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Observer Design for a class of uniformly observable MIMO nonlinear systems with coupled structure

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Abstract: A high gain observer is synthesized from a canonical form that characterizes the class of uniformly observable systems. Two main contributions are to be emphasized: the first is related to the considered structure of the canonical form which does not assume a complete triangular structure. That is, each block may contain nonlinearities which depend on the whole state. The second main contribution lies in the simplicity of the observer gain synthesis since the expression of this gain is given and its calibration is reduced to the choice of a single design parameter. Moreover, this involves a design function that has to satisfy a mild condition which is given. Different expressions of such a function are proposed. Of particular interest, it is shown that high gain observers and sliding mode like observers can be derived by considering particular expressions of the design function. An example with simulation results is given for illustration purposes.

Keywords: Nonlinear systems. High gain observers. Sliding mode observers. MIMO systems.

1. INTRODUCTION

This paper exhibits a state observer, with global error convergence, for nonlinear systems which are diffeomorphic to:

$$\begin{cases} \dot{x} = Ax + \varphi(u,x) \\ y = Cx \end{cases} \tag{1}$$

where the state $x = \begin{pmatrix} x^1 \\ x^2 \\ \vdots \\ x^q \end{pmatrix} \in \mathbb{R}^n$, with $x^k = \begin{pmatrix} x^k_1 \\ x^k_2 \\ \vdots \\ x^k_{\lambda_k} \end{pmatrix} \in \mathbb{R}^{n_k}$, $x^k_i \in \mathbb{R}^{p_k}$, $i = 1,\dots,\lambda_k$, $k = 1,\dots,q$, $\sum_{k=1}^{q} n_k = \sum_{k=1}^{q} p_k\lambda_k = n$; the output $y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_q \end{pmatrix} \in \mathbb{R}^p$ with $y_k \in \mathbb{R}^{p_k}$, $k = 1,\dots,q$ and $\sum_{k=1}^{q} p_k = p$; $A = \begin{bmatrix} A_1 & & \\ & \ddots & \\ & & A_q \end{bmatrix}$, $A_k = \begin{bmatrix} 0 & I_{p_k} & & 0 \\ \vdots & & \ddots & \\ 0 & \dots & 0 & I_{p_k} \\ 0 & \dots & 0 & 0 \end{bmatrix}$, $C = \begin{bmatrix} C_1 & & \\ & \ddots & \\ & & C_q \end{bmatrix}$, $C_k = [\, I_{p_k} \; 0 \dots 0 \,]$ and the nonlinear function field $\varphi(u,x) = \begin{pmatrix} \varphi^1(u,x) \\ \varphi^2(u,x) \\ \vdots \\ \varphi^q(u,x) \end{pmatrix} \in \mathbb{R}^n$; $\varphi^k(u,x) = \begin{pmatrix} \varphi^k_1(u,x) \\ \varphi^k_2(u,x) \\ \vdots \\ \varphi^k_{\lambda_k}(u,x) \end{pmatrix} \in \mathbb{R}^{n_k}$ where for $k = 1,\dots,q$, the element $\varphi^k_i(u,x) \in \mathbb{R}^{p_k}$ has the structural dependence on the states:

- for $1 \le i \le \lambda_k - 1$:

$$\varphi^k_i(u,x) = \varphi^k_i(u, x^1, x^2, \dots, x^{k-1}, x^k_1, x^k_2, \dots, x^k_i, x^{k+1}_1, x^{k+2}_1, \dots, x^q_1) \tag{2}$$

- for $i = \lambda_k$:

$$\varphi^k_{\lambda_k}(u,x) = \varphi^k_{\lambda_k}(u, x^1, x^2, \dots, x^q) \tag{3}$$

The need to study the observer design problem for nonlinear dynamical systems is, from a control point of view, well understood by now. The list of references herein covers part of the recent works done in that area. Roughly speaking, the methods to design observers for nonlinear systems can be classified into four varieties. The first one which has met a great success in the past is based on the Kalman filter which is used as a nonlinear observer (Kalman and

Bucy [1961]). The success of such a method is mainly due to the simplicity of the implementation even for big systems with many outputs. Nevertheless, a major drawback of this method still be the lack of guaranteed stability. Another known approach is that based on linearizable error dynamics where state transformations are exhibited in order to put the considered systems under a form where the nonlinearities depends only on the inputs and the outputs (Krener and Isidori [1983], Krener and Respondek [1985], Xia and W.B.Gao [1989], Hou and Pugh [1999], Guay [2002], Souleiman et al. [2003]). Thus, the resulting class of systems constitutes a subclass of system (1) with the particularity that the function φ only depends on u and y. A third method which is the subject of many studies in the last decade is that based on LMI techniques. In this approach, the gain of the observer is designed through the resolution of a LMI problem and as a consequence an observer exits only of the considered LMI problem is feasible Rajamani [1998], Fan and Arcak [2003]. As noticed in Arcak and Kokotovič [2001], the feasibility of the LMI problems considered in observer design are generally not known *a priori* and are to be determined numerically. The last and forth method is based on the observable canonical form and the first main contribution which falls in this class is that of Gauthier et al. [1992] where the authors gave a necessary and sufficient condition giving rise to the well known single output triangular canonical form. This canonical form is composed of a fixed linear dynamics component and a nonlinear triangular controlled one. Using this canonical form, the authors have designed a high gain observer under some global Lipschitz assumption on the controlled part. The gain of the proposed observer is issued from an algebraic Lyapunov equation. Many generalizations of this result to systems with many outputs have been proposed in Deza et al. [1992], Gauthier and Kupka [1994], Rudolph and Zeitz [1994], Busawon et al. [1998], Hou et al. [2000], Shim et al. [2001], Farza et al. [2004], Hammouri and Farza [2003]. Some of these works considered the extended case where the matrix A of (1) is time varying or depends on u and/or y and the gain of the proposed observer is then issued from a differential Lyapunov equation. In the sequel, one shall focus on the design of nonlinear observers where the gain does not necessitate the resolution of any dynamical system and is explicitly given.

Notice that the class of considered systems generalizes that considered in Shim et al. [2001] in two directions. Indeed, here the output x_1^k of each sub-block k is not a scalar as in Shim et al. [2001] but belongs to $\mathbb{R}^{p_k}$. The second more important generalization lies in the fact that in Shim et al. [2001], the nonlinearity intervening in the last equation of each sub-block, namely $\varphi^k_{\lambda_k}(u,x)$, assumes the same triangular state dependence as the previous variables of the same sub-block that is $\varphi^k_{\lambda_k}(u,x)$ satisfies (2) with $i = \lambda_k$ and not the more general state dependence (3) as assumed in this paper.

It has been also shown in Hammouri and Farza [2003] that the class of systems the authors considered is diffeomorphic to system (1) with $q = 1$ which means that in Hammouri and Farza [2003], all the output belong to the same block.

In a relatively old work Bornard and Hammouri [1991], the authors proposed an observer for a relatively general class of uniformly observable nonlinear systems. However, the design of the observer assumes the existence of a set of integers which are needed to construct the observer gain. In a recent work Bornard and Hammouri [2002], the same authors reintroduced the same observer by considering a graph approach. Nevertheless, the finding of the integers, necessary to construct the observer gain, still be the major drawback of the proposed approach.

In this paper, we propose to design a high gain observer, with global exponential error convergence, for a class of nonlinear systems satisfying some regularity assumptions. The general framework of this observer design is based on the works of Bornard and Hammouri [1991, 2002], Gauthier et al. [1992], Hammouri and Farza [2003]. Indeed, the gain of the proposed observer is issued from the resolution of a constant Lyapunov algebaric equation and it is explicitly given. Its tune is achieved through the choice of a single parameter whatever the dimension of the considered system.

This paper is organized as follows. In section 2, one introduces the class of nonlinear MIMO systems. In section 3, the observer design is given and a full convergence analysis is detailed. In section 4, different expressions of the observer design function are proposed giving rise to different observers. An example with simulation results are given in section 4 for illustration purposes.

2. OBSERVER DESIGN

As generally used in the high gain observer methodology Bornard and Hammouri [1991], Gauthier et al. [1992], Bornard and Hammouri [2002], Hammouri and Farza [2003], Farza et al. [2004], one assumes that system (1) satisfies the following Lipschitz assumption:

Assumption *1.* $\varphi(u,x)$ is a globally Lipschitz nonlinear function with respect to x uniformly to u.

Remark *2.1.* Assumption (1) may be very restrictive since the lipschitz conditions are in general locally satisfied. However, these conditions can be omitted in the case where the trajectory $x(t)$ of system (1) lies in a bounded set Ω (notice that such a situation always occurs in practical situations since physical models take sense only on bounded physical sets). In such a case, we can respectively extend the nonlinearities $\varphi(u,x)$ into $\widetilde{\varphi}(u,x)$ in such a way that the restriction of $\widetilde{\varphi}(u,x)$ coincides with $\varphi(u,x)$ on Ω and $\widetilde{\varphi}(u,x)$ becomes global Lipschitz on the whole state $\mathbb{R}^n$. The prolongations techniques were initially used by Bornard and Hammouri [1991] and Gauthier et al. [1992] and they have been recently detailed by Shim [2000], Shim et al. [2001]. The description of these techniques is not the subject of this work. In the sequel, we shall assume that the prolongations are achieved if necessary and system (1) will be considered on $\mathbb{R}^n$. To avoid redundancy of symbols, we maintain the notation $\varphi(u,x)$ to refer its prolongation.

Before giving a candidate observer, one introduces the following notations:

- Let $\Delta_k(\theta)$ be the diagonal matrix defined by:

$$\Delta_k(\theta) = diag\left(I_{p_k}, \frac{1}{\theta^{\delta_k}} I_{p_k}, \ldots, \frac{1}{\theta^{\delta_k(\lambda_k-1)}} I_{p_k}\right) \qquad (4)$$

where $\theta > 0$ is a real number and one defines δ_k which indicates the power of θ as follows:

$$\begin{cases} \delta_k = \prod_{i=k+1}^{q} (\lambda_i - 1) \quad \text{for} \quad 1 \leq k \leq q-1; \\ \delta_q = 1 \end{cases} \tag{5}$$

• Let S_k be the unique solution of the algebraic Lyapunov equation

$$S_k + A_k^T S_k + S_k A_k = C_k^T C_k \tag{6}$$

where A_k and C_k are defined in system (1). It can be shown that the explicit solution of (6) is symmetric positive definite for every $\theta > 0$ and in particular, one has

$$S_k^{-1} C_k^T = (C_{n_k}^1 I_{p_k}, \ldots, C_{n_k}^{n_k} I_{p_k})^T$$

• $\forall \xi^k = \begin{pmatrix} \xi_1^k \\ \xi_2^k \\ \vdots \\ \xi_{\lambda_k}^k \end{pmatrix} \in \mathbb{R}^{n_k}$ with $\xi_i^k \in \mathbb{R}^{p_k}$, $i = 1, \ldots, \lambda_k$, let $K_k(\xi_1^k) \in \mathbb{R}^{n_k}$ be a vector of smooth functions satisfying:

$$\xi^{k^T} C_k^T C_k K_k(\xi_1^k) \geq \frac{1}{2} \xi^{k^T} C_k^T C_k \xi^k \tag{7}$$

A candidate observer for system (1) is described by the following dynamical system:

$$\dot{\hat{x}}^k = A_k \hat{x}^k + \varphi^k(u, \hat{x}) - \theta^{\delta_k} \Delta_k^{-1}(\theta) S_k^{-1} C_k^T C_k K_k(e_1^k) \tag{8}$$

for $k = 1, \ldots, q$ with

- $\hat{x} = \begin{pmatrix} \hat{x}^1 \\ \hat{x}^2 \\ \vdots \\ \hat{x}^q \end{pmatrix} \in \mathbb{R}^n$, $\hat{x}^k = \begin{pmatrix} \hat{x}_1^k \\ \hat{x}_2^k \\ \vdots \\ \hat{x}_{\lambda_k}^k \end{pmatrix} \in \mathbb{R}^{n_k}$, $\hat{x}_i^k \in \mathbb{R}^{p_k}$,

  $i = 1, \ldots, \lambda_k$, $k = 1, \ldots, q$, $\sum_{k=1}^{q} n_k = n$.
- $\hat{x}_1^k = x_1^k$ (output injection) for $k = 1, \ldots, q$.
- $e_1^k = \hat{x}_1^k - x_1^k$.
- u and y are known inputs and outputs of system (1).

One states the following:

Theorem *2.1.* Assume that system (1) satisfies assumption (1), then:
$\forall M > 0$; $\exists \theta_0 > 0$; $\forall \theta \geq \theta_0$; $\exists \lambda_\theta > 0$; $\mu_\theta > 0$ such that $\|\hat{x}(t) - x(t)\|^2 \leq \lambda_\theta e^{-\mu_\theta t} \|\hat{x}(0) - x(0)\|^2$ for every admissible control u s.t. $Essup\|u(t)\| \leq M$. Moreover, $\lim_{\theta \to \infty} \mu_\theta = +\infty$. This means that system (8) is an exponential observer for system (1) with bounded inputs.

Proof of Theorem (2.1):
Set the estimation error $e(t) = \hat{x}(t) - x(t)$ and let $e^k(t)$ be the $k'th$ subcomponent of $e(t)$. For writing convenience and as long as there is no ambiguity, one shall omit the time t for each variable. One has:

$$\begin{aligned} \dot{e}^k &= A_k e^k + \varphi^k(u, \hat{x}) - \varphi^k(u, x) \\ &\quad - \theta^{\delta_k} \Delta_k^{-1}(\theta) S_k^{-1} C_k^T C_k K_k(e_1^k) \end{aligned} \tag{9}$$

where u is an admissible control such that $\|u\|_\infty \leq M$, $M > 0$ is a given constant.

Before giving the Lyapunov candidate function, one needs to introduce the following diagonal matrices:

$$\Lambda_k(\theta) = \begin{bmatrix} \frac{1}{\theta^{\sigma_1^k}} I_{p_k} & & \\ & \ddots & \\ & & \frac{1}{\theta^{\sigma_{\lambda_k}^k}} I_{p_k} \end{bmatrix} \tag{10}$$

where $\sigma_i^k = \sigma_1^k + (i-1)\delta_k;\ i = 1, \cdots, \lambda_k; k = 1, \cdots, q;$ (11)

and σ_1^k, $k = 1, \ldots, q$ are integers which will be specified later.
It is easy to show that the following identities hold:

- $\Lambda_k(\theta) \Delta_k^{-1}(\theta) \quad = \theta^{-\sigma_1^k} I_{n_k}$
- $\Lambda_k(\theta) A_k \Lambda_k^{-1}(\theta) \ = \theta^{\delta_k} A_k$

Set $\bar{e}^k = \Lambda_k(\theta) e^k$ for $k = 1, \cdots, q$. From equation (9) and using the above identities, one gets:

$$\begin{aligned} \dot{\bar{e}}^k &= \Lambda_k(\theta) A_k \Lambda_k(\theta)^{-1} \bar{e}^k + \Lambda_k(\theta)(\varphi^k(u, \hat{x}) - \varphi^k(u, x)) \\ &\quad - \theta^{\delta_k} \Lambda_k(\theta) \Delta_k^{-1}(\theta) S_k^{-1} C_k^T C_k K_k(\theta^{\sigma_1^k} \bar{e}_1^k) \\ &= \theta^{\delta_k} A_k \bar{e}^k - \theta^{\delta_k - \sigma_1^k} S_k^{-1} C_k^T C_k K_k(\theta^{\sigma_1^k} \bar{e}_1^k) \\ &\quad + \Lambda_k(\theta)(\varphi^k(u, \hat{x}) - \varphi^k(u, x)) \end{aligned} \tag{12}$$

Let $V(\bar{e}) = \bar{e}^T S \bar{e} = \sum_{k=1}^{q} V_k(\bar{e}^k)$ where $V_k(\bar{e}^k) = \bar{e}^{k^T} S_k \bar{e}^k$ and $S = diag(S_1, \cdots, S_q)$, be the candidate Lyapunov function. One has:

$$\begin{aligned} \dot{V}_k &= \dot{\bar{e}}^{k^T} S_k \bar{e}^k + \bar{e}^{k^T} S_k \dot{\bar{e}}^k \\ &= 2\theta^{\delta_k} \bar{e}^{k^T} S_k A_k \bar{e}^k - 2\theta^{\delta_k - \sigma_1^k} \bar{e}^{k^T} C_k^T C_k K_k(\theta^{\sigma_1^k} \bar{e}_1^k) \\ &\quad + 2\bar{e}^{k^T} S_k \Lambda_k(\theta)(\varphi^k(u, \hat{x}) - \varphi^k(u, x)) \end{aligned}$$

Using the algebraic Lyapunov equation (6), one gets:

$$\begin{aligned} \dot{V}_k &= -\theta^{\delta_k} \bar{e}^{k^T} S_k \bar{e}^k + \theta^{\delta_k} \bar{e}^{k^T} C_k^T C_k \bar{e}^k \\ &\quad - 2\theta^{\delta_k - \sigma_1^k} \bar{e}^{k^T} C_k^T C_k K_k(\theta^{\sigma_1^k} \bar{e}_1^k) \\ &\quad + 2\bar{e}^{k^T} S_k \Lambda_k(\theta)(\varphi^k(u, \hat{x}) - \varphi^k(u, x)) \\ &= -\theta^{\delta_k} \bar{e}^{k^T} S_k \bar{e}^k + \theta^{\delta_k} \bar{e}^{k^T} C_k^T C_k \bar{e}^k \\ &\quad - 2\theta^{\delta_k - 2\sigma_1^k} \left(\theta^{\sigma_1^k} \bar{e}^k\right)^T C_k^T C_k K_k(\theta^{\sigma_1^k} \bar{e}_1^k) \\ &\quad + 2\bar{e}^{k^T} S_k \Lambda_k(\theta)(\varphi^k(u, \hat{x}) - \varphi^k(u, x)) \end{aligned}$$

Using property (7) satisfied by the function K_k, one obtains:

$$\begin{aligned} \dot{V}_k &\leq -\theta^{\delta_k} \bar{e}^{k^T} S_k \bar{e}^k + \theta^{\delta_k} \bar{e}^{k^T} C_k^T C_k \bar{e}^k \\ &\quad - 2\theta^{\delta_k - 2\sigma_1^k} \left(\theta^{2\sigma_1^k} \bar{e}^k\right)^T C_k^T C_k \bar{e}^k \\ &\quad + 2\bar{e}^{k^T} S_k \Lambda_k(\theta)(\varphi^k(u, \hat{x}) - \varphi^k(u, x)) \\ &= -\theta^{\delta_k} \bar{e}^{k^T} S_k \bar{e}^k - \theta^{\delta_k} \bar{e}^{k^T} C_k^T C_k \bar{e}^k \\ &\quad + 2\bar{e}^{k^T} S_k \Lambda_k(\theta)(\varphi^k(u, \hat{x}) - \varphi^k(u, x)) \\ &\leq -\theta^{\delta_k} V_k + 2\|S_k \bar{e}^k\| \|\Lambda_k(\theta)(\varphi^k(u, \hat{x}) - \varphi^k(u, x))\| \end{aligned}$$

$$\leq -\theta^{\delta_k} V_k + \\ +2\sqrt{\lambda_{max}^k}\sqrt{V_k}\sum_{i=1}^{\lambda_k}\frac{1}{\theta^{\sigma_i^k}}\|(\varphi_i^k(u,\hat{x})-\varphi_i^k(u,x))\|$$

where λ_{max}^k is the maximum eigenvalue of S_k.
Therefore,

$$\dot{V}_k \leq -\theta^{\delta_k}V_k + 2\rho_k\sqrt{\lambda_{max}^k}\sqrt{V_k}\sum_{i=1}^{\lambda_k}\sum_{l=1}^{q}\sum_{j=1}^{\lambda_l}\chi_{l,j}^{k,i}\theta^{\sigma_j^l-\sigma_i^k}\|\bar{e}_j^l\|$$

where ρ_k=sup$\{\left\|\frac{\partial\varphi_i^k}{\partial x_j^l}(u,x)\right\|; x \in \mathbb{R}^n$and $\|u\|_\infty \leq M\}$ and $\chi_{l,j}^{k,i} = 0$ if $\frac{\partial\varphi_i^k}{\partial x_j^l}(u,x) \equiv 0$, $\chi_{l,j}^{k,i} = 1$ otherwise.
Now, one has

$$\dot{V}_k \leq -\theta^{\delta_k}V_k + 2\rho_k\sqrt{\lambda_{max}^k}\sqrt{V_k}\sum_{i=1}^{\lambda_k}\sum_{l=1}^{q}\sum_{j=1}^{\lambda_l}\chi_{l,j}^{k,i}\theta^{\sigma_j^l-\sigma_i^k}\|\bar{e}^l\|$$

$$\leq -\theta^{\delta_k}V_k \\ + 2\rho_k\sqrt{\lambda_{max}^k}\sqrt{V_k}\sum_{i=1}^{\lambda_k}\sum_{l=1}^{q}\sum_{j=1}^{\lambda_l}\chi_{l,j}^{k,i}\theta^{\sigma_j^l-\sigma_i^k}\frac{\sqrt{V_l}}{\sqrt{\lambda_{min}^l}}$$

where λ_{min}^l is the minimum eigenvalue of S_l.
Thus,

$$\dot{V}_k \leq -(\sqrt{\theta^{\delta_k}V_k})^2 + 2\rho_k\mu_S\sqrt{\theta^{\delta_k}V_k} \\ \sum_{i=1}^{\lambda_k}\sum_{l=1}^{q}\sum_{j=1}^{\lambda_l}\chi_{l,j}^{k,i}\theta^{\sigma_j^l-\sigma_i^k-\frac{\delta_k}{2}-\frac{\delta_l}{2}}\sqrt{\theta^{\delta_l}V_l}$$

where $\mu_S = \sqrt{\frac{\lambda_{max}(S)}{\lambda_{min}(S)}}$.
The remaining of the proof is as follows. We shall firstly suppose that the integers σ_1^k, $k = 1,\ldots,q$ are chosen such that the following condition is satisfied:

$$\text{if } \chi_{l,j}^{k,i} = 1 \text{ then } \sigma_j^l - \sigma_i^k - \frac{\delta_k}{2} - \frac{\delta_l}{2} \leq -\varepsilon < 0 \quad (13)$$

Then, we shall show that such a choice is possible and we shall indeed give a set of σ_1^k and a real number $\varepsilon > 0$ which satisfy such a condition. Thus, suppose that condition (13) holds and assume that $\theta \geq 1$. Then, one gets:

$$\dot{V}_k \leq -(\sqrt{\theta^{\delta_k}V_k})^2 + 2\rho_k\mu_S\sqrt{\theta^{\delta_k}V_k}\sum_{i=1}^{\lambda_k}\sum_{l=1}^{q}\sum_{j=1}^{\lambda_l}\theta^{-\varepsilon}\sqrt{\theta^{\delta_l}V_l}$$

$$\leq -(\sqrt{\theta^{\delta_k}V_k})^2 + 2\lambda_k\rho_k\mu_S\theta^{-\varepsilon}\sqrt{\theta^{\delta_k}V_k}\sum_{l=1}^{q}\sum_{j=1}^{\lambda_l}\sqrt{\theta^{\delta_l}V_l}$$

Now, set $V_k^* = \theta^{\delta_k}V_k$ for $k = 1,\cdots,q$ and $V^* = \sum_{k=1}^{q}V_k^*$.
Notice that $\theta V \leq V^* \leq \theta^{\delta_1}V$, where δ_1 is given by (5).
Then,

$$\dot{V}_k \leq -V_k^* + 2\lambda_k\rho_k\mu_S\theta^{-\varepsilon}\sqrt{V_k^*}\sum_{l=1}^{q}\sum_{j=1}^{\lambda_l}\sqrt{V_l^*}$$

$$\leq -V_k^* + 2\lambda_k n\rho_k\mu_S\theta^{-\varepsilon}\sqrt{V_k^*}\sqrt{V^*} \\ \leq -V_k^* + 2\lambda_k n\rho_k\mu_S\theta^{-\varepsilon}V^*$$

Hence,

$$\dot{V} \leq -V^* + 2n^2\rho\mu_S\theta^{-\varepsilon}V^* \\ \leq -(1 - 2n^2\rho\mu_S\theta^{-\varepsilon})V^*$$

where ρ=max$\{\rho_k, 1 \leq k \leq q\}$.
Finally, $\dot{V} \leq -\theta(1 - 2n^2\rho\mu_S\theta^{-\varepsilon})V$
Now, choosing θ_0 such that $1 - 2n^2\rho\mu_S\theta_0^{-\varepsilon} > 0$, one obtains,

$$\forall\theta \geq \theta_0;\ \bar{e}(t)^T S\bar{e}(t) \leq exp(-\mu_\theta t)\bar{e}(0)^T S\bar{e}(0)$$

where $\mu_\theta = \theta(1 - 2n^2\rho\mu_S\theta^{-\varepsilon})$.
Otherwise, $\|\bar{e}(t)\|^2 \leq \mu_S^2 exp(-\mu_\theta t)\|\bar{e}(0)\|^2$ and consequently $\|e(t)\|^2 \leq \lambda_\theta exp(-\mu_\theta t)\|e(0)\|^2$ where $\lambda_\theta = c_0^2(\theta)\mu_S^2$ with $c_0(\theta) = \theta^{\sigma_{\lambda_q}^q}$.
To end the proof of the theorem, we shall exhibit a set of σ_1^k and a real number $\varepsilon > 0$ satisfying condition (13). Before giving such a set, we firstly note that according to the state dependence given by (2) and (3), the case where $\chi_{l,j}^{k,i} = 1$, $k,l \in \{1,\ldots,q\}$ and $j \in \{2,\ldots,\lambda_l\}$, occurs if and only if one of the following three situations is met:

- $k > l$ and i takes any value in $\{1,\ldots,\lambda_k\}$.
- $k = l$ and $i \in \{1,\ldots,\lambda_k\}$ with $i \geq j$.
- $k < l$ and $i = \lambda_k$.

Now, for $k = 1,\ldots,q$, set

$$\sigma_1^k = (\lambda_1 - \frac{1}{2})\delta_1 - (\lambda_k - \frac{1}{2})\delta_k + (1 - \frac{1}{2^{k-1}}) \quad (14)$$

One has:

$$\sigma_j^l - \sigma_i^k - \frac{\delta_k}{2} - \frac{\delta_l}{2} = \sigma_1^l - \sigma_1^k + (j - \frac{3}{2})\delta_l - (i - \frac{1}{2})\delta_k \\ = (\lambda_k - i)\delta_k + (j - \lambda_l - 1)\delta_l \\ - (\frac{1}{2^{l-1}} - \frac{1}{2^{k-1}}) \quad (15)$$

To check condition (13), it suffices to consider the three situations listed above. Indeed:
• $k > l$ and i takes any value in $\{1,\ldots,\lambda_k\}$:
Using equation (15), one gets:

$$\sigma_j^l - \sigma_i^k - \frac{\delta_k}{2} - \frac{\delta_l}{2} \leq (\lambda_k - 1)\delta_k - \delta_l - (\frac{1}{2^{l-1}} - \frac{1}{2^{k-1}}) \\ = \delta_{k-1} - \delta_l - (\frac{1}{2^{l-1}} - \frac{1}{2^{k-1}}) \quad (16)$$

Now, on one hand and from the expression of δ_k (equation (5)) and since $k - 1 \geq l$, one has

$$\delta_{k-1} - \delta_l \leq 0 \quad (17)$$

On another hand, one has:

$$(\frac{1}{2^{l-1}} - \frac{1}{2^{k-1}}) = \frac{1}{2^{k-1}}(\frac{1}{2^{l-k}} - 1) = \frac{1}{2^{k-1}}(2^{k-l} - 1) \\ \geq \frac{1}{2^{k-1}}(2 - 1) \quad \text{for} \quad k \geq l + 1 \\ = \frac{1}{2^{k-1}} \quad (18)$$

Combining inequalities (16), (17) and (18), one obtains:

$$\sigma_j^l - \sigma_i^k - \frac{\delta_k}{2} - \frac{\delta_l}{2} \leq -\frac{1}{2^{k-1}} \leq -\frac{1}{2^{q-1}}$$

• $k = l$ and $i \in \{1, \ldots, \lambda_k\}$ with $i \geq j$:
Equation (15) specializes as follows:

$$\begin{aligned}\sigma_j^l - \sigma_i^k - \frac{\delta_k}{2} - \frac{\delta_l}{2} &= (j - i - 1)\delta_k \\ &\leq -\delta_k \leq -1 \leq -\frac{1}{2^{q-1}}\end{aligned}$$

• $k < l$ and $i = \lambda_k$:
Equation (15) becomes:

$$\begin{aligned}\sigma_j^l - \sigma_i^k - \frac{\delta_k}{2} - \frac{\delta_l}{2} &= (j - \lambda_l - 1)\delta_l - (\frac{1}{2^{l-1}} - \frac{1}{2^{k-1}}) \\ &\leq -\delta_l + (\frac{1}{2^{k-1}} - \frac{1}{2^{l-1}}) \\ &\leq -\delta_l + (1 - \frac{1}{2^{q-1}}) = -\frac{1}{2^{q-1}}\end{aligned}$$

Now, it is suffices to take $\varepsilon = \frac{1}{2^{q-1}}$. This ends the proof of the theorem.

3. SOME PARTICULAR DESIGN FUNCTIONS

Some particular expressions of the design function K_k that satisfy condition (7) shall be given. It will be shown that each of these expressions yields a well known observer. Indeed, one has:

- The usual high gain design function given by
$$K_k(e_1^k) \;=\; \rho\, C_k^T C_k e^k = \rho\, C_k^T e_1^k \qquad (19)$$
where ρ is a positive scalar satisfying $\rho \geq \frac{1}{2}$. Notice that the function K_k is bounded as soon as the state x_1^k is.
- The design function involved in the actual sliding mode framework
$$K_k(e_1^k) \;=\; \rho\, C_k^T C_k sign(e^k) = \rho\, C_k^T sign(e_1^k) \qquad (20)$$
where ρ is a positive scalar and 'sign' is the usual signum function. It is worth mentioning that the required property (7) holds in the case of bounded input bounded state systems for relatively high values of ρ. However, this design function induces a chattering phenomena which is by no means suitable in practical situations.
- The design functions that are commonly used in the sliding mode practice, namely
$$\begin{aligned}K_k(e_1^k) \;&=\; \rho\, C_k^T C_k tanh(k_o e^k) \\ &= \rho\, C_k^T tanh(k_o e_1^k) \qquad (21)\end{aligned}$$
where $tanh$ denotes the hyperbolic tangent function and ρ and k_o are positive scalars. One can easily show that the design function (21) satisfies the property (7) for relatively high values of k_o. More particularly, recall that one has $\lim_{k_o \longrightarrow +\infty} tanh(k_o e^k) = sign(e^k)$.

4. EXAMPLE

Consider the following dynamical system

$$\begin{cases} \dot{x}_1 = (a-u)x_3 - ux_4 - x_1 \\ \dot{x}_2 = ux_3 + ux_4 - x_2 \\ \dot{x}_3 = -x_3 - x_6^3 - arctan(x_6) - sin(x_5) \\ \dot{x}_4 = x_5^3 + x_5 + \dfrac{x_3}{1+x_3^2} - x_4 \\ \dot{x}_5 = x_6^3 + tanh(x_6) + x_3^2 - x_5 sin(x_5) \\ \dot{x}_6 = -x_6 - 20cos(x_3) \\ y \;= (x_1 \;\; x_2 \;\; x_4)^T \end{cases} \qquad (22)$$

Consider the following injective map (Farza et al. [2004]): $\Phi : \mathbb{R}^6 \mapsto \mathbb{R}^7$, $x \mapsto z$ with

$$\begin{cases} z_1 = x_1 \\ z_2 = x_2 \\ z_3 = (a-u)x_3 \\ z_4 = ux_3 \\ z_5 = x_4 \\ z_6 = x_5^3 + x_5 \\ z_7 = (1+3x_5^2)(x_6^3 + tanh(x_6)) \end{cases} \qquad (23)$$

One can show that the map Φ puts system (22) under form (1) with $x^1 = \begin{pmatrix} x_1^1 \\ x_2^1 \end{pmatrix}$ with $x_1^1 = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$, $x_2^1 = \begin{pmatrix} z_3 \\ z_4 \end{pmatrix}$; $x^2 = \begin{pmatrix} x_1^2 \\ x_2^2 \\ x_3^2 \end{pmatrix}$ with $x_1^2 = z_5$, $x_2^2 = z_6$ and $x_3^2 = z_7$; the output y is then partitioned as $y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ with $y_1 = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$ and $y_2 = z_5$. Accordingly, one has $\lambda_1 = 2$ and $\lambda_2 = 3$. Thus, an observer of the form (8) can be synthesized in the z coordinates. Then, the equations of the observer in the x coordinates can be derived by considering the left inverse of the transformation jacobian (see e.g. Farza et al. [2004]). One can show that when the design function is chosen as in (19), the observer equations in the original coordinates x can be written as follows:

$$\begin{cases} \dot{\hat{x}}_1 = (a-u)\hat{x}_3 - ux_4 - x_1 - 2\theta^{\delta_1}(\hat{x}_1 - x_1) \\ \dot{\hat{x}}_2 = u\hat{x}_3 + ux_4 - x_2 - 2\theta^{\delta_1}(\hat{x}_2 - x_2) \\ \dot{\hat{x}}_3 = -\hat{x}_3 - \hat{x}_6^3 - arctan(\hat{x}_6) - sin(\hat{x}_5) \\ \quad - \;\theta^{2\delta_1}\left(\dfrac{(a-u)(\hat{x}_1 - x_1)}{(a-u)^2 + u^2} + \dfrac{u(\hat{x}_2 - x_2)}{(a-u)^2 + u^2}\right) \\ \dot{\hat{x}}_4 = \hat{x}_5^3 + \hat{x}_5 + \dfrac{\hat{x}_3}{1+\hat{x}_3^2} - x_4 - 3\theta(\hat{x}_4 - x_4) \\ \dot{\hat{x}}_5 = \hat{x}_6^3 + tanh(\hat{x}_6) + \hat{x}_3^2 - \hat{x}_5 sin(\hat{x}_5) \\ \quad - \;3\theta^2\left(\dfrac{\hat{x}_4 - x_4}{1+3\hat{x}_5^2}\right) \\ \dot{\hat{x}}_6 = -\hat{x}_6 - 20cos(\hat{x}_3) \\ \quad - \;\theta^3\left(\dfrac{\hat{x}_4 - x_4}{(1+3\hat{x}_5^2)(3\hat{x}_6^2 + 1 - tanh^2(\hat{x}_6))}\right) \end{cases} \qquad (24)$$

where $\delta_1 = (\lambda_2 - 1) = 2$ according to (5).

In fact, observer (24) has been simulated using different expressions of the design functions chosen amongst these ones listed above. Since, the obtained results were very similar, we only presented here those obtained with equation (19).

The numerical simulations have been carried out using the following initial conditions $x_i(0) = 1$ for the model and $\hat{x}_i = -1$ for the observer, $i = 1, \ldots, 6$. The value of the design parameter θ used in simulation was equal to

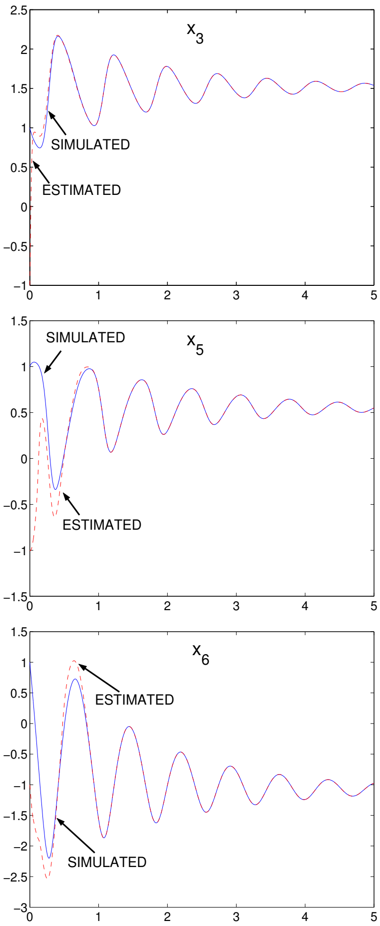

Fig. 1. Simulation results for x_i, $i = 3, 5$ and 6

10. Simulation results corresponding to the non measured state variables are given in figure 1 which clearly shows the quick convergence to zero of the estimation error.

5. CONCLUSION

In this note, we have presented a high gain observer for a large class of nonlinear MIMO systems which are observable for every input. Unlike previous works related to high gain observers synthesis, a complete triangular structure is not assumed. Moreover, the gain of the proposed observer involves a well defined design function which provides a unified framework for the high gain observer design, namely classical high gain observers and several versions of sliding mode observers are obtained by considering particular expressions of the design function. In a forthcoming work, one shall give sufficient conditions under which uniformly observable systems can be put under form (1).

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