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A Multi–Parametric Optimization Strategy for Multilevel Hierarchical Control problems ⋆

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Abstract: In this work the three–level hierarchical control problem and the decentralised control problem are investigated and a general optimisation strategy is developed for solving these problems based on recent developments on multi–parametric programming. The main idea is to recast each optimisation subproblem in the multilevel hierarchy as a multi–parametric programming problem and then transform the multilevel problem into a single-level optimisation problem. This allows for the control policies (decisions) at each level of the multilevel optimisation problem to be obtained as explicit functions of the state of the dynamic systems involved in each level and the control policies of the higher levels. A three person dynamic optimisation problem is presented to illustrate the mathematical developments.

Keywords: Hierarchical control; decentralised control; multilevel and multi-follower optimization; multi–parametric programming.

1. INTRODUCTION

In optimisation and control of large–scale dynamic systems, hierarchical and decentralised control allow for the decomposition of the original problem into smaller, interconnected problems which are typical arranged into a multilevel hierarchy (Mesarovic et al. [1970], Cohen [1977], Morari et al. [1980], N.R. Sandell et al. [1978], Stephanopoulos and Ng [2000], Venkat et al. [2005]). Various applications of hierarchical and decentralised control arise in process systems engineering (Morari et al. [1980], Stephanopoulos and Ng [2000]), mechanical and power systems (Delaleau and Stanković [2004]), aeronautics (Tolani et al. [2004], Li et al. [2002]), traffic control (Shimizu et al. [1995]) and large-scale systems control (N.R. Sandell et al. [1978], Roberts and Becerra [2001]). In most of these problems the general formulations of the decomposed multilevel hierarchy is given in Figures 1-2.

Examples of the hierarchical structure in Figure 1 can be found in Delaleau and Stanković [2004], Singh et al. [1975], Stanković and Šiljak [1989]. In Delaleau and Stanković [2004] the control problem of a PM synchronous motor is decomposed into a bi–level hierarchical control problem (similar to Figure 1 without the third level). A high–level controller, corresponding to the slow dynamics of the motor's mechanical system, is designed to obtain the right set points for the low–level controller which controls the fast dynamics of the electrical system. In [Singh et al., 1975, Section 5] and Stanković and Šiljak [1989] the same hierarchical decomposition was used to deal with optimal LQG and optimal LQR control of sequentially (or serially) interconnected linear dynamical systems. Decentralised control of large-scale systems also yields a two–level structure where in the lower level more than one subproblems are considered (Figure 2) (N.R. Sandell et al. [1978], Venkat et al. [2005]).

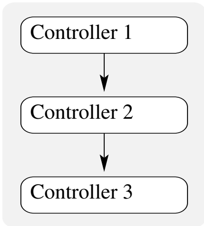

Fig. 1. Three-level controller structure

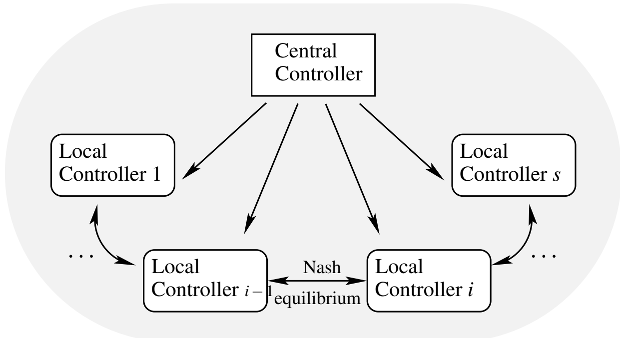

Fig. 2. Hierarchical control configuration.

Nash equilibrium is often a preferred strategy to coordinate such decentralised systems Venkat et al. [2005]. Similarly, this is also the hierarchical structure that is found in a typical leader – multi–follower problem Li et al. [2002].

Generally, it is widely recognised that the successful design of large and complex systems involves some type of decomposition of the original problem into smaller and intercommunicating subsystems, typically arranged in one of the multilevel hierarchies described in Figures 1 and 2. Multilevel and decentralised optimisation problems, which typically arise in many engineering Clark [1983], Morari et al. [1980], Stephanopoulos

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and Ng [2000], Venkat et al. [2005] and financial applications Anandalingman [1988], Nie et al. [2006], involve such an hierarchy of optimisation levels, where each optimisation level (or subproblem) controls a subset of the overall optimisation variables.

Despite their significance, general solution strategies for solving such complex problems are limited, especially due to the multi-layer nature, non-linearities and non-convexities Vicente and Calamai [1994]. In addition, the potential presence of logical decisions (which requires the inclusion of binary variables) increases further the complexity of the problem. Therefore, it is widely accepted that a global optimisation approach is needed for the solution of such multilevel optimisation problems Floudas [2000].

Recently, Pistikopoulos and co-workers Acevedo and Pistikopoulos [1997], Dua et al. [2002], Faísca et al. [2007, 2008], Pistikopoulos et al. [2002] have proposed novel solution algorithms, based on multi–parametric programming theory Fiacco [1983], Pistikopoulos et al. [2007a], which open the possibility to address general classes of multilevel programming problems. In an optimisation problem where the objective is to minimise (or maximise) a cost criterion subject to constraints and which includes a number of parameters varying between specified lower and upper bounds, multi–parametric programming is a method to obtain i) the objective function and the optimisation variables as functions of the parameters and ii) the region in the parameter space where these functions are valid (Fiacco [1983], Dua et al. [2002], Pistikopoulos et al. [2002]). Multi–parametric programming has found many applications in model–based predictive control (Bemporad et al. [2002], Pistikopoulos et al. [2002, 2007b]) where the on–line optimisation problem involved is solved off–line by multi–parametric programming and the control policy is obtained as a set of explicit functions of the state measurements. The control action is then implemented on–line by function evaluations, reducing the complexity of the implementation of the controller and improving the computational speed.

Our approach is to recast each optimisation subproblem as a multi–parametric programming problem, and hence obtain an analytical solution for the rational reaction set for each of the subproblems. Each of the control policies at each level of the three–level hierarchical control problem are then obtained as explicit functions of the states of the system/systems involved in each level and the policies on the higher levels, with the policy of the highest level obtained as an explicit function of the initial states only. In the decentralised control problem, the policy for each secondary controller is obtained as a function of the overall system and the policy of the central controller. The policy of the central controller is an explicit function of the initial state of the system. The policies at each level then of both problems can then be obtained as simple function evaluations. In the knowledge of the authors there is currently no relevant work on treating multilevel hierarchical control problem with multi–parametric programming, despite the vast research on these problems.

The paper is organised as follows. Section II discusses the methodology for solving three–level hierarchical multilevel control problems by multi–parametric programming techniques. The two–level decentralised control problem (leader–follower problem) is described and solved again by multi–parametric programming in Section III. An example is presented in Section IV that illustrates the two methodologies and the complexity analysis and on–line multilevel optimisation issues are also discussed. The paper finally concludes with the necessary conclusions.

2. THREE LEVEL HIERARCHICAL CONTROL

Consider the following three–level hierarchical control problem

$$J_1 = \min_{U,U^2,U^1} V_1(x_t, x_t^1, x_t^2, u_t, u_t^1, u_t^2) \tag{1a}$$

$$\text{s.t. } x_{t+1} = Ax_t + Bu_t + Dz_t \tag{1b}$$

$$g(x_t, x_t^1, x_t^2, u_t, u_t^1, u_t^2) \leq 0 \tag{1c}$$

$$J_2 = \min_{U^2,U^1} V_2(x_t, x_t^1, x_t^2, u_t, u_t^1, u_t^2) \tag{1d}$$

$$\text{s.t. } x_{t+1}^2 = A_2 x_t^2 + B_2 u_t^2 + D_2 z_t^2 \tag{1e}$$

$$g_2(x_t, x_t^1, x_t^2, u_t, u_t^1, u_t^2) \leq 0 \tag{1f}$$

$$J_3 = \min_{U^1} V_3(x_t, x_t^1, x_t^2, u_t, u_t^1, u_t^2) \tag{1g}$$

$$\text{s.t. } x_{t+1}^1 = A_1 x_t^1 + B_1 u_t^1 + D_1 z_t^1 \tag{1h}$$

$$g_1(x_t, x_t^1, x_t^2, u_t, u_t^1, u_t^2) \leq 0 \tag{1i}$$

$$x_t \in \mathbb{R}^n,\ x_t^1 \in \mathbb{R}^{n_1},\ x_t^2 \in \mathbb{R}^{n_2},\ t = 1, \ldots, N \tag{1j}$$

$$u \in \mathbb{R}^m,\ u^1 \in \mathbb{R}^{m_1},\ u^2 \in \mathbb{R}^{m_2},\ t = 1, \ldots, N-1 \tag{1k}$$

where V_1, V_2, V_3 are all strictly convex quadratic functions of x_t, x_t^1, x_t^2 and the control sequences (control policies) $U = \{u_0, \ldots, u_{N-1}\} \in \mathbb{R}^{mN}$, $U^1 = \{u_0^1, \ldots, u_{N-1}^1\} \in \mathbb{R}^{m_1 N}$ and $U^2 = \{u_0^2, \ldots, u_{N-1}^2\} \in \mathbb{R}^{m_2 N}$. The matrices A, A_1, A_2, B, B_1, B_2, D, D_1 and D_2 are of appropriate dimensions while g, g_1, g_2 are vectors of linear functions of x_t, x_t^1, x_t^2, U, U^1 and U^2 defining a set of linear inequality constraints in (1c), (1f) and (1i). Finally, the vectors $z = [(x^1)^T\ (x^2)^T]^T$, $z^2 = [x^T\ (x^1)^T]^T$ and $z^1 = [x^T\ (x^2)^T]^T$ represent the interconnection between the linear discrete–time dynamic system in the different hierarchical levels of problem (1). Problem (1) is a multilevel optimization problem of the form shown in Figure 1 where in each level a different dynamic system, objective and set of constraints are considered.

A special subcase of problem (1) is the multilevel hierarchical control of sequentially interconnected dynamical systems (Singh et al. [1975], Stanković and Šiljak [1989]) where $z^1 = 0$, $z^2 = x$ and* $z = [{x^1}^T\ \ {x^2}^T]^T$ i.e. the dynamical system at each level of (1) is independent of the ones in the higher levels (it does not depent on the states and inputs of the dynamic systems in the higher levels). In this case the constraints of the problem at each level are also independent of the states and inputs of the higher levels. The structure of the problem for sequentially interconnected dynamic systems is thus simpler than the one considered in (1). However, the more complex multilevel optimisation problem (1) is examined here to cover a more broad case of hierarchical control problems as those arising for example in Delaleau and Stanković [2004]. Finally, extension to smaller bi–level optimization problems (with no third optimisation level in (1)) should be easy.

The objective is, given the initial states x_0, x_0^1, x_0^2 of each linear system, to obtain the optimal control policies $U^*, {U^1}^*, {U^2}^*$ for the three–level hierarchical control problem (1). Substituting the linear discrete–time model equations (1b), (1e) and (1h) into (1a), (1d), (1g) and (1c), (1f), (1i) the following multilevel optimisation problem is formulated

$$J_1 = \min_{U,U^2,U^1} V_1(x_0, x_0^1, x_0^2, U, U^1, U^2) \quad (2a)$$

$$\text{s.t. } g(x_0, x_0^1, x_0^2, U, U^1, U^2) \leq 0 \quad (2b)$$

$$J_2 = \min_{U^2,U^1} V_2(x_0, x_0^1, x_0^2, U, U^1, U^2) \quad (2c)$$

$$\text{s.t. } g_2(x_0, x_0^1, x_0^2, U, U^1, U^2) \leq 0 \quad (2d)$$

$$J_3 = \min_{U^1} V_3(x_0, x_0^1, x_0^2, U, U^1, U^2) \quad (2e)$$

$$\text{s.t. } g_1(x_0, x_0^1, x_0^2, U, U^1, U^2) \leq 0 \quad (2f)$$

where the objective functions V_1, V_2, V_3 are strictly convex quadratic and the constraints g, g_1, g_2 are linear functions of x_0, x_0^1, x_0^2 and U, U^1, U^2. The feasible and rational reaction sets for the third and second levels are defined as

$$\Omega_3(x_0, x_0^1, x_0^2, U, U^2) = \{U^1 \in \mathbb{R}^{Nm_1} \mid g_1 \leq 0\} \quad (3)$$

$$\Phi_3(x_0, x_0^1, x_0^2, U, U^2) = \{U^1 \in \mathbb{R}^{Nm_1} \mid U^1 \in \operatorname{argmin}\{V_3 \mid U^1 \in \Omega_3\}\} \quad (4)$$

$$\Omega_2(x_0, x_0^1, x_0^2, U) = \{(U^1, U^2) \in \mathbb{R}^{Nm_1} \times \mathbb{R}^{Nm_2} \mid g_1 \leq 0, g_2 \leq 0\} \quad (5)$$

$$\Phi_2(x_0, x_0^1, x_0^2, U) = \{(U^1, U^2) \in \mathbb{R}^{Nm_1} \times \mathbb{R}^{Nm_2} \mid U^2 \in \operatorname{argmin}\{V_2 \mid (U^1, U^2) \in \Omega_2, U^1 \in \Phi_3\}\} \quad (6)$$

respectively, where (3), (5) are the feasible sets for the third and second level, and (4) and (6) are the rational reaction sets of the third and second level. Equations (4), (6) show the dependence of the policies obtained on the upper level on the policies obtained at the lower levels.

Each of the optimisation problems in each level can be recast as multi–parametric programming problem (Pistikopoulos et al. [2002], Dua et al. [2002], Faísca et al. [2007]) with x_0, x_0^1, x_0^2 being the overall parameters of the system and U, U^2, U^1 being the optimisation variables of the first, second and third optimisation levels respectively. Problem (2) can then be recast as a multi–parametric multilevel optimisation problem (Faísca et al. [2007]). An algorithm for solving (2) can be obtained, following the technique for solving bilevel programming problems in Faísca et al. [2007]. The steps of this algorithm are described as follows

Algorithm 1:

(1) Recast the third optimisation level as a multi–parametric programming problem with U^1 being the optimisation variable and $x_0, x_0^1, x_0^2, U, U^2$ the parameters.

(2) Solve by a multi–parametric programming algorithm (Pistikopoulos et al. [2002], Dua et al. [2002]) to obtain U^1 as a set of functions of the parameters $x_0, x_0^1, x_0^2, U, U^2$ and the critical regions where these functions are valid i.e. the parametric solution

$$\begin{aligned} &U^1 = K_{1i}^1 x_0 + K_{2i}^1 x_0^1 + K_{3i}^1 x_0^2 + K_{4i}^1 U + K_{5i}^1 U^2 + c_i^1 \\ &\text{if } A_{1i}^1 x_0 + A_{2i}^1 x_0^1 + A_{3i}^1 x_0^2 + A_{4i}^1 U + A_{5i}^1 U^2 \leq b_i^1 \\ &i = 1, \ldots, s_1 \end{aligned} \quad (7)$$

Equation (7) for all $i = 1, \ldots, s_1$ gives the rational reaction set of the third level.

(3) Substitute the rational reaction set in the optimisation problem in the second level of (2) and formulate s_1 multi–parametric programming problems in which U^2 is the optimisation variable and x_0, x_0^1, x_0^2, U are the parameters.

(4) Solve the s_1 multi–parametric programming problems with a multi–parametric programming algorithm to obtain the parametric solution of U^2 as a function of x_0, x_0^1, x_0^2, U

$$\begin{aligned} &U^2 = K_{1i}^2 x_0 + K_{2i}^2 x_0^1 + K_{3i}^2 x_0^2 + K_{4i}^2 U + c_i^2 \\ &\text{if } A_{1i}^2 x_0 + A_{2i}^2 x_0^1 + A_{3i}^2 x_0^2 + A_{4i}^2 U \leq b_i^2 \\ &i = 1, \ldots, s_2 \end{aligned} \quad (8)$$

(5) Substitute the s_2 functions and critical regions of the previous step in the optimisation problem of the first level and recast it as a parametric optimisation problem where U is the optimisation variable and x_0, x_0^1, x_0^2 are the parameters to obtain the parametric solution

$$\begin{aligned} &U = K_{1i} x_0 + K_{2i} x_0^1 + K_{3i} x_0^2 + c_i \\ &\text{if } A_{1i} x_0 + A_{2i} x_0^1 + A_{3i} x_0^2 + A_{4i} U \leq b_i^2 \\ &i = 1, \ldots, s \end{aligned} \quad (9)$$

In Step 5 of Algorithm 1 the control policy U is obtained as an explicit function of the initial states x_0, x_0^1, x_0^2 of the linear systems (1b), (1e), (1h). Thus, when the initial states x_0, x_0^1, x_0^2 are given the optimal control policy U can be obtain by simple function evaluations from (9). The optimal control policy U^2 for the second optimisation level is obtained from (8) given x_0, x_0^1, x_0^2, U and finally U^1 is obtained from (7). It can be noticed that the control policy at each hierarchical level is a function of the control policies of the higher levels. The rational reaction set for each level is thus obtained analytically.

Remark 1. It is possible that in Steps 4 and 5 overlapping regions might occur since quadratic objective functions are considered. In that case one can employ the comparison method described in Acevedo and Pistikopoulos [1997] to compare the explicit solutions in the overlapping of the regions.

3. DECENTRALISED HIERARCHICAL CONTROL

Consider the decentralised control scheme of Figure (2) that consist of a central controller (the leader) and m secondary controllers (the followers) on the second level (Başar and Selbuz [1994]). The objective is the optimal control of the overall linear discrete–time system

$$x_{t+1} = Ax_t + B_0 u_t + \sum_{i=1}^{s} B_i u_t^i \quad (10)$$

where $x_t \in \mathbb{R}^n$, $u_t \in \mathbb{R}^m$, $u_t^i \in \mathbb{R}^{m_i}$, $i = 1, \ldots, s$ are the states of the system, the input to the system controlled by the leader and the inputs to the system controlled by the followers respectively, with respect to the following quadratic optimal control problem

$$\begin{aligned} J_0 &= \min_{U,U^1,\ldots,U^s} V(x_t, U) \quad (11a) \\ &= \min_{U} x_N^T Q_N^0 x_N + \sum_{t=0}^{N-1} \left[x_t^T Q_t^0 x_t + u_t^T R_t^0 u_t + \sum_{i=1}^{s} u_t^i R_t^{0i} u_t^i \right] \end{aligned}$$

$$\text{s.t. } x_{t+1} = Ax_t + B_0 u_t + \sum_{i=1}^{s} B_i u_t^i \quad (11b)$$

$$g(x_t, U, U^i) \leq 0 \quad (11c)$$

$$Q_t^0 \geq 0,\ R_t^0 > 0,\ R_t^{0i} \geq 0 \quad (11d)$$

where $U = \{u_0, \ldots, u_{N-1}\}$, $U^i = \{u_0^i, \ldots, u_{N-1}^i\}$, $i = 1, \ldots, s$ are the control policies of the leader and i-th follower respectively, g is a vector of linear functions corresponding to a set of linear inequalities (11c). It is assumed that the secondary controllers do not have access to the minimisation of (11a) but they select the optimal policies U^i based on the *announced* policy U of the central controller. Only the central controller has access over the complete set of optimisation variables.

Each of the secondary controllers solves a local optimisation subproblem

$$J_i = \min_{U^i} V_i(x_t, U, U^i, U^j) \tag{12a}$$
$$= \min_{U^i} x_N^T Q_N^{i0} x_N + \sum_{t=0}^{N-1} \left[x_t^T Q_t^{i0} x_t + u_t^T R_t^{i0} u_t + \sum_{k=1}^{s} u_t^i R_t^{ik} u_t^i \right]$$
$$\text{s.t. } x_{t+1} = A_i x_t + B_0 u_t + \sum_{j=1}^{s} B_j u_t^j \tag{12b}$$
$$g_i(x_t, U, U^i, U^j) \le 0 \tag{12c}$$
$$Q_t^{i0} \ge 0, R_t^{i0} \ge 0,\ R_t^{ik} \ge 0,\ R_t^{ii} > 0 \tag{12d}$$

for the i–th secondary controller, $i = 1, \dots, s$. Only the i-th follower has access to the i-th minimisation problem (12) i.e. the minimisation in (12) is only with respect to U^i.

Remark 2. It is possible that each of the secondary controllers could include its own local linear dynamic model. The problem can then be formulated by including all linear dynamic models altogether in (11)–(12) and by adding extra quadratic terms in the objective functions (11a) and (12a) to accommodate for the new states introduced. The objectives and the constraints in (11) and (12) are then all functions of all states and inputs

The multilevel optimisation problem based on the leader–follower formulation of (11)–(12) has been treated for the case where no constraints are present N.R. Sandell et al. [1978], Başar and Selbuz [1994], while in Venkat et al. [2005] a similar distributed control problem was treated with input constraints. Here, the more general problem is treated where both state and input constraints are present as is the case in most engineering problems where any real actions is restricted by physical or operational constraints. The problem is solved to obtain the global optimum of the central controller and the best possible optima for the local controllers.

Replacing (11b) in (11), (12) and considering we seek for the optimal policies U^*, U^{i*} then the problem reformulates in the following multilevel optimisation problem

$$J_0 = \min_{U, U^1, \dots, U^s} V(x_0, U, U^1, \dots, U^s) \tag{13a}$$
$$\text{s.t. } g(x_0, U, U^1, \dots, U^s) \le 0 \tag{13b}$$
$$\cdots \left\{ \begin{array}{c} J_i = \min_{U^i} V_i(x_0, x_0^i, U, U^1, \dots, U^s) \\ \text{s.t. } g_i(x_0, U, U^1, \dots, U^s) \le 0 \end{array} \right\} \cdots \tag{13c}$$

where in the second level (13c) the secondary controller optimisation problems are considered altogether. Since V, V_i are quadratic functions and g, g_i are linear functions of x_0, U, U^i, both optimisation problems in both levels are quadratic programming problems with respect to the x_0, U, U^i. The feasible and rational reaction set of each of the secondary controllers are respectively

$$\Omega_i(x_0, U, U^1, \dots, U^{i-1}, U^{i+1}, \dots, U^s) = \{U^i \in \mathbb{R}^{Nm_i} \mid g_i \le 0\} \tag{14}$$

$$\Phi_i(x_0, U, U^1, \dots, U^{i-1}, U^{i+1}, \dots, U^s) = \{U^i \in \mathbb{R}^{Nm_i} \mid U^i \in \operatorname{argmin}\{V_i \,|\, U^i \in \Omega_i\}\} \tag{15}$$

One can notice that the central controller problem as well as each of the secondary controller problems can be recast as multi–parametric programming problems where x_0 is the parameter and U, U^i are the optimisation variable. More specifically, each of the secondary controller problem are multi–parametric programming problems where the control policy U^i for the i–th controller is the optimisation variable and x_0 and the control policies U, U^j, $j \ne i$ are the parameters.

Each secondary controller has access to the optimisation problem of the rest of the secondary controllers either via the state x_t or via the announced policy of the central controller. In that case it is most natural to introduce an equilibrium concept between the secondary controllers.

$$V(x_0, U^*, U^{1*}, \dots, U^{s*}) \le V(x_0, U, U^{1*}, \dots, U^{s*}) \tag{16a}$$
$$V_1(x_0, U^*, U^{1*}, \dots, U^{s*}) \le V_1(x_0, U^*, U^1, U^{2*} \dots, U^{s*}) \tag{16b}$$
$$\vdots$$
$$V_s(x_0, U^*, U^{1*}, \dots, U^{s*}) \le V_1(x_0, U^*, U^{1*}, \dots, U^{s-1^*}, U^s) \tag{16c}$$

similar to the Nash equilibrium concept. The equilibrium can be computed by direct comparison as in Faísca et al. [2007], Liu [1998].

As was shown previously each of the optimisation problems (11) and (12) can be recast as multi–parametric programming problems. Hence, one can solve the problem by employing multi–parametric programming techniques (Dua et al. [2002], Pistikopoulos et al. [2002]). This is given in detail in the following algorithm which applies multi–parametric programming to solve the leader – follower problem (11)–(12)

Algorithm 2:

(1) Recast each of the secondary controller subproblems (12) as a parametric optimisation problem where U^i is the optimisation variable and x_0, U, U^j, $j \ne i$ are the parameters.
(2) Solve each of the multi–parametric problems using a multi–parametric programming algorithm and obtain the multi–parametric solution of each of the controllers as

$$\begin{aligned} & U^i = K_\ell^i x_0 + K_{0\ell}^i U + \sum_{j \ne i} K_{j\ell}^i U^j + c_\ell^i \\ & \text{if } L_\ell^i x_0 + L_{0\ell}^i U + \sum_{j \ne i} L_{j\ell}^i U^j \le b_\ell^j \\ & \ell = 1, \dots, p^i \end{aligned} \tag{17}$$

where ℓ is the number of critical regions for the i–th multi–parametric programming problem corresponding to the i–th secondary controller. The set of p^i linear functions of U^i in (17) defines the raitonal reaction set for the i–th subproblem.
(3) Compute the equilibrium point (16).
(4) Substitute the the expressions of U^i into the central controller problem (11) and recast the problem as a multi–parametric programming problem where U is the optimisation variable and x_0 is the parameter.
(5) Solve the resulting multi–parametric programming problem and obtain the parametric solution as an explicit functions of x_0

$$\begin{aligned} & U = K_\ell x_0 + c_\ell \\ & \text{if } L_\ell x_0 \le b_\ell \\ & \ell = 1, \dots, p \end{aligned} \tag{18}$$

Algorithm 2 obtains the central controller control policy as an explicit function of the initial state x_0 and the secondary control policies as explicit functions of x_0 and the central controllers policy U i.e. obtains analytically the rational reaction set for the i–th controller. Each time x_0 is given then the central controller can obtain its optimal policy by function evaluation from (18).

The policy U is then *announced* to the secondary controllers which obtain their policy based in x_0 and U.

4. EXAMPLE

In this section the two multilevel optimisation methods for hierarchical and decentralised control are illustrated by means of an example. For this, the multiple person dynamic linear quadratic optimisation problem presented in Nie et al. [2006] is considered, which involves the coordination of three controllers within a complex environment.

We assume first that the three controllers are structured as a three level hierarchical control problem ((1), Section II). The objectives (1a), (1d) and (1g) for each of the three levels in (1) are given respectively by

$$J_1 = \min_{u_0,u_1,u_2} 4x_3 + 3x_3^1 + 2x_3^2 + \sum_{t=0}^{2} \left\{ (u_t)^2 + \left(u_t^1\right)^2 - \left(u_t^2\right)^2 + 2u_t x_t + (x_t)^2 \right\} \tag{19}$$

$$J_2 = \min_{u_0^2,u_1^2,u_2^2} 2x_3 + 3x_3^2 + \sum_{t=0}^{2} \left\{ 2 \cdot u_t u_t^2 + \left(u_t^1+1\right)^2 + \left(u_t^2+1\right)^2 \right\} \tag{20}$$

$$J_3 = \min_{u_0^1,u_1^1,u_2^1} x_3 + 2x_3^1 - 10x_3^2 + \sum_{t=0}^{2} \left\{ -15u_t + \left(u_t^1-1\right)^2 - 2u_t^1 u_t^2 + \left(u_t^2\right)^2 \right\} \tag{21}$$

the corresponding linear systems (1b), (1e) and (1h) are

$$x_{t+1} = x_t + u_t - 2u_t^1 + u_t^2, \tag{22}$$

$$x_{t+1}^2 = x_t^2 + 2u_t^2 \tag{23}$$

$$x_{t+1}^1 = x_t^1 + 2u_t^1 \tag{24}$$

$$t = 0,1,2$$

respectively and the state and input constraints for each of the systems and hence for each of the subproblems in (1) are

$$-20 \le u_t \le 20,\ -10 \le x_t \le 10 \tag{25}$$

$$-30 \le u_t^2 \le 30,\ -10 \le x_t^2 \le 10 \tag{26}$$

$$-30 \le u_t^1 \le 30,\ -10 \le x_0^1 \le 10 \tag{27}$$

where $t = 0,1,2$. By following the steps of Algorithm 1, we obtain the explicit multi–parametric solution to the three level hierarchical control problem which is given in Table 1. The policy $U = \{u_0, u_1, u_2\}$ to the highest level of (1) for this problem is given by four linear functions, each defined in a different set of the space of x_0. The policies of the two other levels are given by a simple linear function of U. Given a value for x_0, let say $x_0 = 10$ one can obtain the control policy U by looking in which critical region x_0 belongs i.e. critical region 4. Then the policy is obtained by evaluating the expressions of U in this region for the given value $x_0 = 10$

$$u_0 = -16.2732,\ u_1 = 20,\ u_2 = -20$$

These values of the policy are then used to obtain the policies U^2 and U^1 of the lower levels from Table 1

$$u_0^1 = -10.1366,\ u_1^1 = -12,\ u_2^1 = 8,$$
$$u_0^2 = -10.1366,\ u_1^2 = -12,\ u_2^2 = 8.$$

The same problem of the three controller coordination is formulated in the leader – follower structure (Figure 2, Section III). The optimisation problem of the leader (central controller) is formulated by considering (19) as the objective and (22) and (25) the constraints. Similarly, (20), (23), (26) and (21), (24), (27) form the optimisation problems of the two followers. One should notice that in this problem each of the secondary controllers also includes its own dynamics in the problem. Then, as we discussed in Remark 1, the linear models of all three systems

Table 1. Solution to the hierarchical problem

Critical Region 1	**Critical Region 2**
$u_0 = 6.84615 - 0.76928x_0$	$u_0 = -0.333333 - 1.8519x_0$
$u_1 = -20$	$u_1 = -1.33333 + 2.8148x_0$
$u_2 = 15.2308 + 0.15388x_0$	$u_2 = -2 - 2.4444x_0$
$-10 \le x_0 \le -6.63161$	$-6.63161 \le x_0 \le 7.36377$
Critical Region 3	**Critical Region 4**
$u_0 = -1.53333 - 1.6889x_0$	$u_0 = -9 - 0.72732x_0$
$u_1 = 8.26667 + 1.5111x_0$	$u_1 = 20$
$u_2 = -20$	$u_2 = -20$
$7.36377 \le x_0 \le 7.76466$	$7.76466 \le x_0 \le 10$
$u_0^1 = u_0^2 = -2 - 0.5u_0;\ u_1^1 = u_1^2 = -2 - 0.5u_1;\ u_2^1 = u_2^2 = -2 - 0.5u_2$	

Table 2. Solution to the decentralised problem

Critical Region 1
$u_0 = 1 - x_0;\ u_1 = -8 + x_0;\ u_2 = 5 - x_0$
$v_0^1 = v_0^2 = -6 + x_0;\ v_1^1 = v_1^2 = 3 - x_0$
$v_2^1 = v_2^2 = -10 + x_0;\ -10 \le x_0 \le 10$

(22), (23), (24) have to be included in the three optimisation problems (19), (20), (21) of the leader and the two followers respectively. The objective functions in (19), (20), (21) have already included terms corresponding to the states of each of the systems (22), (23), (24). The problem is then solved as a leader–follower multilevel optimisation problem by following the steps Algorithm 2. The results are given in Table 2. The policies of the central controller and the secondary controllers are again obtained as linear functions of the initial condition x_0. The solution yields only one critical region in which each of the policies of the central and the secondary controllers are explicit linear functions of the initial state x_0. For $x_0 = 10$ the control policies are $u_0 = -9$, $u_1 = 2$, $u_2 = -5$, $u_0^1 = u_0^2 = 4$, $u_1^1 = u_1^2 = -7$, $u_2^1 = u_2^2 = 0$.

4.1 Complexity analysis

In this section we briefly discuss a few complexity issues of the two algorithms introduced in this work for the solution of three–level hierarchical and decentralised control problems. It is important to notice that in both Algorithms 1 and 2, the number of critical regions depends on the size of the problem that we encounter. With a significant increase in the number of state and inputs of the systems and number of the constraints involved in the two problems (1) and (11)–(12), it is possible that the number of critical regions will also increase substantially, thus increasing the computational cost. The methodology described here involves convex objectives and linear systems and constraints. The complexity of the problem increases once nonlinear, nonconvex objectives, system dynamics and constraints are considered.

4.2 On–line multilevel optimisation

In many control applications it is rather possible that the hierarchical and decentralised control problems described by (1) and (11)–(12) will have to be solved on–line to maintain a constant level of control and supervision of the system under consideration. In that case, the methodology described in Section II and III moves the solution of the on–line multilevel optimisation problems off–line and solve by employing multi–parametric programming techniques. The control policies, considered in each of the subproblems of the multilevel problem, are implemented on–line by performing function evaluations on the explicit expressions given in the Step 2,4 and 5 of Algorithm 1 and the Steps 2,3 and 5 of Algorithm 2, each time the initial states of the system are available. For example, if problem (1) represents model–based predictive control of a sequentially connected system where in each level a smaller subsystem of the overall system is considered (Stanković and Šiljak [1989]), then each time the control policies U, U^1, U^2 are obtained only u_0, u_0^1 and u_0^2 have to be applied to the system.

5. CONCLUSIONS

In this work the three–level hierarchical control and multilevel decentralised control problems were treated. Two algorithms were developed that solve the problems by recasting each optimisation problem in each level as a multi–parametric programming problem. It was shown that the control policies at each level and the rational reaction sets can be obtained analytically as a set of explicit functions. There is currently non known work relevant to the solution of multi–level optimisation problem, such as the ones in Sections II and III, in the knowledge of the authors, despite the vast research in the area of multilevel optimisation and hierarchical control. This work attempts, possibly for the first time, to address the issue and motivate further developments on the problems of hierarchical control and multilevel optimisation.

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