

$$y = y_d + y_n,$$
$$y_d = CE^+Ex = C_d Ex,$$
$$y_n = C(I - E^+E)x = C_n x,$$

where $C_d = CE^+$, $C_n = C(I - E^+E)$. Then, the nonlinearity f can be decomposed into

$$f(y) = f(y_d + y_n) = f(y_d) + \psi(y_d, y_n), \tag{2}$$
$$\psi(y_d, y_n) \triangleq f(y_d + y_n) - f(y_d), \tag{3}$$

where $f(y_d) = f(C_d Ex)$ depends only on Ex and $\psi(y_d, y_n)$ is its residual.

Therefore, the MLDS (1) can be expressed by

$$E\dot{x} = Ax + Bf(y_d) + B\psi(y_d, y_n) \tag{4a}$$
$$y_d = C_d Ex, \quad y_n = C_n x, \tag{4b}$$

or equivalently

$$E\dot{x} = Ax + Bf(C_d Ex) + B\psi(C_d Ex, C_n x). \tag{4}$$

Remark 1. Let us define

$$x_d = E^+Ex$$
$$x_n = x - x_d = (I - E^+E)x.$$

Then, $x_n \in \operatorname{Null} E$ and $x_d^T x_n = 0$ since $(E^+E)^T = E^+E$. Thus, y_d and y_n can be expressed by the other forms

$$y_d = Cx_d, \quad x_d \in (\operatorname{Null} E)^\perp$$
$$y_n = Cx_n, \quad x_n \in \operatorname{Null} E\,.$$

2.2 Uncertain Nonlinearities

In this paper, the nonlinearity f is assumed to be uncertain, but satisfy some conditions. Let us introduce nonlinear function classes.

Multivariable Sector Conditions

Let us define the spaces $Y = \{y = Cx : \ x \in \mathbf{R}^n\}$, $Y_d = \{y_d = C_d Ex : \ x \in \mathbf{R}^n\}$ and $Y_n = \{y_n = C_n x : x \in \mathbf{R}^n\} = \{y_n = Cx_n : \ x_n \in \operatorname{Null} E\}$.

Class $\mathsf{Sec}[0 \in Y_d]$ *functions:* We say that the nonlinearity f satisfies the *multivariable sector condition at the origin* $y_d = 0$ *in the space* Y_d and denote $f \in \mathsf{Sec}[0 \in Y_d]$ if there exists a positive definite matrix K_d such that for any $y_d \in Y_d$, the inequality

$$\{f(y_d)\}^T K_d^{-1} f(y_d) \le y_d^T f(y_d), \tag{5}$$

holds.

Class $\mathsf{Sec}[(y_d, 0) \in Y_d \times Y_n, Y_n]$ *functions:* We say that the nonlinearity f satisfies the *multivariable sector condition at each* $(y_d, 0) \in Y_d \times Y_n$ *in the* Y_n *direction* and denote $f \in \mathsf{Sec}[(y_d, 0) \in Y_d \times Y_n, Y_n]$ if there exists a positive definite matrix K_n such that for any $y_d \in Y_d$ and any $y_n \in Y_n$, the inequality

$$\{f(y_d + y_n) - f(y_d)\}^T K_n^{-1}\{f(y_d + y_n) - f(y_d)\}$$
$$\le y_n^T \{f(y_d + y_n) - f(y_d)\}, \tag{6}$$

holds.

If $f \in \mathsf{Sec}[(y_d, 0) \in Y_d \times Y_n, Y_n]$, then from (3), for arbitrarily fixed $y_d \in Y_d$, $\psi(y_d, \cdot)$ satisfies multivariable sector condition at the origin $y_n = 0$ in the space Y_n, that is, for any $y_n \in Y_n$,

$$\{\psi(y_d, y_n)\}^T K_n^{-1} \psi(y_d, y_n) \le y_n^T \psi(y_d, y_n). \tag{7}$$

Class $\mathsf{Sec}[y \in Y, Y_n]$ *functions:* We say that the nonlinearity f satisfies the *multivariable sector condition at each* $y \in Y$ *in the* Y_n *direction* and denote $f \in \mathsf{Sec}[y \in Y, Y_n]$ if there exists a positive definite matrix K_n such that for any $y \in Y$ and any $y_n \in Y_n$, the inequality

$$\begin{aligned}&\{f(y+y_n) - f(y)\}^T K_n^{-1} \{f(y+y_n) - f(y)\} \\ &\quad \le y_n^T \{f(y+y_n) - f(y)\},\end{aligned} \tag{8}$$

holds.

If $f \in \mathsf{Sec}[y \in Y, Y_n]$, then $f \in \mathsf{Sec}[(y_d, 0) \in Y_d \times Y_n, Y_n]$ is satisfied.

Symmetric Nonlinear Functions

Class Sym *functions:* We say that the function f is symmetric and denote $f \in \mathsf{Sym}$ if f satisfies the condition

$$\text{grad} \int_0^1 y^T f(\theta y) d\theta = f(y), \quad \forall y \in \mathbf{R}^m. \tag{9}$$

This equality (9) holds under the condition that for any $i \neq k$ $(i, k = 1, 2, \ldots, m)$, $\dfrac{\partial f_i}{\partial y_k}(\cdot)$ is continuous and satisfies the equality (Wada and Ikeda [1993, 2004])

$$\frac{\partial f_i}{\partial y_k}(y) = \frac{\partial f_k}{\partial y_i}(y), \quad \forall y \in \mathbf{R}^m. \tag{10}$$

That is, the off-diagonal elements of the Jacobian matrix of f are symmetric.

Remark 2. In the equality (10), we do not assume the existence of the diagonal elements of the Jacobian matrix of f. This means that the continuous differentiability of f is not required in the case when each component of $f(y)$ is a single-variable function $f_i(y) = f_i(y_i)$ $(i = 1, 2, \ldots, m)$, that is, $f(y) = [f_1(y_1)\ f_2(y_2)\ \ldots\ f_m(y_m)]^T$.

If $f \notin \mathsf{Sym}$, then let us define the functions

$$f_s(y) = \text{grad} \int_0^1 y^T f(\theta y) d\theta, \tag{11}$$

$$f_r(y) = f(y) - f_s(y), \ \forall y \in \mathbf{R}^m, \tag{12}$$

and we say that f_s is the *symmetric part of* f and f_r is the *unsymmetric part of* f.

The symmetric property (9) or (11) is significant for stability analysis since the equality (9) or (11) makes it possible to derive stability conditions by using the Lur'e-Postnicov type Liapunov function as stated in the next section.

Norm Bounded Unsymmetric Parts

Class $\mathsf{Nrbdd}[\gamma]$ *functions:* We say that the function f has the *norm bounded* unsymmetric part and denote $f \in \mathsf{Nrbdd}[\gamma]$ if

$$\|f_r(y)\| \le \gamma \|y\|, \ \forall y \in \mathbf{R}^m \tag{13}$$

for some constant $\gamma > 0$.

2.3 Absolute Stability of the MLDS

In this paper, we asuume that the descriptor equation (1) holds at the initial moment $t = 0$, that is, all the solutions $x(\cdot)$ of (1) must be continuous at the initial moment $t = 0$. We say that an initial value is *admissible* if all the solutions produced by the initial value are continuous at the initial time $t = 0$.

Pre-multiplying (1) by U^T, we obtain the nonlinear algebraic equation

$$U^T [Ax + Bf(Cx)] = 0, \tag{14}$$

since $U^T E = 0$. This static constraint must be satisfied by any solution $x(t)$ of (1) for all $t \ge 0$. Thus, any admissible initial value must be satisfy the algebraic equation (14).

In the following definition, we assume that for any admissible initial value $x_0 \in \mathbf{R}^n$, there exists a solution $x(t)$ ($t \ge 0$) of (1) satisfying $x(0) = x_0$.

Let $\mathsf{H_i}$ denotes an intersection of some of the above-defined function classes for f, where $\mathsf{H_1} \triangleq \mathsf{Sym} \cap \mathsf{Sec}[0 \in Y_d] \cap \mathsf{Sec}[(y_d, 0) \in Y_d \times Y_n, Y_n]$, and $\mathsf{H_2} \triangleq \mathsf{Sec}[0 \in Y_d] \cap \mathsf{Sec}[(y_d, 0) \in Y_d \times Y_n, Y_n] \cap \mathsf{Nrbdd}[\gamma]$.

Definition 1. The MLDS (1) is said to be *absolutely stable with the function class* $\mathsf{H_i}$ if for any $f \in \mathsf{H_i}$, the equilibrium $x = 0$ of (1) is globally asymptotically stable.

3. ABSOLUTE STABILITY CRITERIA

3.1 Stability Conditions

Let us give absolute stability conditions of the MLDS (1) under the existence assumption of solutions of (1) for any admissible initial value.

Case 1: Symmetric Nonlinearity ($\mathsf{H_1} = \mathsf{Sym} \cap \mathsf{Sec}[0 \in Y_d] \cap \mathsf{Sec}[(y_d, 0) \in Y_d \times Y_n, Y_n]$)

It is assumed that the nonlinearity f is symmetric and satisfies the multivariable sector condition (5) at $y_d = 0$ in Y_d as well as (6) at each $(y_d, 0) \in Y_d \times Y_n$ in the Y_n direction.

Theorem 1. (Case 1). The MLDS (1) is *absolutely stable with the function class* $\mathsf{H_1}$ if there exist real numbers $q \in \mathbf{R}$, $\tau > 0$, a matrix $R \in \mathbf{R}^{(n-r)\times(n-r)}$, and a symmetric matrix $P \in \mathbf{R}^{n \times n}$ satisfying $x^T E^T P E x > 0$ for $Ex \neq 0$ and the LMI

$$L_1(q, \tau, P, R) \triangleq \begin{bmatrix} \Theta_{11} & \Theta_{12} & \Theta_{13} \\ \Theta_{12}^T & \Theta_{22} & \Theta_{23} \\ \Theta_{13}^T & \Theta_{23}^T & -\tau K_n^{-1} \end{bmatrix} < 0, \tag{15}$$

where

$$\Theta_{11} = A^T(PE + URV^T) + (PE + URV^T)^T A \tag{16a}$$

$$\Theta_{12} = (PE + URV^T)^T B + \frac{1}{2}(qA + E)^T C_d^T \tag{16b}$$

$$\Theta_{13} = (PE + URV^T)^T B + \frac{\tau}{2} C_n^T \tag{16c}$$

$$\Theta_{22} = -K_d^{-1} + \frac{q}{2}(C_d B + B^T C_d^T) \tag{16d}$$

$$\Theta_{23} = \frac{q}{2} C_d B. \tag{16e}$$

Case 2: Asymmetric Nonlinearity ($\mathsf{H_2} = \mathsf{Sec}[0 \in Y_d] \cap \mathsf{Sec}[(y_d, 0) \in Y_d \times Y_n, Y_n] \cap \mathsf{Nrbdd}[\gamma]$)

The difference of this case from Case 1 is the asymmetricalness of f, that is, in this case we do not assume the symmetricalness of f, but assume that f_r is norm bounded.

Theorem 2. (Case 2). The MLDS (1) is *absolutely stable with the function class* H_2 if there exist real numbers $q \in \mathbf{R}$, $\tau > 0$, $\delta > 0$, a matrix $R \in \mathbf{R}^{(n-r)\times(n-r)}$, and a symmetric matrix $P \in \mathbf{R}^{n\times n}$ satisfying $x^T E^T PEx > 0$ for $Ex \neq 0$ and the LMI

$$L_2(q,\tau,\delta,P,R) \triangleq \begin{bmatrix} \Theta_{11} + \delta\gamma E^T C_d^T C_d E & \Theta_{12} & \Theta_{13} & -\frac{\gamma^{\frac{1}{2}} q}{2} A^T C_d^T \\ \Theta_{12}^T & \Theta_{22} & \Theta_{23} & -\gamma^{\frac{1}{2}}\Theta_{23}^T \\ \Theta_{13}^T & \Theta_{23}^T & -\tau K_n^{-1} & -\gamma^{\frac{1}{2}}\Theta_{23}^T \\ -\frac{\gamma^{\frac{1}{2}} q}{2} C_d A & -\gamma^{\frac{1}{2}}\Theta_{23} & -\gamma^{\frac{1}{2}}\Theta_{23} & -\delta I_m \end{bmatrix} < 0. \quad (17)$$

Remark 3. If E is equal to the identity matrix I, then the MLDS (1) is reduced to an unconstrained Lur'e systems. In this case, the conditions in Theorems 1 and 2 are also reduced to the absolute stability conditions for an unconstrained Lur'e systems given by Wada and Ikeda [2004] since $U = V = 0$, $C_n = 0$, and $K_n = 0$.

If we set $\gamma = 0$, then Case 2 and Theorem 2 reduce to Case 1 and Theorem 1, respectively. This means that Theorem 1 is a special case of Theorem 2. Thus, we give the proof of Theorem 2.

Proof of Theorem 2: To prove Theorem 2, we introduce the following lemma given as a stability condition for a general nonlinear system

$$E\dot{x} = \mathcal{F}(x), \quad (18)$$

where $x \in \mathbf{R}^n$ is the descriptor variable, $\mathcal{F} : \mathbf{R}^n \to \mathbf{R}^n$ is a continuous function.

Lemma 1. (Wada et al. [2006]). Suppose that there exist a continuously differentiable function $\mathcal{V} : \mathbf{R}^n \to \mathbf{R}^+$ satisfying $\mathcal{V}_\eta(0) = 0$, where $\mathcal{V}_\eta$ denotes the gradient of $\mathcal{V}$. Furthermore, suppose that there exist a continuous function $\mathcal{W} : \mathbf{R}^{n-r} \times \mathbf{R}^{n-r} \to \mathbf{R}$ satisfying $\mathcal{W}(w_1, 0) = 0$ for any $w_1 \in \mathbf{R}^{n-r}$, and class $\mathcal{K}$ functions a, b, c, $d : \mathbf{R}^+ \to \mathbf{R}^+$ such that the following conditions (i)–(iii) hold.

(i) $a(\|Ex\|) \leq \mathcal{V}(Ex) \leq b(\|Ex\|)$ for $\forall x \in \mathbf{R}^n$, and $a(\gamma) \to \infty$ as $\gamma \to \infty$.
(ii) $\{\mathcal{V}_\eta(Ex)\}^T \mathcal{F}(x) + \mathcal{W}\left(V^T x, U^T \mathcal{F}(x)\right) \leq -c(\|x\|)$ for $\forall x \in \mathbf{R}^n$.
(iii) $d(\|x\|)\|\mathcal{F}(x)\| \leq c(\|x\|)$ for $\forall x \in \{x \in \mathbf{R}^n : \|Ex\| < \rho_\eta\}$, where ρ_η is some positive number.

Then the zero solution of the descriptor system (18) is globally asymptotically stable.

Under the conditions of Theorem 2, we show that all the assumptions of Lemma 1 are satisfied.

From (1) and (4), we set

$$\begin{aligned}\mathcal{F}(x) &= Ax + Bf(Cx) \\ &= Ax + Bf(C_d Ex) + B\psi(C_d Ex, C_n x). \quad (19)\end{aligned}$$

Let us use the Lur'e-Postnikov type Liapunov function

$$\mathcal{V}(Ex) = x^T E^T PEx + q\int_0^1 (C_d Ex)^T f(\theta C_d Ex)d\theta. \quad (20)$$

Since (11) is satisfied and $\eta = Ex$, the gradient of $\mathcal{V}(\eta)$ is calculated as

$$\mathcal{V}_\eta(\eta) = \mathrm{grad}\,\mathcal{V}(\eta) = 2P\eta + qC_d^T\{f_s(C_d\eta)\}. \quad (21)$$

Thus, $\mathcal{V}_\eta(0) = 0$ is satisfied.

Let us define the function $\mathcal{W}$ as

$$\begin{aligned}\mathcal{W}\left(V^T x, U^T\mathcal{F}(x)\right) &= 2\left(V^T x\right)^T R^T U^T \mathcal{F}(x) \\ &= 2x^T(URV^T)^T\{Ax + B(f_d + \psi)\}, \quad (22)\end{aligned}$$

where we abbreviated $f(C_d Ex)$ and $\psi(C_d Ex, C_n x)$ to f_d and ψ, respectively. Then, for any $w_1 \in \mathbf{R}^{n-r}$, $\mathcal{W}(w_1, 0) = 0$ is satisfied.

To prove (i), we use the inequality

$$0 \leq y_d^T f(y_d) \leq y_d^T K_d y_d, \quad \forall y_d \in Y_d, \quad (23)$$

which is derived from the sector condition (5).

If $q \geq 0$, then by using the inequality (23) to the second term of the right hand side of (20), we obtain the inequality

$$\begin{aligned}x^T E^T PEx &\leq \mathcal{V}(Ex) \\ &\leq x^T E^T PEx + q\int_0^1 \theta(C_d Ex)^T K_d C_d Ex d\theta \\ &= x^T E^T \left(P + \frac{q}{2}C_d^T K_d C_d\right) Ex. \quad (24)\end{aligned}$$

Thus, there exists a class $\mathcal{K}$ functions a and b satisfying (i) since $x^T E^T PEx > 0$ for $Ex \neq 0$. In the case when $q < 0$, the proof is left to the appendix.

From (21) and (22), the left hand side (LHS) of the inequality in (ii) is written as

$$\begin{aligned}\text{LHS of (ii)} &= \{\mathcal{V}_\eta(Ex)\}^T\mathcal{F}(x) + \mathcal{W}\left(V^T x, U^T\mathcal{F}(x)\right) \\ &= x^T\{A^T(PE + URV^T) + (PE + URV^T)^T A\}x \\ &\quad + 2x^T\{(PE + URV)^T B(f_d + \psi)\} \\ &\quad + q(f_d - f_r)^T C_d\{Ax + B(f_d + \psi)\}, \quad (25)\end{aligned}$$

where the abbreviation $f_r = f_r(C_d Ex)$ and the relation $f_s(C_d Ex) = f_d - f_r$ are used.

Applying the S-procedure [Aizerman and Gantmacher, 1964, Boyd et al., 1994] with the sector conditions (5), (7), and the norm bounded condition (13) to (25), we have the inequality

$$\begin{aligned}&\text{LHS of (ii)} \leq \\ &\quad x^T\{A^T(PE + URV^T) + (PE + URV^T)^T A\}x \\ &\quad + 2x^T\{(PE + URV)^T B(f_d + \psi)\} \\ &\quad + q(f_d - f_r)^T C_d\{Ax + B(f_d + \psi)\} \\ &\quad + (C_d Ex)^T f_d - f_d^T K_d^{-1} f_d \\ &\quad + \tau\{(C_n x)^T\psi - \psi^T K_n^{-1}\psi\} \\ &\quad + \delta\{\gamma\|C_d Ex\|^2 - \gamma^{-1}\|f_r\|^2\}. \quad (26)\end{aligned}$$

The LMI (17) condition ensures the negative definiteness of the right hand side of (26). Thus (ii) in Lemma 1 is satisfied.

From (19), we have

$$\begin{aligned}\|\mathcal{F}(x)\| &\leq \|A\|\|x\| \\ &\quad + \|B\|(\lambda_{\max}[K_d]\|C_d E\| + \lambda_{\max}[K_n]\|C_n\|)\|x\|, \quad (27)\end{aligned}$$

since the sector conditions (5) and (7) yield the inequalities

$$\begin{aligned}\|f(C_d Ex)\| &\leq \lambda_{\max}[K_d]\|C_d E\|\|x\|, \\ \|\psi(C_d Ex, C_n x)\| &\leq \lambda_{\max}[K_n]\|C_n\|\|x\|. \quad (28)\end{aligned}$$

Thus, (iii) in Lemma 1 is satisfied.

Therefore, for $q \geq 0$, the assumptions of Lemma 1 are satisfied and the MLDS (1) or (4) is absolutely stable with the function class H_2. □

3.2 Conditions for Existence and Absolute Stability

In the previous section, the existence of solutions of (1) is assumed, while in this section we give a theorem to ensure the absolute stability as well as the existence of solutions for any admissible initial value.

Let us define the function class $\mathsf{H}_3 \triangleq \mathsf{Sec}[0 \in Y_d] \cap \mathsf{Sec}[y \in Y, Y_n] \cap \mathsf{Nrbdd}[\gamma]$. The existence and absolute stability theorem for (1) is stated for this function class H_3. It is noted that $\mathsf{H}_2 \subset \mathsf{H}_3$ since $\mathsf{Sec}[(y_d, 0) \in Y_d \times Y_n, Y_n] \subset \mathsf{Sec}[y \in Y, Y_n]$.

Theorem 3. In the MLDS (1), let f be an arbitrary function in the class H_3. Suppose that the assumptions of Theorem 2 are satisfied. Then, for any admissible initial value $x_0 \in \mathbf{R}^n$, there exists a solution $x(\cdot)$ ($t \geq 0$) of the MLDS (1), and (1) is absolutely stable with the function class H_3.

Proof: Let us recall the results given in the previous work Wada et al. [2006], which ensure the stability as well as the existence of solutions for non-smooth nonlinear descriptor systems.

Lemma 2. (Wada et al. [2006]). In Lemma 1, suppose that there exists a square matrix $R \in \mathbf{R}^{(n-r)\times(n-r)}$ and the function $\mathcal{W}$ is chosen as $\mathcal{W}\left(V^T x, U^T \mathcal{F}(x)\right) = 2\left(V^T x\right)^T R^T U^T \mathcal{F}(x)$. Furthermore, in addition to the assumptions of Lemma 1, suppose that $d(\rho) \to \infty$ as $\rho \to \infty$, and the following (iv) hold.

(iv) For any $x \in \mathbf{R}^n$ and any $\Delta \in \mathbf{R}^{n-r} \setminus \{0\}$,
$$U^T \mathcal{F}(x + V\Delta) \neq U^T \mathcal{F}(x).$$

Then, for any admissible initial value $x_0 \in \mathbf{R}^n$, there exists a solution $x(\cdot)$ ($t \geq 0$) of the descriptor system (18), and the zero solution of (18) is globally asymptotically stable.

From the proof of Theorem 2, the conditions (i)–(iii) with $d(\rho) \to \infty$ as $\rho \to \infty$ are satisfied under the assumptions of Theorem 2. Thus, we investigate the condition (iv) in Lemma 2 for the MLDS (1). From (19), we have for any $x \in \mathbf{R}^n$ and any $\Delta \in \mathbf{R}^{n-r} \setminus \{0\}$,
$$\begin{aligned} &U^T \mathcal{F}(x + V\Delta) - U^T \mathcal{F}(x) \\ &= U^T AV\Delta + U^T B\{f(Cx + C_n V\Delta) - f(Cx)\}, \end{aligned} \quad (29)$$
where we have used the relation $C_n V\Delta = CV\Delta$. Let us assume that the sector condition (8) is satisfied, that is, $f \in \mathsf{Sec}[y \in Y, Y_n]$. Then, pre-multiplying (29) by $2\Delta^T V^T V R^T$, we obtain
$$\begin{aligned} &-2\|RV^T V\Delta\| \|U^T \mathcal{F}(x + V\Delta) - U^T \mathcal{F}(x)\| \\ &\leq 2\Delta^T V^T V R^T \{U^T \mathcal{F}(x + V\Delta) - U^T \mathcal{F}(x)\} \\ &\leq [\Delta^T V^T \quad \{f(Cx + C_n V\Delta) - f(Cx)\}^T] \\ &\cdot \begin{bmatrix} A^T URV^T + (URV^T)^T A & (URV^T)^T B + \frac{\tau}{2} C_n^T \\ B^T URV^T + \frac{\tau}{2} C_n & -\tau K_n^{-1} \end{bmatrix} \\ &\cdot \begin{bmatrix} V\Delta \\ f(Cx + C_n V\Delta) - f(Cx) \end{bmatrix}. \end{aligned} \quad (30)$$
If the LMI (17) is satisfied, then the right hand side of the above inequality is negative for $\Delta \neq 0$, which can be obtained by pre-multiplying and post-multiplying the LMI (17) by $[\Delta^T V^T \ 0^T \ \{f(Cx + C_n V\Delta) - f(Cx)\}^T \ 0^T]$ and its transpose, respectively. Thus, if the LMI (17) is satisfied and $f \in \mathsf{Sec}[y \in Y, Y_n]$, then (iv) is satisfied. This completes the proof. □

4. A NUMERICAL EXAMPLE

As an application of Theorem 1, we consider a maximization sector problem.

Let the matrix K_d in the sector condition (5) be written as
$$K_d = \mu K_{d0}, \quad (31)$$
where K_{d0} is a given positive definite matrix and $\mu > 0$ is a parameter scaling the sector. For a given MLDS (1), we consider the problem to find the maximum parameter $\mu = \mu_{\max}$ which satisfies the assumptions of Theorem 1 for (31) [1]. Thus, for any parameter value $0 < \mu \leq \mu_{\max}$, the absolute stability of (1) with the function class H_1 is ensured.

Let the MLDS (1) be given with the coefficient matrices
$$E = \begin{bmatrix} 1&0&0&0 \\ 0&1&0&0 \\ 0&0&1&0 \\ 0&0&0&0 \end{bmatrix}, \qquad A = \begin{bmatrix} -5&-1&-3&1 \\ 0&-3&3&0 \\ 0&-4&-5&0 \\ -1&-2&-3&1 \end{bmatrix},$$
$$B = \begin{bmatrix} 1&0 \\ 0&0.9 \\ 0&1 \\ 0&0 \end{bmatrix}, \qquad C = \begin{bmatrix} 1&1&0&0 \\ 0&0&1&1 \end{bmatrix},$$
and the nonlinearity $f \in \mathsf{H}_1$, where the matrix K_d in (5) satisfies (31), and K_{d0}, K_n are given as
$$K_{d0} = \begin{bmatrix} 1.48 & 0.18 \\ 0.18 & 0.98 \end{bmatrix}, \ K_n = \begin{bmatrix} 1 & 0.1 \\ 0.1 & 0.5 \end{bmatrix}.$$
Then, the matrices E^+, C_d, C_n, U, and V are calculated as
$$E^+ = \begin{bmatrix} 1&0&0&0 \\ 0&1&0&0 \\ 0&0&1&0 \\ 0&0&0&0 \end{bmatrix}, \ C_d = \begin{bmatrix} 1&1&0&0 \\ 0&0&1&0 \end{bmatrix}, \ C_n = \begin{bmatrix} 1&0&0&0 \\ 0&0&0&1 \end{bmatrix},$$
$$U = V = [\,0 \ 0 \ 0 \ 1\,]^T.$$
Solving this problem by using the well-known software package LMITOOL in SCILAB [Nikoukhah et al., 1995], we obtain the maximum $\mu_{\max} = 1.38$.

5. CONCLUSIONS

Sufficient conditions for absolute stability of multivariable Lur'e-type descriptor systems with multivariable sector bounded nonlinearities have been derived. Since the variable Ex represents the dynamical behavior of the system, we have decomposed the output of the linear system into the part $y_d \in Y_d$ depending only on Ex and the residual part $y_n \in Y_n$ in the Null E direction. With this decomposition, we have assumed that the nonlinearity shoud satisfy both of the multivariable sector conditions in the Y_d direction and in the Y_n direction, and have introduced the Lur'e-Postnikov type Liapunov function depending only on Ex, which has enabled us to derive LMI conditions for absolute stability by applying S-procedure with the sector conditions.

[1] This problem is motivated by the work [Šiljak and Stipanovic, 2000].

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Appendix A.

Proof of Theorem 2 in the case when $q < 0$

All the proof for $q \geq 0$ except the satisfaction of (i) in Lemma 1 are valid for $q < 0$. Thus, here we show that the sector condition (5) and the LMI (17) ensure (i) in Lemma 1 in the case when $q < 0$.

By applying the inequality (23) derived from the sector condition (5) to the Liapunov function (20), we obtain for any $q < 0$,

$$\begin{aligned}\mathcal{V}(Ex) &\geq x^T E^T P E x + q \int_0^1 \theta (C_d E x)^T K_d C_d E x d\theta \\ &= x^T E^T \left(P + \frac{q}{2} C_d^T K_d C_d \right) Ex. \qquad \text{(A.1)}\end{aligned}$$

Thus, if the last quadratic form of (A.1) is positive for $Ex \neq 0$, then (i) in Lemma 1 is satisfied. To prove this, let us introduce the matrices $\tilde{A}(\alpha) = A + \alpha B K_d C_d E$ and $\tilde{P}(\alpha) = P + \frac{\alpha q}{2} C_d^T K_d C_d$ for $0 \leq \alpha \leq 1$. The last quadratic form of (A.1) is written as $x^T E^T \tilde{P}(1) E x$.

When $\alpha = 0$, from the assumptions of Theorem 2, the inequalities

$$x^T E^T \tilde{P}(0) E x > 0, \quad \text{for } Ex \neq 0 \qquad \text{(A.2)}$$

$$\{\tilde{A}(0)\}^T \{\tilde{P}(0)E + URV^T\} + \{\tilde{P}(0)E + URV^T\}^T \tilde{A}(0) < 0 \qquad \text{(A.3)}$$

hold since $\tilde{A}(0) = A$ and $\tilde{P}(0) = P$.

Pre-multiplying and post-multiplying the LMI (17) by $\left[I_n \ \alpha E^T C_d^T K_d \ 0 \ 0 \right]$ ($0 \leq \alpha \leq 1$) and its transpose, respectively, we obtain the matrix inequality

$$\begin{aligned}&\{\tilde{A}(\alpha)\}^T \{\tilde{P}(\alpha)E + URV^T\} + \{\tilde{P}(\alpha)E + URV^T\}^T \tilde{A}(\alpha) \\ &+ \delta\gamma E^T C_d^T C_d E + \alpha(1-\alpha) E^T C_d K_d C_d E < 0. \qquad \text{(A.4)}\end{aligned}$$

Since E_L, E_R are $n \times r$ full column rank matrices and V is an $n \times (n-r)$ full column rank matrix satisfying $E_L E_R^T = E$ and $V^T E_R = 0$, for any $\zeta_c \in \mathbf{R}^r$ and any $\alpha \in [0,1]$, pre-multiplying and post-multiplying (A.4) by $(E_R \zeta_c)^T$ and its transpose $E_R \zeta_c$, respectively, we can derive the inequality

$$\zeta_c^T E_R^T \{\tilde{A}(\alpha)\}^T \tilde{P}(\alpha) E E_R \zeta_c < 0. \qquad \text{(A.5)}$$

Thus, the $r \times r$ matrix $E_R^T \{\tilde{A}(\alpha)\}^T \tilde{P}(\alpha) E E_R$ is nonsingular and

$$\operatorname{rank}\tilde{A}(\alpha) = \operatorname{rank}\tilde{P}(\alpha) = r \quad \forall \alpha \in [0,1]. \qquad \text{(A.6)}$$

On the other hand, we assume that there exist $\alpha_0 \in (0,1]$ and $x_1 \in \mathbf{R}^n$ such that $Ex_1 \neq 0$ and $x_1^T E^T \tilde{P}(\alpha_0) E x_1 \leq 0$. Then, from (A.2) and the continuity of $x_1^T E^T \tilde{P}(\alpha) E x_1$ with respect to α, there exists $\alpha_1 \in (0, \alpha_0)$ such that

$$x_1^T E^T \tilde{P}(\alpha_1) E x = 0. \qquad \text{(A.7)}$$

Since $E_R^T V = 0$ implies that the matrix $[E_R \ V]$ is nonsingular, there exist $\zeta_{c1} \in \mathbf{R}^r$ and $\zeta_{n1} \in \mathbf{R}^{n-r}$ satisfying $x_1 = E_R \zeta_{c1} + V \zeta_{n1}$. From $EV = 0$ and $Ex_1 \neq 0$, we have $0 \neq Ex_1 = EE_R \zeta_{c1}$. Substituting $Ex_1 = EE_R \zeta_{c1}$ to (A.7), we obtain

$$\zeta_{c1}^T E_R^T E^T \tilde{P}(\alpha_1) E E_R \zeta_{c1} = 0. \qquad \text{(A.8)}$$

Since $\zeta_{c1} \neq 0$ and $\operatorname{rank} E = \operatorname{rank} E_R = r$, we have $\operatorname{rank}\tilde{P}(\alpha_1) < r$. This contradicts (A.6). Therefore, for any $\alpha \in [0,1]$ and any $x \in \mathbf{R}^n$ satisfying $Ex \neq 0$, we obtain $x^T E^T \tilde{P}(\alpha) E x > 0$, and thus, for $\alpha = 1$, $x^T E^T \tilde{P}(1) E x > 0$. Consequently, (i) in Lemma 1 is satisfied. □