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Positional procedures for solving dynamical optimization problems of prescribed duration ⋆

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Abstract: Optimal control problems under conflict or uncertainties are considered on a finite time interval. Resolving dynamical procedures are suggested. They are based either on constructions of stable bridges produced with the help of step-by-step programmed absorptions or on applications of the backward dynamic programming method. Results of simulations illustrating the suggested methods are exposed.

1. INTRODUCTION

The paper deals with optimal control problems and problems of the theory of differential games considered on a finite time interval. Solution technique is presented for problems formed in the framework of the approach, which is developed in the scientific school of N.N. Krasovskii. A distinctive feature of these methods is defined by the use of backward dynamic constructions in controlled systems and application of the theory of generalized solutions of Hamilton-Jacobi equations, which was created by A.I. Subbotin in the last decades of the 20-th century.

2. SOLUTION OF DIFFERENTIAL GAMES OF PRESCRIBED DURATION

The first part of the paper deals with game pursuit problems on a finite time interval. The pursuit problem is formulated as a positional differential game. The proposed solution method is based on the concept of extremal shift, developed by N.N. Krasovskii and his pupils (Krasovskii [1970], Krasovskii, and Subbotin [1988], Kurzhanski et al. [2002], Kryazhimskii, and Osipov [1973], Subbotin, and Chentsov [1981], Subbotina et al. [1986]). According to this concept, the central element of the resolving construction is the maximal stable bridge, i.e. the set W^0 of all positions (t_*, x_*) of the game from which the pursuit problem is solvable. The solving control $U^*(t,x)$ can be constructed as a positional strategy extremal to W^0 (Krasovskii, and Subbotin [1988]).

Efficient analytical descriptions of W^0 are possible for simple problems only. So, development of numerical methods for constructing W^0 is of importance. The definition of the set W^0 is not constructive itself. However, it is important that the set W^0 is stable and this property can be used for numerical finding of W^0. The paper (Tarasyev, and Ushakov [1983]) presents a method of approximate calculation of W^0 based on backward step-by-step constructions of stability properties. Examples of calculating W^0 for some plane problems of pursuit are given.

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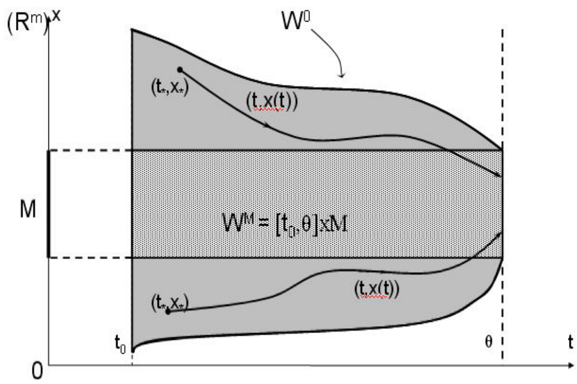


Fig. 1. Stable bridge.

2.1 *Statement of the encounter game problem*

On $[t_0, \vartheta]$ a conflict controlled system is given

$$\dot{x} = f(t,x,u,v), \quad x(t_0) = x_0, \quad u \in P, \quad v \in Q \qquad (1)$$

$x \in R^m$ is a phase vector, u and v are the vectors of control and disturbance (control of the 1st and 2nd players), $P \subset R^p$, $Q \subset R^q$ are compact. System (1) satisfies the traditional assumptions of the theory of differential games In the phase space R^m, a compact set M is given, which is the target for the 1st player. Taking into account the conditions imposed on system (1) and the set M, we can assume that the game takes place in a compact set $D \subset [t_0, \vartheta] \times R^m$, which is large enough.

Let $W^M = [t_0, \vartheta] \times R^m$ be a cylinder in $[t_0, \vartheta] \times R^m$.

Problem 1. The first player needs to find a strategy $U^*(t,x)$ such that all motions $x(t)$ of system (1) generated by this strategy together with various admissible controls $v(t)$ of the second player satisfy the condition: $(\tau, x(\tau)) \in W^M$ at some moment $\tau \in [t_0, \vartheta]$.

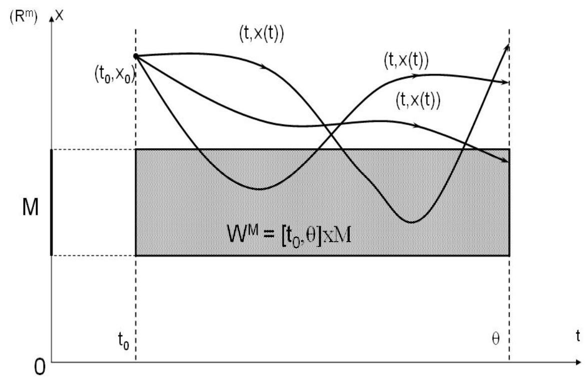


Fig. 2. An illustrative scheme of problem 1.

2.2 Unification in problems of pursuit

The basic property used to construct the solution of problem 1 is the property of stability of the set W^0. The stability property admits wide variety of formulations. The efficiency of approximate calculation of W^0 depends on what formulation of stability is chosen. In the mid of 1970-s in articles by N.N. Krasovskii (Krasovskii [1976, 1977]) a unification formulation of the stability property was proposed. In this formulation, a family of differential inclusions (d.i.) was considered

$$\dot{x} \in F_s(t,x), \quad s \in S, \tag{2}$$
$$F_s(t,x) = F(t,x) \cap \Pi_s(t,x),$$
$$F(t,x) = co\{f(t,x,u,v) : u \in P, v \in Q\},$$
$$\Pi_s(t,x) = \{f \in R^m : \langle s, f \rangle \leqslant H(t,x,s)\}.$$

The stability of the set W in the space of positions (t_*, x_*) of the game is defined as the property of weak invariance of W with respect to differential inclusions (2).

The unification formulation of stability has proved to be convenient in many purposes, and in particular, for elaboration of algorithms for constructing solutions of differential games. Various aspects of unification are studied in articles (Alekseichik [1970], Ushakov [1980], Grigorieva et al. [1996]). In the 1980-s, a generalized unification model of differential game was introduced.

2.3 A generalized unification model

Let us fix a set $\Psi = \{\psi\}$, a family of multivalued maps $(t,x) \mapsto F_\psi(t,x)$, $\psi \in \Psi$, and a ball $G = \{g \in R^m : \|g\| \leqslant \leqslant K\}$. Assume

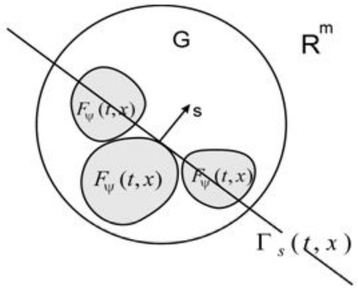
A.1. $F_\psi(t,x) \subset G$ is closed and convex for $\forall (t,x,\psi) \in \in D \times \Psi$

A.2. for $\forall (t,x,s) \in D \times S$

$$\min_{\psi \in \Psi} h_{F_\psi(t,x)}(l) = H(t,x,l),$$

where $h_F(l)$ is the support function of the set $F = F_\psi(t,x)$.

A.3. The maps $(t,x) \mapsto F_\psi(t,x)$ are continuous on D uniformly with respect to $\psi \in \Psi$.



$$\Gamma_s(t,x) = \{f \in R^m : \langle s, f \rangle = H(t,x,s)\}$$

Fig. 3. A family of maps $(t,x) \mapsto F_\psi(t,x)$.

For some systems of the form (1), one can introduce a family of maps $(t,x) \mapsto F_psi(t,x)$, $\psi \in \Psi$, satisfying A.1, A.2, A.3, where Ψ is finite. Namely, it takes place for the system

$$\dot{x} = f(t,x,u) + C(t,x)v, \quad u \in P, \quad v \in Q \tag{3}$$

where Q is a convex polyhedron in R^m. For Ψ, one can take a set $\{v^*\}$ of vertices of the polyhedron Q, and for $F_\psi(t,x)$ one can take the sets $F_{v^*}(t,x) = co\{f(t,x,u) : u \in P\} + C(t,x)v^*$.

Let us recall the definition of u-stable bridge in terms of the generalized unification scheme. Let $X_\psi(t; t_*, x_*)$ be the attainability set at the moment t of the differential inclusion $\dot{x} \in F_\psi(t,x)$, $x(t_*) = x_*$.

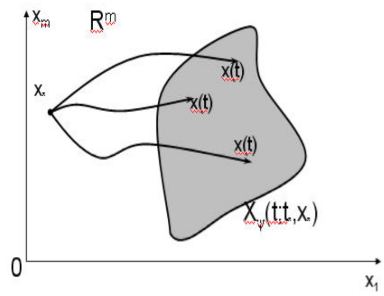


Fig. 4. An attainability set of d. i. $\dot{x} \in F_\psi(t,x)$, $x(t_*) = x_*$.

Definition 1. A set $W \subset [t_0, \vartheta] \times R^m$ is called u-stable bridge in the problem 1 if

1. $W(\vartheta) = \{x \in R^m : (\vartheta, x) \in W\} \subset M$;

2. For any t_*, t^* $(t_0 \leqslant t_* < t^* \leqslant \vartheta)$, $(t_*, x_*) \in W$, $\psi \in \Psi$ one has $X_\psi(t^*; t_*, x_*) \cap W(t^*) \neq \varnothing$ or $X_\psi(\tau; t_*, x_*) \cap \cap M \neq \varnothing$ for some $\tau \in [t_*, t^*]$.

2.4 An approximating system of sets

Let us consider the maximal u-stable bridge W^0. For approximate backward construction of W^0, let us introduce a partition $\Gamma = \{t_0, t_1, \ldots, t_i, t_{i+1}, \ldots, t_N = \vartheta\}$ of the interval $[t^0, \vartheta]$. For each interval $[t_i, t_{i+1})$, we substitute the differential inclusion

$$\dot{x} \in F_\psi(t,x) \tag{4}$$

with the initial value $x(t_i)$, by the differential inclusion

$$\dot{x} \in F_\psi(t_i, x(t_i)) \tag{5}$$

with the initial value $x(t_i)$.

This substitution implies the corresponding transformation of the attainability sets $X_\psi(t_{i+1}; t_i, x(t_i))$ for differential inclusion (4) to the following convex closed sets for differential inclusion (5):

$$\tilde{X}_\psi(t_{i+1}; t_i, x(t_i)) = x(t_i) + \Delta_i F_\psi(t_i, x(t_i)),$$
$$\Delta_i = t_{i+1} - t_i > 0.$$

This transformation turns out to be convenient. For an arbitrary set Φ from R^m, we introduce the set

$$\tilde{X}_\psi^{-1}(t_i; t_{i+1}, \Phi) =$$
$$= \{x(t_i) \in R^m : \tilde{X}_\psi(t_{i+1}; t_i, x(t_i)) \cap \Phi \neq \varnothing\}.$$

In the phase space R^m, let us consider a system of sets $\{\tilde{W}(t_i) : t_i \in \Gamma\}$:

$$\tilde{W}(t_N) = M_{\varepsilon_N}, \quad \tilde{W}(t_i) = M_{\varepsilon_{i+1}} \cup \pi(t_i; t_{i+1}, \tilde{W}(t_{i+1})),$$
$$i = N-1, N-2, \ldots, 1, 0.$$

Here $\pi(t_i; t_{i+1}, \tilde{W}(t_{i+1})) = \bigcap\limits_{\psi \in \Psi} \tilde{X}_\psi^{-1}(t_i; t_{i+1}, \tilde{W}(t_{i+1}))$, ε_0, ε_1, …, ε_N ($\varepsilon_0 = 0$) is a monotone increasing sequence.

To the system $\{\tilde{W}(t_i) : t_i \in \Gamma\}$ in the phase space R^m, one can put into correspondence a system $\{(t_i, \tilde{W}(t_i)) : t_i \in \Gamma\}$ in the space of positions (t, x) of the game. For the last system, one can introduce a suitable definition of the limit

$$\lim_{\Delta(\Gamma) \to 0} \{(t_i, \tilde{W}(t_i)) : t_i \in \Gamma\} = W^*$$

Condition 1. Let the set M in R^m be a union of closed balls whose radiuses are bounded from below by a positive number.

Theorem 1. Let the target set M in problem 1 satisfy condition 1. Then $W^* = W^0$.

Thus, the system $(t_i, \tilde{W}(t_i)) : t_i \in \Gamma)$ approximates the set W^0.

2.5 A modification of the homicidal chauffeur game

Example 1. Consider the system

$$\begin{cases} \dot{x}_1 = -\dfrac{w_1}{R} x_2 u_1 + w_2 v_1 \\ \dot{x}_2 = \dfrac{w_1}{R} x_1 u_1 + w_2 v_2 + w_1 u_2 + \gamma \end{cases}$$

where $x = (x_1, x_2) \in R^2$ is a phase vector, $u \in P = \{u^* = (u_1, u_2) : \|u^*\| \leqslant 1\}$ is a vector of control of the 1-st player, $v \in Q = \{v^* = (v_1, v_2) : \|v^*\| \leqslant 1\}$ is a vector of control of the 2-nd player. The target set M is the circle of radius 1. The approximations of u-stable bridges are presented on Fig.5-6 for the following parameters

A. $[t_0, \vartheta] = [0, 10]$, $R = 1$, $w_1 = 1.2$, $w_2 = 1$, $\gamma = 0$

B. $[t_0, \vartheta] = [0, 32]$, $R = 4$, $w_1 = 1.5$, $w_2 = 1$, $\gamma = 0.5$

3. OPTIMAL CONTROL PROBLEMS OF PRESCRIBED DURATION

The section deals with a new numerical method for constructing the value functions to optimal control problems

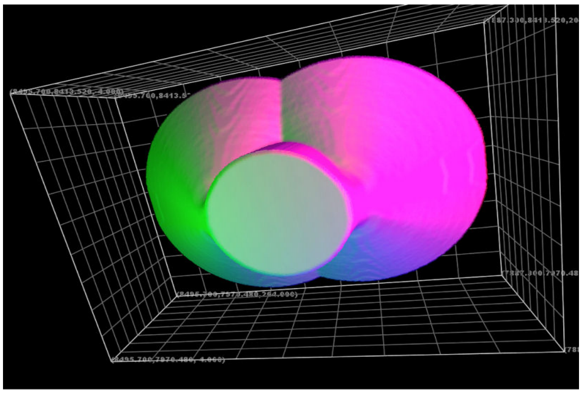


Fig. 5. The stable bridge in the modified homicidal chauffeur game, case A.

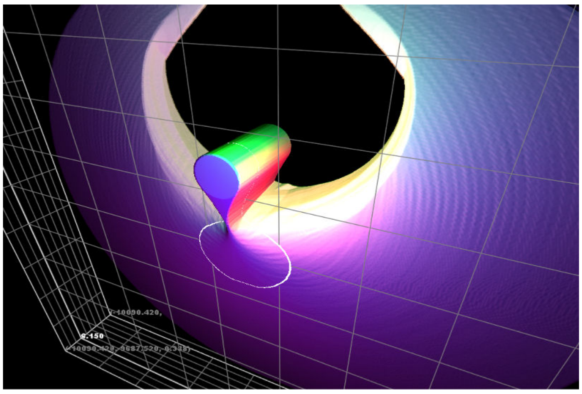


Fig. 6. The stable bridge in the modified homicidal chauffeur game, case B.

of prescribed duration. We consider the problems with positional running cost over motions of controlled dynamical systems on a finite time interval. The presenting algorithm is based on a backward procedure involved generalized characteristics of the Bellman equation. Note that the value function takes the key part in constructions of optimal feedbacks solving the problems. The optimal synthesis can be constructed as aiming in directions of generalized gradients of the value function.

3.1 Statement of the optimal control problem and basic notions

Problem 2. We consider optimal control problems where dynamics are described by the equation

$$\dot{x}(t) = f(t, x, u), \quad x[t_0] = x_0, \quad u \in P. \tag{6}$$

Here $t \in [0, T]$ is time, $x \in R^n$ is a phase vector, controls u belong to the given compact set $P \subset R^m$. Assume that the cost functional has the form

$$I_{t_0, x_0}(x(\cdot), u(\cdot)) = \tag{7}$$

$$= \min_{\theta \in [t_0, T]} \{\sigma(\theta,\, x(\theta; t_0, x_0, u(\cdot))) + \int\limits_{t_0}^{\theta} g(t, x(t), u(t))dt\}, \tag{8}$$

where $x(\cdot) = x(\cdot; t_0, x_0, u(\cdot))$: $[t_0, T] \mapsto R^n$ is a trajectory of the system (6) started at an initial state (t_0, x_0) under an admissible control $u(\cdot)$: $[t_0, T] \mapsto P$.

We consider the problems of prescribed duration $(T - t_0)$ provided the optimal result $V(t_0, x_0)$ (the value) is equal to

$$V(t_0, x_0) = \inf_{u(\cdot)\in U_{t_0}} I_{t_0, x_0}(x(\cdot; t_0, x_0, u(\cdot)), u(\cdot)) \quad (9)$$

at any initial state $(t_0, x_0) \in [0, T] \times R^n$. The symbol U_{t_0} denotes the set of all measurable functions $u(\cdot)\colon [t_0, T] \mapsto P$ called programs.

Let symbols Π_T and $cl\,\Pi_T$ denote the regions in space R^{n+1}:

$$\Pi_T = (0, T) \times R^n, \quad cl\,\Pi_T = [0, T] \times R^n.$$

We assume that the data are Lipschitz continuous with respect to the phase variables (t, x), and satisfy extendability conditions. The sets

$$\mathrm{Arg} \min_{(f,g)\in F(t,x)} [\langle p, f\rangle + g] = \{(f^0(t, x, p), g^0(t, x, p))\},$$

are assumed to be singletons, for any $(t, x) \in P$, $p \in R^n$. Namely, each set contains the unique element equal to $(f^0(t, x, p), g^0(t, x, p))$. Here

$$E(t, x) = (f(t, x, P), g(t, x, P)) \subset R^n \times R.$$

The symbol $\langle\cdot,\cdot\rangle$ denotes the inner product.

The value function $V(t, x)$ is nonsmooth, as a rule. It is known (Subbotin [1995]), that the value function $V(t, x)$ of the considered optimal control problem coincides with the unique minimax solution of the following Cauchy problem for the Bellman equation

$$\partial V(t, x)/\partial t + \min_{u\in P}\big[\langle D_x V(t, x), f(t, x, u)\rangle + g(t, x, u)\big] = 0, \quad (t, x) \in \Pi_T, \quad (10)$$

with the boundary condition

$$V(T, x) = \sigma(T, x), \quad \forall x \in R^n, \quad (11)$$

and with the additional restrictions

$$V(t, x) \le \sigma(t, x), \quad \forall (t, x) \in \Pi_T. \quad (12)$$

Here

$$D_x V(t, x) = \left(\frac{\partial V(t, x)}{\partial x_1}, \ldots, \frac{\partial V(t, x)}{\partial x_n}\right).$$

3.2 Generalized characteristics for the Bellman equation

Consider the Hamiltonian $H(t, x, s)$

$$H(t, x, s) = \min_{u\in P}\big[\langle s, f(t, x, u)\rangle + g(t, x, u)\big]. \quad (13)$$

Let us recall that the classical characteristics for the Bellman equation (10) are solutions of the system of ODEs:

$$\begin{cases} \dfrac{d\hat{x}}{dt} = D_{\hat{p}} H(t, \hat{x}, \hat{p}), \\ \dfrac{d\hat{p}}{dt} = -D_{\hat{x}} H(t, \hat{x}, \hat{p}), \\ \dfrac{d\hat{z}}{dt} = \langle \hat{p}, D_{\hat{p}} H(t, \hat{x}, \hat{p})\rangle - H(t, \hat{x}, \hat{p}), \end{cases} \quad (14)$$

and satisfy the boundary conditions:

$$\hat{x}(T, y) = y,\ \hat{p}(T, y) = D_y\sigma(T, y),\ \hat{z}(T, y) = \sigma(T, y),$$

at $t = T$; here vectors $y \in R^n$ are parameters.

The symbols $D_p H(t, \hat{x}, \hat{p})$, $D_x H(t, \hat{x}, \hat{p})$ denote the vectors

$$D_p H(t, \hat{x}, \hat{p}) = (\partial H(t, \hat{x}, \hat{p})/\partial p_1, \ldots, \partial H(t, \hat{x}, \hat{p})/\partial p_n),$$
$$D_x H(t, \hat{x}, \hat{p}) = (\partial H(t, \hat{x}, \hat{p})/\partial x_1, \ldots, \partial H(t, \hat{x}, \hat{p})/\partial x_n).$$

Due to the Lipschitz continuity of the data of the considered optimal control problem, the Hamiltonian $H(t, x, s)$ is Lipshitz continuous, too. Notions of generalized characteristics were introduced in different manners, for example, (Clarke [1983], Blagodatskikh [1984], Melikyan [1998], Bardi et al. [1999]). We introduce the following notion.

The set $\partial_x^{Cl} H(t, \hat{x}, \hat{p}) \in R^n$ of the form

$$\partial_x^{Cl} H(t, \hat{x}, \hat{p}) = \mathrm{co}\{\forall \lim_{(t',x')\to(t,\hat{x})} D_x H(t', x', \hat{p})\}. \quad (15)$$

is called the Clarke partial subdifferential in x of the Hamiltonian (see, Clarke [1983]).

Here the symbol co denotes convex hull, the points $(t', x', \hat{p})$ are regular points of the Hamiltonian $H(t, x, s)$, where $H(t, x, s)$ is differentiable in variables (t, x).

Let us introduce *characteristic inclusions* as follows

$$\begin{cases} \dfrac{d\hat{x}}{dt} = D_p H(t, \hat{x}, \hat{p}) = f^0(t, \hat{x}, \hat{p}), \\ \dfrac{d\hat{p}}{dt} \in -\partial_x^{Cl} H(t, \hat{x}, \hat{p}), \\ \dfrac{d\hat{z}}{dt} = \langle \hat{p}, D_p H(t, \hat{x}, \hat{p})\rangle - H(t, \hat{x}, \hat{p}) = -g^0(t, \hat{x}, \hat{p}), \end{cases} \quad (16)$$

Let us also introduce the boundary conditions:

$$\hat{x}(t, y) = y,\ \hat{p}(t, y) \in \partial_x^{Cl}\sigma(t, y),\ \hat{z}(T, y) = \sigma(t, y), \quad (17)$$

for any $(t, y) \in \Pi_T$, where $V(t, y) = \sigma(t, y)$. The variables $y \in R^n$ play roles of parameters.

Definition 2. Absolutely continuous functions

$$(\hat{x}(\cdot), \hat{p}(\cdot), \hat{z}(\cdot)) :\ [0, T] \to R^n \times R^n \times R,$$

satisfying the differential inclusions (16) and the boundary condition (17) are called *generalized characteristics* for the Bellman equation (10).

3.3 An auxiliary optimal control problem

Generalized characteristics (16)-(17) can be used to calculate the optimal result (9). A justification of the calculations is based on connections between extremals and co-extremals of the Pontryagin maximum principles and characteristics of the Bellman equation (Pontryagin et al. [1962], Bellman [1957]). A key part of the construction is the following auxiliary optimal control problem on time interval $[t_*, t^*] \subset [0, T]$:

Problem 3.

$$\dot{x}(t) = f(t, x, u), \quad x[t_*] = x_*, \quad u \in P; \quad (18)$$

$$I_{t_*, x_*}(x(\cdot), u(\cdot)) = \quad (19)$$

$$= \sigma(t^*,\ x(t^*; t_*, x_*, u(\cdot))) + \int_{t_*}^{t^*} g(t, x(t), u(t))dt; \quad (20)$$

$$V^*(t_*, x_*) = \inf_{u(\cdot)\in U_{t_*}} I_{t_*, x_*}(x(\cdot; t_*, x_*, u(\cdot)), u(\cdot)) \quad (21)$$

Results of the monographs (Clarke [1983], Subbotina [2006]) imply validity of the following propositions.

Theorem 2. Let the data of the problem (18)-(21) provide existence, boundedness and continuity of partial derivatives of the Hamiltonian $H(t,x,s)$ (13). Then, for any point $(t_*,x_*) \in \Pi_T$, the set of all extremals $x^e(\cdot) = x(\cdot;t_*,x_*,u^e(\cdot))$ satisfying the Pontryagin maximum principle coincides with the set $X(t_*,x_*,t^*)$ of components $\hat{x}(\cdot,y)$ of classical characteristics for the Bellman equation crossing at the point (t_*,x_*), i.e.

$$X(t_*,x_*,t^*) = \{\hat{x}(\cdot,y)\colon \hat{x}(t_*,y) = x_*,\ \hat{x}(t^*,y) = y\}.$$

The set of corresponding co-extremals $p^e(\cdot) = p(\cdot;t^*,x^e(t^*), u^e(\cdot))$ coincides with the set of components $\hat{p}(\cdot,y)$ of classical characteristics satisfying the boundary conditions:

$$y \in Y(t_*,x_*,t^*) = \{y\colon \hat{x}(t^*,y) = y,\ \hat{x}(t_*,y) = x_*\},$$
$$\hat{p}(t^*,y) = D_y\sigma(t^*,y).$$

Theorem 3. Let the data of the problem (18)-(21) provide existence, boundedness and continuity of partial derivatives of the Hamiltonian $H(t,x,s)$ (13). Then, for any point $(t_*,x_*) \in \Pi_T$, the optimal result $V^*(t_*,x_*,t^*)$ for the controlled system (18) on the interval $[t_*,t^*]$ can be represented as follows

$$V^*(t_*,x_*,t^*) = \inf_{y_0,\ p_0} V(t^*,y_0) + \int t_*t^* g^0(\tau,\hat{x}(\tau),\hat{p}(\tau))d\tau,$$

where

$$y_0 \in Y(t_*,x_*,t^*), \quad p_0 = \hat{p}(t^*,y_0) = D_y\sigma(t^*,y_0).$$

3.4 A representation formula of the value function

Let us introduce a new tool of nonsmooth analysis needed to get necessary and sufficient optimality conditions to the problem (2), and apply it to a representation formula of optimal result $V(t,x)$ (21).

Let us consider a local Lipschitz continuous function $V'(\cdot): \Pi_T \to R$, a point $(t,x) \in \Pi_T$, a vector $h \in R^n$, and regular points $(t+\delta_k, x+\delta_k h'_k)$ of the function $V'(t,x)$.

Definition 3. The set

$$\partial_{1,h}V'(t,x) = \mathrm{co}\Bigg\{\lim_{\delta_k\downarrow 0, h'_k\to h, k\to\infty}\Bigg(\frac{\partial V'(t+\delta_k,x+\delta_k h'_k)}{\partial t}, D_xV'(t+\delta_k,x+\delta_k h'_k)\Bigg)\Bigg\},$$

is called *the subdifferential* of $V'(t,x)$ *in the direction* $(1,h) \in R^{n+1}$.

Theorems 2, 3, and the notion of the directional subdifferential (3) are the basis of the following proposition (see, Subbotina [2006]).

Theorem 4. For any initial point $(t,x) \in \Pi_T$, the value function $V(t,x)$ (9) has the representation

$$V(t,x) = \min_{x^0(\cdot),p^0(\cdot)} \sigma(y) + \int_t^T g^0(\tau,\hat{x}^0(\tau,y),\hat{p}^0(\tau,y))\, d\tau, \tag{22}$$

where $\hat{x}^0(\cdot,y)$ are generalized characteristics (16), (17) crossing at point $(t,x):\ \hat{x}^0(t,y) = x$. Moreover, for all $\tau \in [t,T]$, the characteristics satisfy the relations:

$$\begin{aligned}&(-H(\tau,\hat{x}^0(\tau,y),\hat{p}^0(\tau,y)),\ \hat{p}^0(\tau,y)) \in \\ &\in \partial_{1,f^0(\tau,\hat{x}^0(\tau,y),\hat{p}^0(\tau,y))}V(\tau,\hat{x}^0(\tau,y)).\end{aligned} \tag{23}$$

3.5 Algorithms

An algorithm for numerical approximation of the value function $V(t,x)$ to the problem (6)-(9), which based on theorems 2-4, is created for the case of the smooth data (see, Subbotina, and Tokmantsev [2006]).

Consider a partition $\Delta = \{t_0,t_1,\ldots,\ t_N = T\}$ of the given finite time interval. The characteristic system (16) is integrated on intervals $[t_i,t_{i+1}]$, $i = \overline{0,N-1}$, in the backward time. According to (22) the numerical approximation $\tilde{V}(t_i,x)$ of the value function is constructed at points $\hat{x}$ on the characteristics at instants $t_i \in \Delta$. Let us stress that *all* boundary conditions are defined on the boundary manifolds (17). So, one can consider the backward dynamic programming procedure as a generalization of the Cauchy characteristics method.

The inequalities (23) are used also to create optimal feedbacks $U^0(t,x)$:

$$\begin{aligned}U^0(t,x) \in \{u^0 \in P: \quad &(f(t,x,u^0),g(t,x,u^0)) = \\ = \big(f^0(t,\hat{x}^0(t,y),\hat{p}^0(t,y))\big)&,g^0(t,\hat{x}^0(t,y),\hat{p}^0(t,y)\big), \\ x = \hat{x}^0(t,y),\ \hat{p}^0(t',y) \in \partial_y V(t',y),&\ \hat{z}^0(t',y) = \sigma(t',y)\}.\end{aligned} \tag{24}$$

Results of realization of the algorithms are exposed in the next subsection.

3.6 Examples

Example 2. Let dynamics of the controlled system be described by the equation

$$\dot{x}_1 = x_2,$$
$$\dot{x}_2 = u,$$
$$\|u\| \le 1;\ t \in [0,1].$$

The cost functional has the form

$$I_{t_0,x_0}(x(\cdot),u(\cdot)) = \min_{\theta\in[t_0,1]}\Big\{(x_1(\theta)+\theta-1)^2/2 + x_2^2(\theta)/2 + \int_{t_0}^{\theta} u^2(t)dt\Big\}.$$

Parameters of approximation are $\Delta t = 0.02$, $\Delta x = 0.027$, $\Delta p = 0.03$. Below on Fig.7 two graphs are exposed at the time moment $t = 0.0$.

Example 3. Consider dynamics of the controlled system in the form

$$\dot{x}_1 = x_2,$$
$$\dot{x}_2 = -\sin x_1 + u,$$
$$\|u\| \le 1;\ t \in [1,2].$$

The cost functional is

$$I_{t_0,x_0}(x(\cdot),u(\cdot)) = \min_{\theta\in[t_0,2]}\Big\{-\frac{(x_1(\theta)+\theta-2)^2}{2} - (x_2^2(\theta))/2 - 3\theta(2-\theta) - \int_{t_0}^{\theta}\sqrt{1-u^2(t)}dt\Big\}.$$

The charascteristic system has the form

$$\begin{cases} d\hat{x}_1/dt = \hat{x}_2, \quad d\hat{x}_2/dt = -\sin\hat{x}_1 - \hat{p}_2/\sqrt{(\hat{p}_2)^2+1}; \\ d\hat{p}_1/dt = \hat{p}_2, \quad d\hat{p}_2/dt = -\hat{p}_1 \quad d\hat{z}/dt = 1/(\sqrt{1+(\hat{p}_2)^2}. \end{cases}$$

The boundary conditions are

$$\hat{x}_1(T,y) = y_1,\ \hat{x}_2(T,y) = y_2,$$

$$\hat{p}_1(T,y) = -y_1,\ \hat{p}_2(T,y) = -y_2,\ \hat{z}(T,y) = -(y_1^2 + y_2^2)/2.$$

Parameters of approximation are $\Delta t = 0.02$, $\Delta x = 0.16$, $\Delta p = 0.02$. Below on Fig.8 two graphs are exposed at the time moment $t = 1.0$.

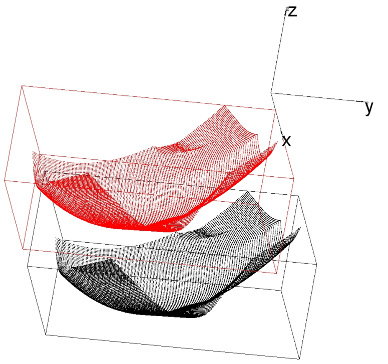


Fig. 7. The upper graph is the graph of the value function, and the lower graph is the graph of the cost functional generated by the optimal feedback.

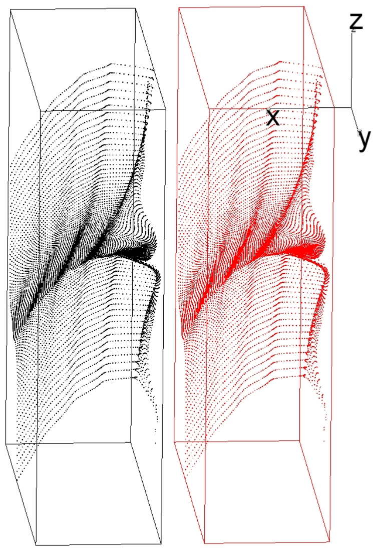


Fig. 8. The right graph is the graph of the value function, and the left graph is the graph of the cost functional generated by the optimal feedback.

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