Proceedings of the 17th World Congress
The International Federation of Automatic Control
Seoul, Korea, July 6-11, 2008

Moment Matching Model Order Reduction in Time Domain via Laguerre Series *

Rudy Eid,* Boris Lohmann*

* *Lehrstuhl für Regelungstechnik, Technische Universität München, Boltzmannstr. 15, 85748 Garching, Germany, (email: {eid,lohmann}@tum.de).*

Abstract: A new time-domain model order reduction method based on the Laguerre function expansion of the impulse response is presented. The Laguerre coefficients of the impulse response of the reduced-order model, which is calculated using a projection whose matrices form basis of appropriate Krylov subspaces, match, up to a given order, those of the original system. In addition, it is shown that the obtained reduced-order model in time-domain, is equivalent to the one obtained by the classical moment matching around a single expansion point in frequency-domain. Accordingly, a new time-domain interpretation for the rational interpolation problem is deduced.

Keywords: Model reduction; Laguerre functions; Laguerre series representation; Large scale systems; Reduced-order models.

1. INTRODUCTION

In numerous fields of engineering, large-scale models are becoming nowadays a common and unavoidable result for the accurate modeling of complex systems. With an order of at least a few hundred thousands, handling such models for the purpose of simulation, optimization or control has become impractical or even unfeasible. One of the most popular approaches to solve this problem is Model Order Reduction (MOR) where, with the aim of using it in place of the original system, a reduced-order model that approximates the behavior of the original model is calculated.

In order reduction of large-scale systems, the various reduction methods (including their modified and improved versions) based on moment matching using the Krylov subspaces are nowadays among the best choices (Freund, 2003; Bai, 2002; Antoulas, 2005; Odabasioglu et al., 1997). Even though the reduced-order model is calculated, via a projection, in a relatively short time with a good numerical accuracy, these methods are restricted to the approximation of the frequency response of the original system, as they match the moments in the frequency-domain. Consequently, they can not guarantee a good approximation of the impulse response, as it is quite hard in most practical cases, to predict the accuracy of the time-domain response of the reduced-order model from its frequency-domain one.

It is then more natural to do order reduction directly in the time-domain through the approximation of the system's impulse response while benefiting from the numerical and computational advantages of the Krylov subspace-based methods. The first work to use the Krylov subspace approach in the time-domain appeared to be (Gunupudi and Nakhla, 1999), where some of the first derivatives of the time response of the large nonlinear system and those of its corresponding reduced-order model are matched.

Lately, based on the various successful methods of approximating the impulse response using orthogonal polynomials, several approaches tried to improve these methods to make them suitable for the reduction of large-scale systems based on projecting the time response of the original system onto a lower order dimensional subspace spanned by an orthogonal basis. Consequently, some of the first coefficients of the orthogonal series' expansion of the impulse response of the reduced order model match those of the original one. For instance, in (Wang et al., 2000), the Chebyshev expansion have been used in time-domain for passive model order reduction of interconnect networks. In (Chen et al., 2002), a time-domain approach involving the Laguerre polynomials and some Krylov subspaces for the approximation of the impulse response has been presented. A disadvantage of this method is that it employed the Laguerre polynomials which are known to form an unbounded basis for the Hilbert space $\mathcal{L}_2(R_+)$.

In this paper, the results of Chen et al. (2002) are generalized to involve the Laguerre functions - instead of the Laguerre polynomials -, which are exponentially decreasing and form a bounded orthonormal basis for $\mathcal{L}_2(R_+)$. This results in a purely time-domain Krylov-based model reduction approach. In addition, and based on the work in (Eid et al., 2007), where the equivalence between the classical moment matching and the Laguerre-based reduction approach in frequency-domain (Knockaert and De Zutter, 2000) has been shown, a similar equivalence between the time-domain Laguerre-based approach and moment matching around a single expansion point is presented. The importance of this equivalence lies in the fact that it allows a first interpretation of the moment matching approach - which is developed and applied in the frequency-domain - in the time-domain. In addition, by

* This work is partially supported by the German Academic Exchange Service (DAAD).

showing that the time scale factor α in the Laguerre functions corresponds to the expansion point in the frequency-domain, the open problem of choosing a suitable expansion point in the rational Krylov subspace reduction methods, is converted into the problem of finding the optimal time scale α in the Laguerre-based reduction methods.

The rest of the paper is organized as follows: In the next section some preliminary facts related to the Laguerre series expansion and the impulse response of state-space models is presented. In section 3, order reduction by moment matching is reviewed. In section 4, the new time-domain approach involving the Laguerre functions is introduced. The equivalence between moment matching and different Laguerre-based order reduction approaches in the time domain is presented in section 4.

2. BACKGROUND

2.1 Problem Formulation

Consider the dynamical system of the form:

$$\begin{cases} \mathbf{E}\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{b}u(t), \\ y(t) = \mathbf{c}^T\mathbf{x}(t), \end{cases} \tag{1}$$

where $\mathbf{E}, \mathbf{A} \in \mathbb{R}^{N\times N}$, $\mathbf{b} \in \mathbb{R}^N$, $\mathbf{c} \in \mathbb{R}^N$ are constant matrices, $u(t) \in \mathbb{R}$, $y(t) \in \mathbb{R}$, and $\mathbf{x}(t) \in \mathbb{R}^N$ are, respectively, the input, output and states vectors of the system. For the simplicity of exposition, the SISO case is only considered, however all the results of this paper can be easily generalized to the MIMO case. After integration and assuming zero initial conditions, the state equation of system (1) becomes

$$\mathbf{E}\mathbf{x}(t) = \mathbf{A}\int_0^t \mathbf{x}(\tau)d\tau + \mathbf{b}\int_0^t u(\tau)d\tau, \tag{2}$$

and its impulse response can be shown to be

$$y(t) = \mathbf{c}^T\mathbf{x}(t) = \mathbf{c}^T e^{(\mathbf{E}^{-1}\mathbf{A})t}\mathbf{E}^{-1}\mathbf{b}. \tag{3}$$

2.2 The Laguerre function expansion

The ith Laguerre polynomial is defined as

$$l_i(t) = \frac{e^t}{i!}\frac{d^i}{dt^i}(e^{-t}t^i) \tag{4}$$

and the scaled Laguerre functions (see Fig. 1) are

$$\phi_i^\alpha(t) = \sqrt{2\alpha}e^{-\alpha t}l_i(2\alpha t) \tag{5}$$

where α is a positive scaling parameter called time-scale factor. These functions, form a uniformly bounded orthonormal basis for the Hilbert space $\mathcal{L}_2(R_+)$ (Szegö, 1959). Hence, every function in $\mathcal{L}_2(R_+)$ admits the Laguerre expansion

$$f(t) = \sum_{i=0}^{\infty} f_i\phi_i^\alpha(t), \tag{6}$$

where f_i are defined as the Laguerre coefficients. When truncated to order k, this series results in an optimal approximation $\hat{f}_k(t) = \sum_{i=0}^k f_i\phi_i^\alpha(t)$ of the function $f(t)$ and minimizes the following integral:

$$\int_0^\infty \left(f(t) - \hat{f}_k(t)\right)^2 dt. \tag{7}$$

This approximation is thus the optimal projection of $f(t)$ into the k-dimensional subspace spanned by the first k Laguerre functions.

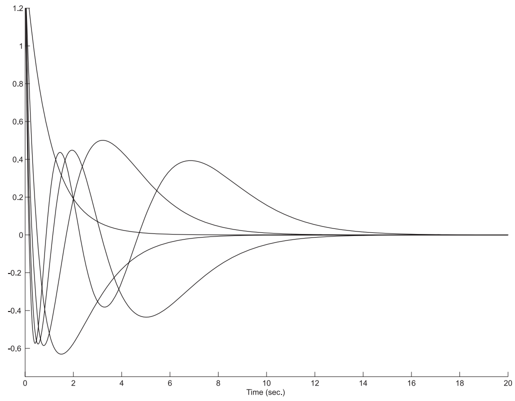

Fig. 1. The first five Laguerre functions with $\alpha = 1$.

3. ORDER REDUCTION BY MOMENT MATCHING

The transfer function of the system (1) is

$$H(s) = \mathbf{c}^T(s\mathbf{E} - \mathbf{A})^{-1}\mathbf{b}, \tag{8}$$

and the moments, which are defined as the negative coefficients of the Taylor series expansion about zero of the system's transfer function,

$$H(s) = -\mathbf{c}^T(\mathbf{I} - s\mathbf{A}^{-1}\mathbf{E})^{-1}\mathbf{A}^{-1}\mathbf{b} = -\mathbf{c}^T\mathbf{A}^{-1}\mathbf{b}$$
$$-\mathbf{c}^T\mathbf{A}^{-1}\mathbf{E}\mathbf{A}^{-1}\mathbf{b}s - \cdots - \mathbf{c}^T(\mathbf{A}^{-1}\mathbf{E})^i\mathbf{A}^{-1}\mathbf{b}s^i - \cdots$$

are calculated as follows (Antoulas, 2005):

$$\mathbf{m}_i = \mathbf{c}^T(\mathbf{A}^{-1}\mathbf{E})^i\mathbf{A}^{-1}\mathbf{b} \qquad i = 0, 1, \cdots. \tag{9}$$

The aim of order reduction by moment matching is to find a reduced order model of order $k \ll N$, whose moments match some of those of the original one (Freund, 2003).

One way to calculate this reduced order model is by applying a projection to the original model,

$$\begin{cases} \mathbf{W}^T\mathbf{E}\mathbf{V}\dot{\mathbf{x}}_r(t) = \mathbf{W}^T\mathbf{A}\mathbf{V}\mathbf{x}_r(t) + \mathbf{W}^T\mathbf{b}\mathbf{u}(t), \\ y(t) = \mathbf{c}^T\mathbf{V}\mathbf{x}_r(t), \end{cases} \tag{10}$$

by means of the so-called projection matrices, $\mathbf{V}$ and $\mathbf{W}$. For the choice of the projection matrices, the Krylov subspace, defined in e.g., (Freund, 2003; Antoulas, 2005) is used,

$$\mathcal{K}_k(\mathbf{A}_1, \mathbf{b}_1) = span\{\mathbf{b}_1, \mathbf{A}_1\mathbf{b}_1, \cdots, \mathbf{A}_1^{k-1}\mathbf{b}_1\} \tag{11}$$

where $\mathbf{A}_1 \in \mathbb{R}^{N\times N}$, and $\mathbf{b}_1 \in \mathbb{R}^N$ is called the starting vector. Now, when the matrices $\mathbf{V}$ and $\mathbf{W}$ form basis of the the input and output Krylov subspaces $\mathcal{K}_k(\mathbf{A}^{-1}\mathbf{E}, \mathbf{A}^{-1}\mathbf{b})$ and $\mathcal{K}_k(\mathbf{A}^{-T}\mathbf{E}^T, \mathbf{A}^{-T}\mathbf{c})$ respectively, the first $2k$ moments around $s_0 = 0$ [1], are matched, and hence, a good approximation of the low-frequency behavior is achieved.

For the numerical computation of the matrices $\mathbf{V}$ and $\mathbf{W}$, the known Lanczos or Arnoldi or one of their numerous improved and modified versions are used. For more details, see (Antoulas, 2005; Salimbahrami, 2005) and the references therein.

Now, in order to produce a reduced-order model that approximates the middle or high frequency behavior of the

[1] In the so called one-sided method, only one Krylov subspace is used with a common choice $\mathbf{W} = \mathbf{V}$ and only k moments match.

original system, matching some of the moments around $s_0 \neq 0$ is required. This is achieved by choosing $\mathbf{V}$ and $\mathbf{W}$ as basis of the following Krylov subspaces:

$$\mathcal{K}_k\left((\mathbf{A}-s_0\mathbf{E})^{-1}\mathbf{E},(\mathbf{A}-s_0\mathbf{E})^{-1}\mathbf{b}\right), \quad (12)$$
$$\mathcal{K}_k\left((\mathbf{A}-s_0\mathbf{E})^{-T}\mathbf{E}^T,(\mathbf{A}-s_0\mathbf{E})^{-T}\mathbf{c}^T\right). \quad (13)$$

4. TIME-DOMAIN ORDER REDUCTION USING LAGUERRE FUNCTIONS

4.1 Approximation of the state vector

As a first step towards the approximation of the impulse response, the state vector $\mathbf{x}(t)$ is approximated as

$$\mathbf{x}(t) \approx \mathbf{x}_k(t) = \sum_{i=0}^{k} \mathbf{f}_i \phi_i^\alpha(t). \quad (14)$$

Based on(Szegö, 1959; Lee et al., 2006), the integral of $\mathbf{x}(t)$ can be expressed as

$$\int_0^t \mathbf{x}(\tau)d\tau = \frac{1}{\alpha}\sum_{i=0}^{\infty}\left(\mathbf{f}_i + 4\sum_{j=0}^{i-1}(-1)^{i+j}\mathbf{f}_j\right)\phi_i^\alpha(t), \quad (15)$$

and thus (2) can be rewritten, after replacing $\mathbf{x}(t)$ by $\mathbf{x}_k$(t), as:

$$\mathbf{E}\sum_{i=0}^{k}\mathbf{f}_i\phi_i^\alpha(t) - \frac{\mathbf{A}}{\alpha}\sum_{i=0}^{k}\mathbf{f}_i\phi_i^\alpha(t) - \frac{4\mathbf{A}}{\alpha}\sum_{i=0}^{k}\sum_{j=0}^{i-1}(-1)^{i+j}\mathbf{f}_j\phi_i^\alpha(t) = \mathbf{b}.$$

It can be easily shown that

$$\mathbf{\Phi}\begin{bmatrix}\mathbf{f}_k\\ \vdots\\ \mathbf{f}_1\\ \mathbf{f}_0\end{bmatrix} = \begin{bmatrix}0\\ \vdots\\ 0\\ \mathbf{b}\end{bmatrix}, \quad (16)$$

where

$$\mathbf{\Phi}=\begin{bmatrix}(\alpha\mathbf{E}-\mathbf{A})^{-1} & -4\mathbf{A} & 4\mathbf{A} & -4\mathbf{A} & 4\mathbf{A}\\ & \cdots & \cdots & & \\ & & (\alpha\mathbf{E}-\mathbf{A})^{-1} & -4\mathbf{A} & 4\mathbf{A}\\ & & & (\alpha\mathbf{E}-\mathbf{A})^{-1} & -4\mathbf{A}\\ & & & & \alpha\left(\alpha\mathbf{E}-\mathbf{A}\right)^{-1}\end{bmatrix},$$

and k is an even integer.

As the first k entries of the vector on the r.h.s. of (16) are zero, the Laguerre coefficients $\mathbf{f}_i$ of $\mathbf{x}_k(t)$ can be expressed using the following recursive formulas:

$$\mathbf{f}_0 = \alpha\left(\alpha\mathbf{E}-\mathbf{A}\right)^{-1}\mathbf{b}, \quad (17)$$
$$\mathbf{f}_i = (\alpha\mathbf{E}-\mathbf{A})^{-1}\sum_{j=0}^{i-1}(-1)^{i+j}4\mathbf{A}\mathbf{f}_j, \quad 1 \le i \le k. \quad (18)$$

Based on the approximation (14), $\mathbf{x}_k(t)$ can be reformulated as:

$$\mathbf{x}_k(t) = \mathbf{F}_k\begin{bmatrix}\phi_0^\alpha(t)\\ \phi_1^\alpha(t)\\ \vdots\\ \phi_k^\alpha(t)\end{bmatrix}, \quad (19)$$

with

$$\mathbf{F}_k = [\,\mathbf{f}_0\ \mathbf{f}_1\ \cdots\ \mathbf{f}_k\,]. \quad (20)$$

Thus, $\mathbf{x}_k(t)$ lies in the subspace spanned by the columns of $\mathbf{F}_k$ for all t.

4.2 Model order reduction

The key idea of the reduction method presented here consists of projecting the state vector $\mathbf{x}(t)$ of system (1) onto the k-th subspace spanned by the first k Laguerre functions $\phi_i^\alpha(t)$, resulting in a reduced order model whose impulse response's Laguerre coefficients match some of the first coefficients of the original response $y(t)$.

The reduced order system is obtained by applying the projection $\mathbf{x} = \mathbf{V}\hat{\mathbf{x}}, \mathbf{V} \in \mathbb{R}^{k\times N}, k < N$ to the system (1) and multiplying the state equation by the transpose of the matrix $\mathbf{V}$,

$$\begin{cases}\overbrace{\mathbf{V}^T\mathbf{E}\mathbf{V}}^{\hat{\mathbf{E}}}\dot{\hat{\mathbf{x}}}(t) = \overbrace{\mathbf{V}^T\mathbf{A}\mathbf{V}}^{\hat{\mathbf{A}}}\hat{\mathbf{x}}(t) + \overbrace{\mathbf{V}^T\mathbf{b}}^{\hat{\mathbf{b}}}u(t),\\ y(t) = \underbrace{\mathbf{c}^T\mathbf{V}}_{\hat{\mathbf{c}}}\overset{T}{}\hat{\mathbf{x}}(t),\end{cases} \quad (21)$$

where k is the order of the reduced system.

Lemma 1. If the columns of $\mathbf{V}$ used in (21), form an orthonormal basis for the subspace spanned by the columns of $\mathbf{F}_k$, then the first k Laguerre coefficients of the Laguerre series expansions of the original and reduced state vectors satisfy

$$\mathbf{f}_i = \mathbf{V}\hat{\mathbf{f}}_i, \qquad 0 \le i \le k. \quad (22)$$

with $\mathbf{x}(t) = \sum_{i=0}^{k}\mathbf{f}_i\phi_i^\alpha(t)$ and $\hat{\mathbf{x}}(t) = \sum_{i=0}^{k}\hat{\mathbf{f}}_i\phi_i^\alpha(t)$.

Proof. After integration, the state equation of system (21) becomes

$$\mathbf{V}^T\mathbf{E}\mathbf{V}\hat{\mathbf{x}}(t) - \mathbf{V}^T\mathbf{A}\mathbf{V}\int_0^t \hat{\mathbf{x}}(\tau)d\tau = \mathbf{V}^T\mathbf{b}, \quad (23)$$

with $\hat{\mathbf{x}}(t)$ as its solution. As the coefficients $\mathbf{f}_i \in$ *colspan*$(\mathbf{V})$, they can be written as a linear combination of the columns of $\mathbf{V}$,

$$\mathbf{f}_i = \mathbf{V}\mathbf{z}_i.$$

By substituting the above equation in (16) and multiplying both sides by $\mathbf{V}^T$, we get

$$\begin{bmatrix}\left(\alpha\hat{\mathbf{E}}-\hat{\mathbf{A}}\right)^{-1} & -4\hat{\mathbf{A}} & 4\hat{\mathbf{A}} & -4\hat{\mathbf{A}} & 4\hat{\mathbf{A}}\\ & \cdots & \cdots & & \\ & & \left(\alpha\hat{\mathbf{E}}-\hat{\mathbf{A}}\right)^{-1} & -4\hat{\mathbf{A}} & 4\hat{\mathbf{A}}\\ & & & \left(\alpha\hat{\mathbf{E}}-\hat{\mathbf{A}}\right)^{-1} & -4\hat{\mathbf{A}}\\ & & & & \alpha\left(\alpha\hat{\mathbf{E}}-\hat{\mathbf{A}}\right)^{-1}\end{bmatrix}$$
$$*\begin{bmatrix}\mathbf{z}_k\\ \vdots\\ \mathbf{z}_1\\ \mathbf{z}_0\end{bmatrix} = \begin{bmatrix}0\\ \vdots\\ 0\\ \hat{\mathbf{b}}\end{bmatrix}. \quad (24)$$

Now, from (23) and (16), the equations for $\hat{\mathbf{f}}_i$ are found to be

$$\begin{bmatrix} \left(\alpha\hat{\mathbf{E}}-\hat{\mathbf{A}}\right)^{-1} & -4\hat{\mathbf{A}} & 4\hat{\mathbf{A}} & -4\hat{\mathbf{A}} & 4\mathbf{A} \\ & \cdots & \cdots & & \\ & & \left(\alpha\hat{\mathbf{E}}-\hat{\mathbf{A}}\right)^{-1} & -4\hat{\mathbf{A}} & 4\hat{\mathbf{A}} \\ & & & \left(\alpha\hat{\mathbf{E}}-\hat{\mathbf{A}}\right)^{-1} & -4\hat{\mathbf{A}} \\ & & & & \alpha\left(\alpha\hat{\mathbf{E}}-\hat{\mathbf{A}}\right)^{-1} \end{bmatrix}$$

$$* \begin{bmatrix} \hat{\mathbf{f}}_k \\ \vdots \\ \hat{\mathbf{f}}_1 \\ \hat{\mathbf{f}}_0 \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \hat{\mathbf{b}} \end{bmatrix}. \tag{25}$$

The proof is completed by comparing (25) and (24).

Based on the previous lemma, the main theorem of this section is stated:

Theorem 2. If the columns of $\mathbf{V}$ used in (21), form an orthonormal basis for the subspace spanned by the columns of $\mathbf{F}_k$, then the first k Laguerre coefficients of the Laguerre series expansions of the impulse response of the original and reduced systems match, i.e.,

$$y_i = \hat{y}_i, \qquad 0 \leq i \leq k. \tag{26}$$

with $y(t) = \sum_{i=0}^{k} y_i \phi_i^\alpha(t)$ and $\hat{y}(t) = \sum_{i=0}^{k} \hat{y}_i \phi_i^\alpha(t)$.

Proof. From (1), we have $y_i = \mathbf{c}^T\mathbf{f}_i$, and similarly from (21), $\hat{y}_i = \hat{\mathbf{c}}^T\hat{\mathbf{f}}_i$. Based on the orthonormality of $\mathbf{V}$ and lemma 1, it can be easily shown that $\hat{\mathbf{f}}_i = \mathbf{V}^T\mathbf{f}_i$. Hence, using Lemma 1, $\mathbf{V}\mathbf{V}^T\mathbf{f}_i = \mathbf{V}\mathbf{V}^T\mathbf{V}\hat{\mathbf{f}}_i = \mathbf{V}\hat{\mathbf{f}}_i = \mathbf{f}_i$ Finally,

$$\hat{y}_i = \hat{\mathbf{c}}^T\hat{\mathbf{f}}_i = \mathbf{c}^T\mathbf{V}\mathbf{V}^T\mathbf{f}_i = \mathbf{c}^T\mathbf{f}_i = y_i.$$

In the following theorem, it is shown that the subspace spanned by the matrix of Laguerre coefficients $\mathbf{F}_k$ can be formulated as Krylov subspace involving the system matrices.

Lemma 3. The subspace spanned by the columns of $\mathbf{F}_k$ is equivalent to the Krylov subspace:

$$\mathcal{K}_k\left((\mathbf{A}-\alpha\mathbf{E})^{-1}\mathbf{A}, (\mathbf{A}-\alpha\mathbf{E})^{-1}\mathbf{b}\right). \tag{27}$$

Proof. Let $\mathbf{P}_i$ and $\hat{\mathbf{P}}_i$ be the basic blocks of the matrix $\mathbf{F}_k$ and the Krylov subspace $\mathcal{K}_k$, respectively. It is shown that the basic blocks of the two subspaces span the same space by proving that the ith basic block of one subspace can be written as a linear combination of the first i blocks of the other.

As α is a constant, and based on (17), it is clear that the starting vectors are the same, i.e., $\mathbf{P}_0 = \hat{\mathbf{P}}_0$. Recall that multiplying any basic block by a minus sign or a constant does not affect the spanned subspace. For the next two basic blocks, we have,

$$\begin{aligned} \mathbf{P}_1 &= (\mathbf{A}-\alpha\mathbf{E})^{-1}(4\mathbf{A}\mathbf{f}_0) \\ &= -4\alpha(\mathbf{A}-\alpha\mathbf{E})^{-1}\mathbf{A}(\mathbf{A}-\alpha\mathbf{E})^{-1}\mathbf{b} \\ &= -4\alpha\hat{\mathbf{P}}_1. \\ \mathbf{P}_2 &= (\mathbf{A}-\alpha\mathbf{E})^{-1}[4\mathbf{A}(-\mathbf{f}_0+\mathbf{f}_1)] \\ &= -\mathbf{P}_1 + 4\alpha(\mathbf{A}-\alpha\mathbf{E})^{-1}\mathbf{A}(\mathbf{A}-\alpha\mathbf{E})^{-1}\mathbf{A}(\mathbf{A}-\alpha\mathbf{E})^{-1}\mathbf{b} \\ &= 4\alpha\hat{\mathbf{P}}_1 + 4\alpha\hat{\mathbf{P}}_2. \end{aligned}$$

Now consider that $\mathbf{P}_n = \sum_{j=1}^{n} \beta_j\hat{\mathbf{P}}_j$ for $n = 2, \cdots, k-1$, where β_n is a constant. For an even [2] $i = k$, we have, based on (18),

$$\begin{aligned} \mathbf{P}_k &= (\mathbf{A}-\alpha\mathbf{E})^{-1}[4\mathbf{A}(-\mathbf{f}_0+\mathbf{f}_1+\cdots-\mathbf{f}_{k-1})] \\ &= -4\mathbf{P}_1 + 4\mathbf{P}_2 + \cdots - 4(\mathbf{A}-\alpha\mathbf{E})^{-1}\mathbf{A}\mathbf{P}_{k-1} \\ &= \sum_{j=1}^{k-2}\beta'_j\hat{\mathbf{P}}_j - 4(\mathbf{A}-\alpha\mathbf{E})^{-1}\mathbf{A}\mathbf{P}_{k-1} \\ &= \sum_{j=1}^{k-2}\beta'_j\hat{\mathbf{P}}_j - (\mathbf{A}-\alpha\mathbf{E})^{-1}\mathbf{A}\sum_{j=1}^{k-1}\beta'_j\hat{\mathbf{P}}_j \\ &= \sum_{j=1}^{k-2}\beta'_j\hat{\mathbf{P}}_j - \sum_{j=2}^{k}\beta'_j\hat{\mathbf{P}}_j \end{aligned}$$

The proof is completed by induction.

The importance of this theorem lies in the fact that it allows a numerically stable and computationally efficient way of calculating the projection matrix $\mathbf{V}$ using the well-known Arnoldi and Lanczos algorithms or one of their improved versions (Antoulas, 2005; Salimbahrami, 2005). In addition, it shows the direct dependance of the projection matrix $\mathbf{V}$, and thus the reduced system, on the parameter α. By varying α, different basis functions $\phi_i^\alpha(t)$ are produced and consequently the error-spreading of the approximation of the impulse response along the temporal axis can be controlled. Due to space limitations, the discussion related to the possible choices of α is kept for further publications.

5. THE EQUIVALENCE

In order to be able to prove the equivalence between the moment matching and both Laguerre-based order reduction approaches presented here and in (Chen et al., 2002), it is important to recall the theorem from (Eid et al., 2007) proving the invariance of the transfer function of the reduced order model to the change of basis of the Krylov subspaces.

Theorem 4. The transfer function of the reduced-order model depends only on the choice of the Krylov subspaces and not on the bases of these subspaces.

Another lemma that is relevant for the results of this section and presenting a main property of the krylov subspaces is the following (Eid et al., 2007):

Lemma 5. The Krylov subspaces $\mathcal{K}_k(\mathbf{M}, \mathbf{v})$ and $\mathcal{K}_k(\mathbf{N}, \mathbf{v})$ with $\mathbf{M} + c\mathbf{N} = \gamma\mathbf{I}$ where $0 \neq c, \gamma \in \mathbb{R}$, are identical.

Consequently,

Theorem 6. The Krylov subspaces $\mathcal{K}_k\left((\mathbf{A}-\alpha\mathbf{E})^{-1}\mathbf{E}, \mathbf{v}\right)$ of (5) and $\mathcal{K}_k\left((\mathbf{A}-\alpha\mathbf{E})^{-1}\mathbf{A}, \mathbf{v}\right)$ of (12) with $\mathbf{v} = (\mathbf{A}-\alpha\mathbf{E})^{-1}\mathbf{b}$, are identical.

Proof. Set $\mathbf{N} = (\mathbf{A}-\alpha\mathbf{E})^{-1}\mathbf{E}$ and $\mathbf{M} = (\mathbf{A}-\alpha\mathbf{E})^{-1}\mathbf{A}$. $\mathbf{M}$ can be rewritten as:

[2] for an odd k the same proof is valid, however the following derivation will have the opposite alternating sign of the coefficients because of the $(-1)^{i+j}$ in (18).

$$\begin{aligned}\mathbf{M} &= (\mathbf{A} - \alpha\mathbf{E})^{-1}(\mathbf{A} - \alpha\mathbf{E} + \alpha\mathbf{E}) \\ &= \mathbf{I} + \alpha(\mathbf{A} - \alpha\mathbf{E})^{-1}\mathbf{E} \\ &= \mathbf{I} + \alpha\mathbf{N}.\end{aligned}$$

By applying Lemma 5 with $\gamma = 1$ and $c = -\alpha$, the proof is completed.

Based on what was presented, the main theorem stating the equivalence of the two approaches is the following:

Theorem 7. Reducing a state space model in time-domain by matching the Laguerre coefficients of the impulse responses of the original and reduced models is exactly equivalent to matching the moments of their transfer functions around $s = \alpha$ in the frequency-domain.

Proof. Using theorem 4, it is shown that the subspaces involved in both approaches are equivalent, and using the fact that the transfer function of the reduced-order model depends only on the choice of the Krylov subspaces as stated in theorem 6, the proof is completed.

Corollary 8. Based on Theorem 7, an important time-domain interpretation of the moment matching approach can deduced. In fact, if order reduction is carried out completely in time-domain to match some of the first Laguerre coefficients with a certain parameter α as proposed in section 4, the same number of moments around $s_0 = \alpha$ in the frequency-domain automatically match. Similarly, if order reduction is carried out completely in frequency-domain to match some of the first moments around s_0, the same number of the first Laguerre coefficients of the Laguerre series expansion of the impulse response with $\alpha = s_0$ automatically match.

It should be noted also, that Theorem 7 indirectly states that the reduced systems obtained by the two different methods have the same input-output behavior, however, they may generally possess different realizations. Now, since the two methods employ equal subspaces, the resulting reduced models will have exactly the same realization when using the same numerical algorithm (e.g. Lanczos or Arnoldi) for the calculation of the projection matrices.

From the numerical point of view, by considering the corresponding subspaces (12) and (27), it can be remarked that both approaches require almost the same computational effort, however, for the case where $\mathbf{E} = \mathbf{I}$, the Laguerre-based method is numerically more expensive.

6. ILLUSTRATIVE EXAMPLE

To illustrate the equivalence results of this paper the following very low order example has been chosen in order to make the calculations more transparent and the reduction steps and results easily visualized:

$$H(s) = \frac{(s+4)(s+5)}{(s+1)(s+3)^2(s+7)(s+12)}.$$

The corresponding state space matrices are:

$$\mathbf{A} = \begin{bmatrix} -1 & 1 & 1 & 0 & 0 \\ 0 & -3 & 1 & 0 & 0 \\ 0 & 0 & -3 & 1 & 0 \\ 0 & 0 & 0 & -7 & 1 \\ 0 & 0 & 0 & 0 & -12 \end{bmatrix},$$

$$\mathbf{b}^T = [\,0\ 0\ 0\ 0\ 2\,],\ \mathbf{c} = [\,0.5\ 0.5\ 0.5\ 0\ 0\,]$$

This system is reduced to order $k = 3$ first by the classical moment matching around $s_0 = 1.5$ and then by the new proposed method with $\alpha = 1.5$.

In order to match the moments around $s_0 = 1.5$, the projection matrix,

$$\mathbf{V}_m = \begin{bmatrix} -0.1894 & 0.1618 & -0.0923 \\ -0.0861 & 0.0548 & -0.0225 \\ -0.3873 & 0.1603 & -0.0465 \\ -1.7429 & 0.3342 & -0.0489 \\ -14.8148 & 1.0974 & -0.0813 \end{bmatrix} \times 10^{-2},$$

whose columns form a basis for the Krylov subspace $\mathcal{K}_3\left((\mathbf{A} - 1.5\mathbf{I})^{-1}, (\mathbf{A} - 1.5\mathbf{I})^{-1}\mathbf{b}\right)$, is used with the choice $\mathbf{W} = \mathbf{V}$, leading to the following reduced system H_m:

$$\mathbf{A}_m = \begin{bmatrix} -21.8158 & 1 & 0 \\ -149.7183 & 1.5 & 1 \\ -230.7410 & 0 & 1.5 \end{bmatrix},$$

$$\mathbf{b}_m^T = [\,-21.8158\ -149.7183\ -230.7410\,],$$

$$\mathbf{c}_m = [\,-0.3314\ 0.1884\ -0.0806\,] \times 10^{-2}.$$

The first six moments of the original and reduced systems denoted as m and m_r are shown in Fig. 2. As expected, the first 3 moments are matching.

In order to match the Laguerre coefficients in time-domain with $\alpha = 1.5$, the projection matrix,

$$\mathbf{V}_L = \begin{bmatrix} -0.1894 & 0.0533 & 0.0883 \\ -0.0861 & -0.0039 & 0.0276 \\ -0.3873 & -0.1468 & -0.0109 \\ -1.7429 & -1.2417 & -0.8504 \\ -14.8148 & -13.1687 & -11.7055 \end{bmatrix} \times 10^{-2},$$

whose columns form a basis for the Krylov subspace $\mathcal{K}_3((\mathbf{A} - 1.5\mathbf{I})^{-1}\,\mathbf{A}, (\mathbf{A} - 1.5\mathbf{I})^{-1}\,\mathbf{b})$, is used with the choice $\mathbf{W} = \mathbf{V}$, leading to the following reduced system H_L:

$$\mathbf{A}_L = \begin{bmatrix} -24.5551 & -24.5551 & -24.5551 \\ 105.2909 & 106.7909 & 106.7909 \\ -102.5515 & -102.5515 & -101.0515 \end{bmatrix},$$

$$\mathbf{b}_L^T = [\,-26.0551\ 105.2909\ -102.5515\,],$$

$$\mathbf{c}_L = [\,-0.3314\ -0.0487\ 0.0525\,] \times 10^{-2}.$$

The first six Laguerre coefficients of the original and reduced systems denoted as f and f_r are shown in Fig. 2. As expected, the first 3 coefficients are matching.

Although the two systems H_m and H_L appears to be different, they are connected by the similarity transformation,

$$\mathbf{Q} = \begin{bmatrix} 1 & -2 & 4 \\ 0 & 2 & -8 \\ 0 & 0 & 4 \end{bmatrix},$$

and have the same transfer function,

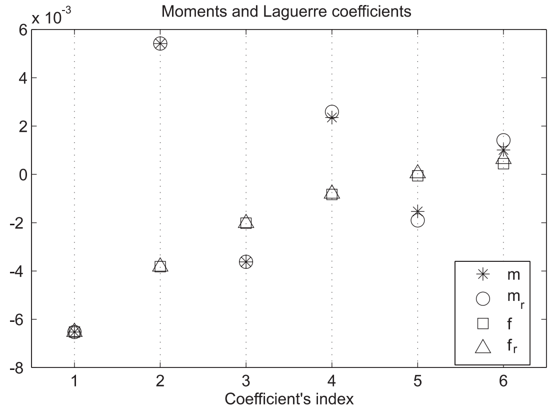

Fig. 2. Moments and Laguerre series coefficients of the original and reduced systems.

$$H_m(s) = H_L(s) = \frac{-0.04389s^2 + 0.1503s + 0.5648}{s^3 + 18.05s^2 + 77.35s + 53}.$$

Moreover, the projection matrices satisfy the relationship $\mathbf{V}_L = \mathbf{V}_m\mathbf{Q}$ due to the proof Theorem 4.

This confirms the results of Theorem 7 and therefore the reduced system H_m matches also the first 3 Laguerre coefficients of the original system in the time-domain and the reduced system H_L matches also the first 3 moments in the frequency-domain.

As illustrated in Fig. 2, the equivalence property does not imply that the Laguerre coefficients are equal to the moments but only states that matching one set of coefficients results in matching the other one.

7. CONCLUSION

A new time-domain order reduction method based on the Laguerre function expansion of the impulse response has been presented. By showing that the subspace spanned by the Laguerre coefficients vectors is a Krylov subspace and thus can be computed very efficiently, the method can be applied for the reduction of large-scale systems. In addition, a time-domain interpretation of the classical moment matching approach has been developed based on the fact that the reduced order model obtained by the new method and the one obtained by order reduction by rational interpolation around a single point, are exactly the same.

Even though the methods were shown to be equivalent, it has to be noted that the results of this paper does not degrade any of the two methods because each of them possess its own advantages and has some very unique properties, but offer new possibilities to improve each of them by importing the properties of the other one.

REFERENCES

A. C. Antoulas. *Approximation of Large-Scale Dynamical Systems*. SIAM, Philadelphia, 2005.

Z. Bai. Krylov Subspace Techniques for Reduced-Order Modeling of Large Scale Dynamical Systems. *Appl. Numer. Math.*, 43:9–44, 2002.

Y. Chen, V. Balakrishnan, C. K. Koh, and K. Roy. Model reduction in time domain using Laguerre polynomial and Krylov methods. In *Proc. Design, Automation and Test in Europe*, 931-935, 2002.

R. Eid, B. Salimbahrami, and B. Lohmann. Equivalence of laguerre-based model order reduction and moment matching. *IEEE Trans. on Automatic Control*, 52(6): 1104–1108, 2007.

R. W. Freund. Model reduction methods based on Krylov subspaces. *Acta Numerica*, 12:267–319, 2003.

P. K. Gunupudi and M. S. Nakhla. Model-reduction of nonlinear circuits using krylov-subspace techniques. In *Proc. 36th DAC*, pages 13–19, June 1999.

L. Knockaert and D. De Zutter. Laguerre-SVD reduced-order modeling. *IEEE Trans. on Microwave Theory and Techniques*, 48(9):1469–1475, 2000.

J. Lee, Y. Chung, J. So, J. Kim, J. Shin, C. Jang, W.and Cheon, H Jung, and T.K.; Sarkar. A new stable solution for td-efie using weighted laguerre polynomials as temporal basis functions. In *IEEE Antennas and Propagation Society International Symposium*, pages 2971–2974, 2006.

A. Odabasioglu, M. Celik, and L.T. Pileggi. PRIMA: Passive Reduced-Order Interconnect Macromodeling Algorithm. In *Int. Conf. Computer-Aided Design (ICCAD)*, pages 58–65, San Jose, CA, November 1997.

B. Salimbahrami. *Structure Preserving Order Reduction of Large Scale Second Order Models*. PhD thesis, Institute of Automatic Control, Technical University of Munich, 2005.

G. Szegö. *Orthogonal Polynomials*. American Mathematical Society, New York City, 1959.

J. M. Wang, Q. Yu, and E. S. Kuh. Passive model order reduction algorithm based on chebychev expansion of impulse response of interconnect networks. *Proc. Design Automation Conf.*, pages 520–524, 2000.