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Delay-Range-Dependent Exponential Stability of Singular Systems with Multiple Time-Varying Delays ⋆

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Abstract: This paper deals with the class of continuous-time singular linear systems with multiple time-varying delays in a range. The global exponential stability problem of this class of systems is addressed. Delay-range-dependent sufficient conditions such that the system is regular, impulse free and α-stable are developed in the linear matrix inequality (LMI) setting. Moreover, an estimate of the convergence rate of such stable systems is presented. A numerical example is employed to show the usefulness of the proposed results.

Keywords: Singular time-delay systems; delay-dependent; stability; α-stability; linear matrix inequality.

1. INTRODUCTION

Singular time-delay systems arise in a variety of practical systems such as networks, circuits, power systems and so on [1]. Since singular time-delay systems are matrix delay differential equations coupled with matrix difference equations, the study for such systems is much more complicated than that for standard state-space time-delay systems or singular systems. The existence and uniqueness of a solution to a given singular time-delay system is not always guaranteed and the system can also have undesired impulsive behavior.

Both delay-independent and delay-dependent stability conditions for singular time-delay systems have been derived using the time domain method, see [2, 3, 4, 13] and references therein. However, most of the delay-dependent results in the literature tackle only the case of constant time delay where two approaches were used to prove the stability of the system. The first approach consists of decomposing the system into fast and slow subsystems and the stability of the slow subsystem is proved using some Lyapunov functional. Then, the fast variables is expressed explicitly by an iterative equation in terms of the slow variables [2]. The second approach introduced by [3] and it consistes of constructing a Lyapunov-Krasovskii functional that corresponds directly to the descriptor form of the system. The extension of these approaches to time-varying delays has not been addressed yet. In [13], where time-varying delays are considered, the response of the fast variables has been bounded by an exponential term using a different approach. Using this approach, it is not possible to give an estimate of the convergence rate of the states of the system.

Recently, a free-weighting matrices method is proposed in [5] and [6] to study the delay-dependent stability for time-delay systems. In 2007, Zhu et al. adopted this technique for singular time-delay systems [4]. Also, delay-range-dependent concept was recently studied for time-delay systems, where the delays are considered to vary in a range and thereby more applicable in practice [7].

Formally speaking, these conditions provide only the asymptotic stability of singular time-delay systems. In [8], the global delay-independent exponential stability for a class of singular systems with multiple constant time delays is investigated and an estimate of the convergence rate of such systems is presented. One may ask if there exists a possibility to use the LMI approach for deriving exponential estimates for solutions of singular time-delay systems. In [13], exponential stability conditions in terms of LMIs are given but no estimate of the convergence rate is presented.

This paper addresses an important problem that has not been fully investigated. Delay-range-dependent exponential stability conditions for singular systems with multiple time-varying delays is established in terms of LMIs. These conditions will guarantee that the system will be regular, impulse-free and exponentially stable. Moreover, an estimate of the convergence rate of such systems is presented. It has been shown also that this rate depends if the system has single or multiple delays. The method used is based on the Lyapunov-Krasovskii approach, and some graph theory terminology has been used to prove the stability of the fast subsystem.

The rest of the paper is organized as follows. In section II, the problem, the goal of the paper and some definitions and Lemmas are stated. In section III, the main results are given. In sections IV and V, a numerical example and the conclusion are given, respectively.

Notation: Throughout this paper, $\lambda_{max}(P)$ and $\lambda_{min}(P)$ denote, respectively, the maximal and minimal eigenvalue of matrix P. $C_\tau = C([-\tau,0],\mathbb{R}^n)$ denotes the Banach space of continuous vector functions mapping the interval $[-\tau,0]$ into $\mathbb{R}^n$ with the topology of uniform convergence. $\|\cdot\|$ refers to the Euclidean vector norm whereas $\|\phi\|_c = sup_{-\tau\leq t\leq 0}\|\phi(t)\|$ stands for the

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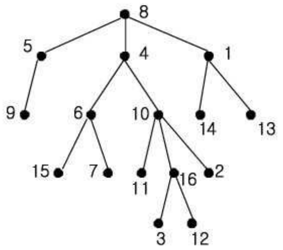

Fig. 1. An example of a tree

norm of a function $\phi \in C_\tau$. C_τ^v is defined by $C_\tau^v = \{\phi \in C_\tau; \|\phi\|_c < v, v > 0\}$.

2. PROBLEM STATEMENT AND DEFINITIONS

Consider the linear singular time-delay system:

$$\begin{cases} E\dot{x}(t) = Ax(t) + \sum_{k=1}^{p} A_k x(t - d_k(t)) \\ x(t) = \phi(t), -\overline{d} \le t \le 0 \end{cases} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state, the matrix $E \in \mathbb{R}^{n\times n}$ may be singular, and we assume that $rank(E) = r \le n$, A and A_k are known real constant matrices, $\phi(t) \in C_\tau$ is a compatible vector valued continuous function and $d_k(t)$, $k = 1,\ldots,p$, is the time delay and that is assumed to satify:

$$\begin{cases} 0 < \underline{d}_k \le d_k(t) \le \overline{d}_k \\ \dot{d}_k(t) \le \mu < 1 \end{cases} \tag{2}$$

with $\underline{d}_k$ and $\overline{d}_k$ are given positive scalars. Also, $\overline{d}$ and $\underline{d}$ are positive scalars with $\overline{d} = max\{\overline{d}_1, \overline{d}_2, \ldots, \overline{d}_p\}$ and $\underline{d} = min\{\underline{d}_1, \underline{d}_2, \ldots, \underline{d}_p\}$.

The following definitions will be used in the rest of this paper:

Definition 1. [12]

i. System (1) is said to be regular if the characteristic polynomial, $det(sE - A)$ is not identically zero.
ii. System (1) is said to be impulse-free if $deg(det(sE - A)) = rank(E)$
iii. System (1) is said to be *exponentially stable* if there exist $\sigma > 0$ and $\gamma > 0$ such that, for any compatible initial conditions $\phi(t)$, the solution $x(t)$ to the singular time-delay system satisfies
$$\|x(t)\| \le \gamma e^{-\sigma t}\|\phi\|_c$$
iv. System (1) is said to be exponentially admissible if it is regular, impulse-free and exponentially stable.

Remark 2. In the rest of this paper, the following terminology borrowed from graph theory will be used.

- A tree structure is a way of representing the hierarchical nature of a structure in a graphical form (see Figure (1)).
- The topmost node in a tree is called the root node.
- A node is a parent of another node (child) if it is one step higher in the hierarchy and closer to the root node.
- Nodes at the bottommost level of the tree are called leaf nodes.

Lemma 3. [10] Suppose that system (1) is regular and impulse-free, then the solution to system (1) exists and it is impulse-free and unique on $(0,\infty)$.

Lemma 4. Given a set of matrices $(D_1,\ldots,D_p)$. Let a set of symmetric and positive-definite matrices $(S_1,\ldots,S_p)$ and a set of positive scalars $(\eta_1,\ldots,\eta_p) \in (0,1)$ exist such that

$$\begin{cases} D_1^\top S_1 D_1 - \eta_1{}^2 S_1 < 0 \\ \quad\vdots \\ D_p^\top S_p D_p - \eta_p{}^2 S_p < 0 \end{cases} \tag{3}$$

then, any random multiplication of these matrices (each matrix can appear more than once) satisfies the bound

$$\|D_{k_1} D_{k_2} \ldots D_{k_c}\| \le \chi e^{-\lambda} \tag{4}$$

with

$$c \in \mathbb{N}, \qquad k_i \in \{1,\ldots,p\}$$

$$\lambda = -ln\left(\prod_{i=1}^{c} \eta_{k_i}\right), \quad \chi = \sqrt{\frac{\lambda_{max}(S_{k1})\lambda_{max}(S_{k2})\ldots\lambda_{max}(S_{kc})}{\lambda_{min}(S_{k1})\lambda_{min}(S_{k2})\ldots\lambda_{min}(S_{kc})}}.$$

3. MAIN RESULTS

3.1 Delay-Range-Dependent Exponential Stability

Theorem 5. Given positive scalars $\underline{d}_k$ and $\overline{d}_k$, with $\underline{d}_k < \overline{d}_k$, $k = 1,\ldots,p$, $\mu < 1$ and $\alpha > \frac{1}{\underline{d}}$. System (1) with time-varying delays $d_k(t)$ satisfying (2) is exponentially admissible with $\sigma > \alpha - \frac{1}{\underline{d}}$ if there exist a nonsingular matrix $P \in \mathbb{R}^{n\times n}$, $n\times n$ symmetric and positive-definite matrices $Q_{k1}, Q_{k2}, Q_{k3}, Z_{k1}$ and Z_{k2}, $k = 1,\ldots,p$, and $n\times n$ matrices M_{ki}, N_{ki} and S_{ki}, $i = 1,2$, $k = 1,\ldots,p$, such that the following LMI hold

$$\begin{bmatrix} \Pi & \Upsilon & \tilde{A}U \\ \star & T & 0 \\ \star & \star & -U \end{bmatrix} < 0 \tag{5}$$

with the following constraint

$$E^\top P = P^\top E \ge 0 \tag{6}$$

where

$$T = diag\left\{-\frac{e^{2\alpha\overline{d}_k} - 1}{2\alpha}Z_{k1}, -\frac{e^{2\alpha\overline{d}_k} - e^{2\alpha\underline{d}_k}}{2\alpha}(Z_{k1} + Z_{k2}), \right.$$
$$\left. -\frac{e^{2\alpha\overline{d}_k} - e^{2\alpha\underline{d}_k}}{2\alpha}Z_{k2}\right\}, \quad k = 1,\ldots,p$$

$$\widetilde{A}^\top = \left[A \ \tilde{A}_1 \ \ldots \ \tilde{A}_p\right], \text{ with } \tilde{A}_k = \left[A_k \ 0 \ 0\right]$$

$$U = \sum_{k=1}^{p}\left\{\left(\overline{d}_k Z_{k1} + \underline{\overline{d}}_k Z_{k2}\right)\right\} \text{ with } \underline{\overline{d}}_k = \overline{d}_k - \underline{d}_k$$

$$\Pi = \begin{bmatrix} \Pi_1 & F \\ \star & G \end{bmatrix} \text{ and } \Upsilon = \left[\tilde{N}_1 \ \tilde{S}_1 \ \tilde{M}_1 \ \ldots \ \tilde{N}_p \ \tilde{S}_p \ \tilde{M}_p\right]$$

$$\widetilde{N}_k^\top = \left[N_{k1}^\top \ \mathbf{0}_{n\times 3n(k-1)} \ N_{k2}^\top \ 0 \ 0 \ \mathbf{0}_{n\times 3n(p-k)}\right], \ k = 1,\ldots,p$$

$$\widetilde{M}_k^\top = \left[M_{k1}^\top \ \mathbf{0}_{n\times 3n(k-1)} \ M_{k2}^\top \ 0 \ 0 \ \mathbf{0}_{n\times 3n(p-k)}\right], \ k = 1,\ldots,p$$

$$\widetilde{S}_k^\top = \left[S_{k1}^\top \ \mathbf{0}_{n\times 3n(k-1)} \ S_{k2}^\top \ 0 \ 0 \ \mathbf{0}_{n\times 3n(p-k)}\right], \ k = 1,\ldots,p$$

$$\Pi_1 = P^\top A + A^\top P + \sum_{k=1}^{p}\left\{\sum_{i=1}^{3} Q_{ki} + N_{k1}E + (N_{k1}E)^\top\right\}$$
$$+ 2\alpha E^\top P$$

$$F = \left[W_1 \ \ldots \ W_p\right] \text{ and } G = diag\left\{J_1,\ldots,J_p\right\}$$

$$J_k = \begin{bmatrix} \Pi_{k3} & e^{\alpha \underline{d}_k} M_{k2}E & -e^{\alpha \bar{d}_k} S_{k2}E \\ \star & -Q_{k1} & \mathbf{0} \\ \star & \star & -Q_{k2} \end{bmatrix} \quad k = 1, \ldots, p$$

$$W_k = \begin{bmatrix} \Pi_{k2} & e^{\alpha \underline{d}_k} M_{k1}E & -e^{\alpha \bar{d}_k} S_{k1}E \end{bmatrix}, \; k = 1, \ldots, p$$

$$\Pi_{k2} = P^\top A_k + (N_{k2}E)^\top - N_{k1}E + S_{k1}E - M_{k1}E \quad k = 1, \ldots, p$$

$$\Pi_{k3} = -(1-\mu)e^{-2\alpha \bar{d}_k} Q_{k3} + S_{k2}E + (S_{k2}E)^\top - N_{k2}E - (N_{k2}E)^\top - M_{k2}E - (M_{k2}E)^\top \quad k = 1, \ldots, p.$$

Proof. The system can be shown to be regular and impulse-free. Therefore, there exist two matrices R, L such that (see [10])

$$\bar{E} = REL = \begin{bmatrix} \mathbb{I}_r & 0 \\ 0 & 0 \end{bmatrix} \qquad \bar{A} = RAL = \begin{bmatrix} \widehat{A} & 0 \\ 0 & \mathbb{I}_{n-r} \end{bmatrix} \tag{7}$$

Now, let $\bar{A}_{kd} = RA_kL$ and $\bar{Q}_{ki} = L^\top Q_{ki} L$,

$$\bar{A}_{kd} = \begin{bmatrix} A_{kd11} & A_{kd12} \\ A_{kd21} & A_{kd22} \end{bmatrix}, \qquad \bar{Q}_{ki} = \begin{bmatrix} Q_{ki11} & Q_{ki12} \\ Q_{ki21} & Q_{ki22} \end{bmatrix}$$

Then, the following relations can be shown

$$A_{kd22}{}^\top Q_{k322} A_{kd22} - e^{-2\alpha \bar{d}_k} Q_{k322} < 0 \quad \forall k = 1, \ldots, p \tag{8}$$

$$\rho(e^{\alpha \bar{d}_k} A_{kd22}) < 1 \quad \forall k = 1, \ldots, p \tag{9}$$

Let $\zeta(t) = L^{-1}x(t) = \begin{bmatrix} \zeta_1(t) \\ \zeta_2(t) \end{bmatrix}$, where $\zeta_1(t) \in \mathbb{R}^r$ and $\zeta_2(t) \in \mathbb{R}^{n-r}$. System (1) is equivalent to the following one

$$\dot{\zeta}_1(t) = \widehat{A}\zeta_1(t) + \sum_{k=1}^{p} \{A_{kd11}\zeta_1(t - d_k(t)) + A_{kd12}\zeta_2(t - d_k(t))\} \tag{10}$$

$$0 = \zeta_2(t) + \sum_{k=1}^{p} \{A_{kd21}\zeta_1(t - d_k(t)) + A_{kd22}\zeta_2(t - d_k(t))\}. \tag{11}$$

Now, choose the Lyapunov functional as follows:

$$\begin{aligned} V(\zeta_t) = \zeta(t)^\top \bar{E}^\top \bar{P} \zeta(t) + \sum_{k=1}^{p} \Big\{ & \int_{t-\underline{d}_k}^{t} \zeta(s)^\top e^{2\alpha(s-t)} \bar{Q}_{k1} \zeta(s) ds \\ & + \int_{t-\bar{d}_k}^{t} \zeta(s)^\top e^{2\alpha(s-t)} \bar{Q}_{k2} \zeta(s) ds \\ & + \int_{t-d_k(t)}^{t} \zeta(s)^\top e^{2\alpha(s-t)} \bar{Q}_{k3} \zeta(s) ds \\ & + \int_{-\bar{d}_k}^{0} \int_{t+\theta}^{t} (\bar{E}\dot{\zeta}(s))^\top e^{2\alpha(s-t)} \bar{Z}_{k1} \bar{E}\dot{\zeta}(s) ds d\theta \\ & + \int_{-\bar{d}_k}^{-\underline{d}_k} \int_{t+\theta}^{t} (\bar{E}\dot{\zeta}(s))^\top e^{2\alpha(s-t)} \bar{Z}_{k2} \bar{E}\dot{\zeta}(s) ds d\theta \Big\} \end{aligned}$$

where $\zeta_t = \zeta(t - \beta)$, $\beta \in (-d_2, 0]$. Then, the following estimation can be obtained

$$\|\zeta_1(t)\| \le \sqrt{\frac{\lambda_2}{\lambda_1}} \|\phi\|_c e^{-\alpha t} \tag{12}$$

where λ_1 and λ_2 are positive integers. In order to proof the exponential stability of the fast system, the relation in (11) should be used. For constant time delay, an explicit equation of $\zeta_2(t)$ can be found easily by an iterative method [2]. It can be seen that ζ_2 at time t depends on ζ_2 at time $t - \tau$, where τ is the constant delay, and ζ_2 at time $t - \tau$ depends on ζ_2 at time $t - 2\tau$, and so on. In the case of multiple time-varying or even single delay, such a simple relation can't be found. Thus, a tree structure will be adopted to model the dependency of $\zeta_2(t)$ on past instances. Now, define

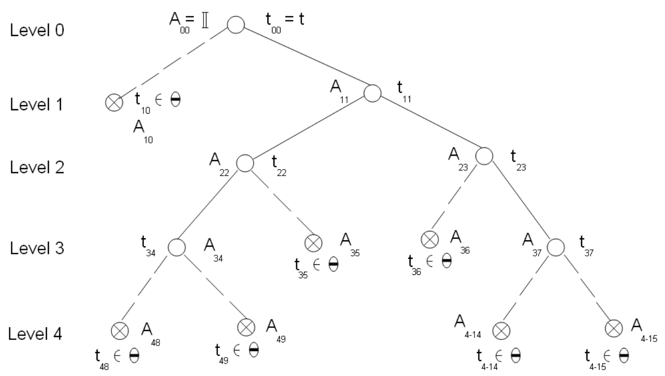

Fig. 2. An example with p = 2. Take note that $\zeta_2(t_{ij})$ depends on the value of ζ at all times indicated by the children of t_{ij} in the tree.

$$t_{00} = t \tag{13}$$

$$t_{ij} = t_{(i-1)v_j} - d_{(\mathrm{mod}(j,p)+1)}(t_{(i-1)v_j}) \tag{14}$$

$$\Theta = \{t_{ij} \mid t_{ij} \in (-\bar{d}, 0] \text{ and } t_{(i-1)v_j} \notin (-\bar{d}, 0]\} \tag{15}$$

$$\hat{A}_{00} = \mathbb{I} \tag{16}$$

$$\hat{A}_{ij} = \left(\hat{A}_{(i-1)v_j}\right) \times \left(-A_{(\mathrm{mod}(j,p)+1)d22}\right) \tag{17}$$

where

$v_j =$ the greatest integer less than or equal to $\dfrac{j}{p}$,

$\mathrm{mod}(j,p) =$ the remainder of the integer division $\dfrac{j}{p}$, and

t_{ij} and $\hat{A}_{ij}$ are undefined if $t_{(i-1)v_j} \in (-\bar{d}, 0]$.

If we let the parents of t_{ij} and $\hat{A}_{ij}$ to be $t_{(i-1)v_j}$ and $\hat{A}_{(i-1)v_j}$, respectively, t_{ij}'s and $\hat{A}_{ij}$'s will represent two trees with the same structure (see Figure 2), with roots t and $\mathbb{I}$, respectively. Note also that the values of the leaf nodes of the t_{ij}'s tree belongs to Θ. Noting that $\mathrm{mod}(j,p) = j$ if $j < p$, then from (11), and using the definitions (13)-(17), we get

$$\begin{aligned} \zeta_2(t) &= -\sum_{k=1}^{p} \{A_{kd21}\zeta_1(t - d_k(t)) + A_{kd22}\zeta_2(t - d_k(t))\} \\ \zeta_2(t) &= \sum_{j=0}^{p-1} \left\{-A_{(j+1)d21}\zeta_1(t_{1j}) + \hat{A}_{1j}\zeta_2(t_{1j})\right\} \\ &= \sum_{t_{1j} \in \Theta} \left\{\hat{A}_{1j}\zeta_2(t_{1j})\right\} + \sum_{j=0}^{p-1} \left\{-A_{(j+1)d21}\zeta_1(t_{1j})\right\} \\ &\quad + \sum_{\substack{j=0 \\ t_{1j} \notin \Theta}}^{p-1} \hat{A}_{1j}\zeta_2(t_{1j}) \end{aligned} \tag{18}$$

if $t_{1j} \notin \Theta$, from (11) and (13)-(17), we get

$$\hat{A}_{1j}\zeta_2(t_{1j}) = \hat{A}_{1j} \sum_{r=jp}^{(j+1)p-1} \left\{-A_{(\mathrm{mod}(r,p)+1)d21}\zeta_1(t_{2r}) - A_{(\mathrm{mod}(r,p)+1)d22}\zeta_2(t_{2r})\right\}$$

$$\hat{A}_{1j}\zeta_2(t_{1j}) = \sum_{r=jp}^{(j+1)p-1} \left\{-\hat{A}_{1j}A_{(\mathrm{mod}(r,p)+1)d21}\zeta_1(t_{2r})\right.$$

$$-\hat{A}_{1j}A_{(\text{mod}(r,p)+1)d22}\zeta_2(t_{2r})\}$$
$$=\sum_{r=jp}^{(j+1)p-1}\{-\hat{A}_{1j}A_{(\text{mod}(r,p)+1)d21}\zeta_1(t_{2r}) +\hat{A}_{2r}\zeta_2(t_{2r})\}$$

thus, $\zeta_2(t)$ in (18) can be computed from

$$\zeta_2(t)=\sum_{t_{1j}\in\Theta}\{\hat{A}_{1j}\zeta_2(t_{1j})\}+\sum_{j=0}^{p-1}\{-A_{(j+1)d21}\zeta_1(t_{1j})\} +\sum_{\substack{j=0\\ t_{1j}\notin\Theta}}^{p-1}\sum_{r=jp}^{(j+1)p-1}\{-\hat{A}_{1j}A_{(\text{mod}(r,p)+1)d21}\zeta_1(t_{2r})+\hat{A}_{2r}\zeta_2(t_{2r})\}$$
$$=\sum_{t_{1j}\in\Theta}\{\hat{A}_{1j}\zeta_2(t_{1j})\}+\sum_{j=0}^{p-1}\{-A_{(j+1)d21}\zeta_1(t_{1j})\} +\sum_{\substack{j=0\\ t_{1v_j}\notin\Theta}}^{p^2-1}\{-\hat{A}_{1j}A_{(\text{mod}(j,p)+1)d21}\zeta_1(t_{2j})+\hat{A}_{2j}\zeta_2(t_{2j})\}$$
$$=\sum_{t_{1j}\in\Theta}\{\hat{A}_{1j}\zeta_2(t_{1j})\}-\sum_{i=0}^{1}\sum_{\substack{j=0\\ t_{iv_j}\notin\Theta}}^{p^{i+1}-1}\hat{A}_{ij}A_{(\text{mod}(j,p)+1)d21}\times \zeta_1(t_{(i+1)j})+\sum_{\substack{j=0\\ t_{1v_j}\notin\Theta}}^{p^2-1}\hat{A}_{2j}\zeta_2(t_{2j})$$
$$=\sum_{i=1}^{2}\sum_{t_{ij}\in\Theta}\{\hat{A}_{ij}\zeta_2(t_{ij})\}-\sum_{i=0}^{1}\sum_{\substack{j=0\\ t_{iv_j}\notin\Theta}}^{p^{i+1}-1}\hat{A}_{ij}A_{(\text{mod}(j,p)+1)d21}\times \zeta_1(t_{(i+1)j})+\sum_{\substack{j=0\\ t_{2j}\notin\Theta}}^{p^2-1}\hat{A}_{2j}\zeta_2(t_{2j})$$

Continuing in the same manner, if $t_{2j}\notin\Theta$,

$$\hat{A}_{2j}\zeta_2(t_{2j})=\sum_{r=jp}^{(j+1)p-1}\{-\hat{A}_{2j}A_{(\text{mod}(r,p)+1)d21}\zeta_1(t_{3r}) -\hat{A}_{2j}A_{(\text{mod}(r,p)+1)d22}\zeta_2(t_{3r})\}$$
$$=\sum_{r=jp}^{(j+1)p-1}\{-\hat{A}_{2j}A_{(\text{mod}(r,p)+1)d21}\zeta_1(t_{3r})+\hat{A}_{3r}\zeta_2(t_{3r})\}$$

we get,

$$\zeta_2(t)=\sum_{i=1}^{3}\sum_{t_{ij}\in\Theta}\{\hat{A}_{ij}\zeta_2(t_{ij})\}-\sum_{i=0}^{2}\sum_{\substack{j=0\\ t_{iv_j}\notin\Theta}}^{p^{i+1}-1}\hat{A}_{ij}A_{(\text{mod}(j,p)+1)d21}\times \zeta_1(t_{(i+1)j})+\sum_{\substack{j=0\\ t_{3j}\notin\Theta}}^{p^3-1}\hat{A}_{3j}\zeta_2(t_{3j})$$

Note that

$$t_{ij}=t_{(i-1)v_j}-d_{(\text{mod}(j,p)+1)}(t_{(i-1)v_j}) \le t_{(i-1)v_j}-\underline{d}_{(\text{mod}(j,p)+1)}<t_{(i-1)v_j}$$

which means that the time of a child is always less than the time of its parent. Therefore, there exists a positive finite integer $k(t)$ such that

$$\zeta_2(t)=\sum_{i=1}^{k(t)}\sum_{t_{ij}\in\Theta}\{\hat{A}_{ij}\zeta_2(t_{ij})\} -\sum_{i=0}^{k(t)-1}\sum_{\substack{j=0\\ t_{iv_j}\notin\Theta}}^{p^{i+1}-1}\hat{A}_{ij}A_{(\text{mod}(j,p)+1)d21}\zeta_1(t_{(i+1)j})$$

and $t_{ij}\in[-\overline{d},0]$. Thus, we get

$$\|\zeta_2(t)\|\le\sum_{i=1}^{k(t)}\sum_{t_{ij}\in\Theta}\{\|\hat{A}_{ij}\|\}\|\phi\|_c +\sum_{i=0}^{k(t)-1}\sum_{\substack{j=0\\ t_{iv_j}\notin\Theta}}^{p^{i+1}-1}\|\hat{A}_{ij}\|\|A_{(\text{mod}(j,p)+1)d21}\|\|\zeta_1(t_{(i+1)j})\| \quad (19)$$

Now, in order to estimate $\|\zeta_2(t)\|$, the two terms in (19) have to be estimated. For the first term, from (16)-(17), $\hat{A}_{ij}$ can be written as

$$\hat{A}_{ij}=\left(\hat{A}_{(i-1)v_j}\right)\times\left(-A_{(\text{mod}(j,p)+1)d22}\right) =\left(\hat{A}_{(i-1)v_j}\right)\times\left(-A_{k_1d22}\right)$$

Iterating on $\hat{A}_{(i-1)v_j}$ gives after i-1 iterations,

$$\hat{A}_{ij}=A_{k_id22}\ldots A_{k_2d22}A_{k_1d22}$$

where $k_1,k_2,\ldots,k_i$ are integers between 1 and p. Using Lemma 4 and (8), $\hat{A}_{ij}$ satisfies the bound

$$\|\hat{A}_{ij}\|=\|A_{k_id22}\ldots A_{k_2d22}A_{k_1d22}\|\le\chi e^{-\alpha\hat{d}_{ij}} \quad (20)$$

with

$$\chi=\sqrt{\frac{\lambda_{max}(Q_{k_i}322)\ldots\lambda_{max}(Q_{k_2}322)\lambda_{max}(Q_{k_1}322)}{\lambda_{min}(Q_{k_i}322)\lambda_{min}\ldots(Q_{k_2}322)\lambda_{min}(Q_{k_1}322)}}$$
$$\hat{d}_{ij}=\sum_{e=1}^{i}\overline{d}_{k_e}$$

Note also that if $t_{ij}\in\Theta$, $\hat{d}_{ij}$ will be greater than or equal to t. Therefore, using (20), the first term in (19) can be estimated as

$$\sum_{i=1}^{k(t)}\sum_{t_{ij}\in\Theta}\{\|\hat{A}_{ij}\|\}\|\phi\|_c\le\sum_{i=1}^{k(t)}\sum_{t_{ij}\in\Theta}\chi e^{-\alpha\hat{d}_{ij}}\|\phi\|_c \quad (21)$$

Noting that $t_{ij}\in\Theta$, which implies that $\hat{d}_{ij}>t$, (21) can be estimated as

$$\sum_{i=1}^{k(t)}\sum_{t_{ij}\in\Theta}\{\|\hat{A}_{ij}\|\}\|\phi\|_c\le\sum_{i=1}^{k(t)}\sum_{t_{ij}\in\Theta}\chi\|\phi\|_c e^{-\alpha t} \le\chi\sum_{i=1}^{k(t)}\sum_{t_{ij}\in\Theta}\{1\}\|\phi\|_c e^{-\alpha t} \quad (22)$$

The summation in (22) equals to the number of leaves in the tree (see Figure f1). It can be seen easily that the worst case when all the leaves are in the level $k(t)$ (i.e. $i=k(t)$), thus the summation can be bouneded by $p^{k(t)}$, and we get

$$\sum_{i=1}^{k(t)}\sum_{t_{ij}\in\Theta}\{\|\hat{A}_{ij}\|\}\|\phi\|_c e^{-\alpha t}\le\chi p^{k(t)}\|\phi\|_c e^{-\alpha t} \quad (23)$$

Now, in order to estimate the second term in (19), define

$$\|\check{A}_1\|=max\{\|A_{1d21}\|,\ldots,\|A_{pd21}\|\} \quad (24)$$
$$\|\check{A}_e\|=max\{\|A_{1d22}e^{\alpha\overline{d}_1}\|,\ldots,\|A_{pd22}e^{\alpha\overline{d}_p}\|\}. \quad (25)$$

From (8), there exist constants $\beta > 1$ and $\gamma \in (0,1)$ such that

$$\|\check{A}_e\| \le \beta\gamma^i,\ i = 1,2,\cdots. \tag{26}$$

Then, from (13)-(17), we get

$$\begin{aligned}
&\|\hat{A}_{ij}\|e^{-\alpha(t_{(i+1)j})}\\
&\quad\le \|\hat{A}_{(i-1)v_j}A_{(\mathrm{mod}(j,p)+1)d22}\|e^{-\alpha(t_{iv_j})}e^{\alpha d_{(\mathrm{mod}(j,p)+1)}(t_{iv_j})}\\
&\quad\le \|\hat{A}_{(i-1)v_j}e^{-\alpha(t_{iv_j})}A_{(\mathrm{mod}(j,p)+1)d22}e^{\alpha\bar{d}_{(\mathrm{mod}(j,p)+1)}}\|\\
&\quad\le \|\hat{A}_{(i-1)v_j}e^{-\alpha(t_{iv_j})}A_{k_1 d22}e^{\alpha\bar{d}_{k_1}}\|
\end{aligned} \tag{27}$$

Iterating on $\hat{A}_{(i-1)v_j}$ and t_{iv_j} gives after i-1 iterations,

$$\begin{aligned}
\|\hat{A}_{ij}\|e^{-\alpha(t_{(i+1)j})} &\le \|\mathbb{I}e^{-\alpha t_{1k_{i+1}}}A_{k_i d22}e^{\alpha\bar{d}_{k_i}}\dots A_{k_1 d22}e^{\alpha\bar{d}_{k_1}}\|\\
&\le \|A_{k_i d22}e^{\alpha\bar{d}_{k_i}}\dots A_{k_1 d22}e^{\alpha\bar{d}_{k_1}}\|e^{-\alpha t}e^{\alpha\bar{d}}\\
&\le \|A_{k_i d22}e^{\alpha\bar{d}_{k_i}}\|\dots\|A_{k_1 d22}e^{\alpha\bar{d}_{k_1}}\|e^{-\alpha t}e^{\alpha\bar{d}}\\
&\le \|\check{A}_e\|^i e^{-\alpha t}e^{\alpha\bar{d}}
\end{aligned} \tag{28}$$

Therefore, using (12) and noting that $\|A_{(\mathrm{mod}(j,p)+1)d21}\| \le \|\check{A}_1\|$ for any integer j, the second term in (19) can be estimated as

$$\begin{aligned}
&\sum_{i=0}^{k(t)-1}\sum_{\substack{j=0\\ t_{iv_j}\notin\Theta}}^{p^{i+1}-1}\|\hat{A}_{ij}\|\|A_{(\mathrm{mod}(j,p)+1)d21}\|\|\zeta_1(t_{(i+1)j})\|\\
&\qquad\le \|\check{A}_1\|\sqrt{\frac{\lambda_2}{\lambda_1}}\|\phi\|_c\sum_{i=0}^{k(t)-1}\sum_{\substack{j=0\\ t_{iv_j}\notin\Theta}}^{p^{i+1}-1}\|\hat{A}_{ij}\|e^{-\alpha(t_{(i+1)j})}
\end{aligned}$$

using (28),

$$\le \|\check{A}_1\|\sqrt{\frac{\lambda_2}{\lambda_1}}e^{\alpha\bar{d}}\sum_{i=0}^{k(t)-1}\sum_{\substack{j=0\\ t_{iv_j}\notin\Theta}}^{p^{i+1}-1}\{\|\check{A}_e\|^i\}\|\phi\|_c e^{-\alpha t} \tag{29}$$

Note that for any i, $\sum_{\substack{j=0\\ t_{iv_j}\notin\Theta}}^{p^{i+1}-1}\|\check{A}_e\|^i = m\|\check{A}_e\|^i$, where m equals to the number of nodes in level $i+1$ (see Figure 2). It can be seen easily that the worst case is when all the nodes exist in the level (i.e. p^{i+1} nodes), and we get

$$\begin{aligned}
\sum_{i=0}^{k(t)-1}\sum_{\substack{j=0\\ t_{iv_j}\notin\Theta}}^{p^{i+1}-1}\|\check{A}_e\|^i &\le \sum_{i=0}^{k(t)-1}p^{i+1}\|\check{A}_e\|^i\\
&\le p^{k(t)}\sum_{i=0}^{k(t)-1}\|\check{A}_e\|^i
\end{aligned}$$

using (26)

$$\le p^{k(t)}M \text{ where } M = \frac{\beta}{1-\gamma} \tag{30}$$

Therefore, using (29) and (30), the second term in (19) can be estimated as

$$\begin{aligned}
&\sum_{i=0}^{k(t)-1}\sum_{\substack{j=0\\ t_{iv_j}\notin\Theta}}^{p^{i+1}-1}\|\hat{A}_{ij}\|\|A_{(\mathrm{mod}(j,p)+1)d21}\|\|\zeta_1(t_{(i+1)j})\|\\
&\qquad\le \|\check{A}_1\|\sqrt{\frac{\lambda_2}{\lambda_1}}e^{\alpha\bar{d}}\|\phi\|_c p^{k(t)}Me^{-\alpha t}
\end{aligned} \tag{31}$$

Now, from (31) and (23), $\|\zeta_2(t)\|$ in (19) is estimated by

$$\|\zeta_2(t)\| \le \left[\chi\|\phi\|_c p^{k(t)} + \|\check{A}_1\|\sqrt{\frac{\lambda_2}{\lambda_1}}e^{\alpha\bar{d}}\|\phi\|_c p^{k(t)}M\right]e^{-\alpha t}$$

Note that,

$$\begin{aligned}
&k(t)-1 \le \frac{t}{\underline{d}}\\
&p^{k(t)-1} \le p^{\frac{t}{\underline{d}}}\\
&\max_{t\ge 0}\{p^{k(t)-1}e^{-at}\} \le \max_{t\ge 0}\{p^{\frac{t}{\underline{d}}}e^{-at}\} = 1 \text{ if } a > \frac{1}{\underline{d}}
\end{aligned}$$

Then,

$$\begin{aligned}
\|\zeta_2(t)\| &\le \Bigg[\chi\|\phi\|_c p^{k(t)}e^{-\beta t}\\
&\quad + \|\check{A}_1\|\sqrt{\frac{\lambda_2}{\lambda_1}}e^{\alpha\bar{d}}\|\phi\|_c p^{k(t)}e^{-\beta t}M\Bigg]e^{-(\alpha-\beta)t} &(32)\\
&\le \left[\chi p + \|\check{A}_1\|\sqrt{\frac{\lambda_2}{\lambda_1}}e^{\alpha\bar{d}}pM\right]\|\phi\|_c e^{-(\alpha-\beta)t} &(33)
\end{aligned}$$

where $\beta > \frac{1}{\underline{d}}$. Thus, system in (10) and (11) is exponentially stable with a minimum decaying rate equals to $(\alpha-\beta)$. Finally, as we have shown that this system is also regular and impulse-free, by Definition (1), we conclude that the system (1) is exponentially admissible. This completes the proof. □

Remark 6. If we perform the slow-fast decomposition presented in [9] to the class of singular systems, we will get two subsystems. These subsystems are referred to in the literature as *fast* and *slow* subsystems. The dynamics of the *slow* one is governed by differential equation while the dynamics of the *fast* one is governed by algebraic equation and it goes to zero immediately (This is why it is called *fast* subsystem). Yet, if we perform the decomposition to our class of systems (i.e. with multiple time-varying delays). It has been shown in the proof of Theorem (5) that the subsystem which is usually called *slow* converge with a guaranteed convergence rate equals to α, while the subsystem which is usually called *fast* converge with a guaranteed convergence rate equals to $\alpha - \frac{1}{\underline{d}}$. In other words, the so called *fast* subsystem usually converge slower than the one called *slow* subsystem. In this sense, we prefer to refer to the two subsystems as differential and algebraic subsystems instead of fast and slow subsystems.

As a special case of our class of systems, we present here the result in the case of single time-varying delay.

Corollary 7. Given positive scalars $\underline{d}_1$ and $\bar{d}_1$ with $\underline{d}_1 \le \bar{d}_1$, $\mu < 1$ and α, system (1) with time-varying delay $d_1(t)$ satisfying (2) is exponentially admissible with $\sigma = \alpha$ if there exist a nonsingular matrix $P \in \mathbb{R}^{n\times n}$, $n\times n$ symmetric and positive-definite matrices Q_1, Q_2, Q_3, Z_1, Z_2, and $n\times n$ matrices M_i, N_i, S_i, $i = 1,2$ such that the following LMI hold:

$$\left[\begin{array}{ccccc}
\Pi_{11} & \Pi_{12} & e^{\alpha d_1}M_1E & -e^{\alpha d_2}S_1E & \frac{e^{2\alpha d_2}-1}{2\alpha}N_1\\
\star & \Pi_{22} & e^{\alpha d_1}M_2E & -e^{\alpha d_2}S_2E & \frac{e^{2\alpha d_2}-1}{2\alpha}N_2\\
\star & \star & -Q_1 & 0 & 0\\
\star & \star & \star & -Q_2 & 0\\
\star & \star & \star & \star & -\frac{e^{2\alpha d_2}-1}{2\alpha}Z_1\\
\star & \star & \star & \star & \star\\
\star & \star & \star & \star & \star\\
\star & \star & \star & \star & \star
\end{array}\right.$$

$$\left.\begin{matrix} \frac{e^{2\alpha d_2}-e^{2\alpha d_1}}{2\alpha}S_1 & \frac{e^{2\alpha d_2}-e^{2\alpha d_1}}{2\alpha}M_1 & A^\top U \\ \frac{e^{2\alpha d_2}-e^{2\alpha d_1}}{2\alpha}S_2 & \frac{e^{2\alpha d_2}-e^{2\alpha d_1}}{2\alpha}M_2 & A_d{}^\top U \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ -\frac{e^{2\alpha d_2}-e^{2\alpha d_1}}{2\alpha}(Z_1+Z_2) & 0 & 0 \\ \star & -\frac{e^{2\alpha d_2}-e^{2\alpha d_1}}{2\alpha}Z_2 & 0 \\ \star & \star & -U \end{matrix}\right] < 0$$

with the following constraint

$$E^\top P = P^\top E \geq 0$$

where

$$\Pi_{11} = P^\top A + A^\top P + \sum_{i=1}^{3} Q_i + N_1 E + (N_1 E)^\top + 2\alpha E^\top P$$
$$\Pi_{12} = P^\top A_d + (N_2 E)^\top - N_1 E + S_1 E - M_1 E$$
$$\Pi_{22} = -(1-\mu)e^{-2\alpha d_2} Q_3 + S_2 E + (S_2 E)^\top - N_2 E - (N_2 E)^\top - M_2 E - (M_2 E)^\top$$
$$\underline{d} = \bar{d}_1 - \underline{d}_1, \quad U = \bar{d}_1 Z_1 + \underline{d} Z_2$$

Note that the guaranteed exponential rate in this special case is α and not $\alpha - \frac{1}{\underline{d}}$ as it may be expected from Theorem (5). This is due to the fact that $p = 1$ to the power any number equals one. Therefore, the steps in (32) and (33) in the proof of Theorem (5) are not needed and the guaranteed exponential rate is going to be α.

4. EXAMPLE

Consider the following singular system with multiple time delays:

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad A = \begin{bmatrix} -3 & 2 & 0 \\ 0 & -5 & 1 \\ 1 & 0 & 2 \end{bmatrix},$$
$$A_{1d} = \begin{bmatrix} 0 & 0.5 & 0 \\ 0 & 0 & 0.1 \\ -0.1 & 0 & -0.1 \end{bmatrix} \quad A_{2d} = \begin{bmatrix} 0.05 & 0.1 & 0 \\ 0 & -0.4 & 0.05 \\ 0.1 & 0.1 & -0.1 \end{bmatrix}$$

Let $\underline{d}_1 = 0.5$, $\bar{d}_1 = 0.6$, $\underline{d}_2 = 0.6$, $\bar{d}_2 = 0.8$ and $\mu = 0.5$. Using Theorem 5, the guaranteed convergence rate of this system is given by $\sigma = 0.4$. Note also that $\alpha = 2.4$, this means that the differential subsystem exponentially converge with a guaranteed convergence rate equals to 2.4. Figure 3 gives the simulation results of x_1, x_2 and x_3 when $d_1(t) = 0.55 + 0.04\sin(5t)$, $d_2(t) = 0.7 + 0.5\sin(5t)$ and the initial function is $\phi(t) = [\,2\ \ -1\ \ -1.0556\,]^\top, t \in [-0.7, 0]$. Note that the algebraic subsystem $x_3(t)$ converge slower than the differential subsystem.

5. CONCLUSION

This paper deals with the control of singular systems with multiple time-varying delays. New sufficient conditions for checking exponential stability are developed. The results can be extended to uncertain systems.

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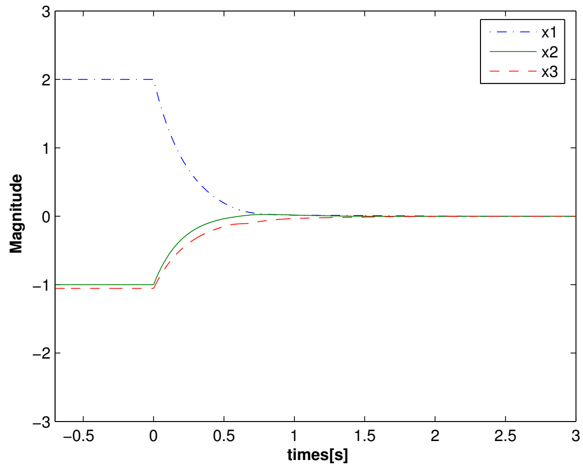

Fig. 3. Simulation results of x_1, x_2 and x_3

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