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Approximate stabilization of a quantum particle in a 1D infinite potential well *

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Abstract: We consider a non relativistic charged particle in a 1D infinite square potential well. This quantum system is subjected to a control, which is a uniform (in space) time depending electric field. It is represented by a complex probability amplitude, solution of a Schrödinger equation on a 1D bounded domain, with Dirichlet boundary conditions. We prove the almost global approximate stabilization of the eigenstates by explicit feedback laws.

Keywords: Quantum systems, PDE control, Schrödinger equation, Lyapunov stabilization

1. INTRODUCTION

As in Rouchon [2002], Beauchard [2005], Beauchard and Coron [2006], we consider a non-relativist charged particle in a one dimensional space, with a potential $V(x)$, in a uniform electric field $t \mapsto u(t) \in \mathbb{R}$. Under dipolar momentum approximation assumption, and after appropriate changes of scales, the evolution of the particle's wave function is given by the following Schrödinger equation

$$i\frac{\partial \Psi}{\partial t}(t,x) = -\frac{\partial^2 \Psi}{\partial x^2}(t,x) + (V(x) - u(t)x)\Psi(t,x).$$

We study the case of an infinite square potential well: $V(x) = 0$ for $x \in I = (-1/2, 1/2)$ and $V(x) = +\infty$ for x outside I. Therefore our system is

$$i\frac{\partial \Psi}{\partial t}(t,x) = -\frac{1}{2}\frac{\partial^2 \Psi}{\partial x^2}(t,x) - u(t)x\Psi(t,x), \qquad x \in I \quad (1)$$

$$\Psi(0,x) = \Psi_0(x), \quad (2)$$

$$\Psi(t, \pm 1/2) = 0. \quad (3)$$

It is a nonlinear control system, denoted by (Σ), in which, u, the electric field, is a scalar control and $\Psi(t,x) : \mathbb{R}_+ \times I \to \mathbb{C}$, the particle's wavefunction, is the state of the system. Furthermore, the self-adjointness of the operators in the evolution equation implies the conservation of the L^2-norm: $\Psi(t) \in \mathbb{S} = \{\varphi \in L^2(I;\mathbb{C}); \|\varphi\|_{L^2} = 1\}$.

For $\sigma \in \mathbb{R}$, we introduce the operator A_σ defined by

$$D(A_\sigma) = (H^2 \cap H_0^1)(I;\mathbb{C}), \ A_\sigma \varphi = -\frac{1}{2}\frac{\partial^2 \varphi}{dx^2} - \sigma x \varphi.$$

It is well known that there exists an orthonormal basis $(\phi_{k,\sigma})_{k \in \mathbb{N}^*}$ of $L^2(I, \mathbb{C})$ of eigenvectors of A_σ :

$$\phi_{k,\sigma} \in H^2 \cap H_0^1(I,\mathbb{C}), \ A_\sigma \phi_{k,\sigma} = \lambda_{k,\sigma} \phi_{k,\sigma}$$

where $(\lambda_{k,\sigma})_{k \in \mathbb{N}^*}$ is a non decreasing sequence of real numbers. For $k \in \mathbb{N}^*$ and $\sigma \in \mathbb{R}$, we define $\mathscr{C}_{k,\sigma} = \{\phi_{k,\sigma}e^{i\theta}; \theta \in [0, 2\pi)\}$. Note that, a change of the global phase does not change the physical state of the system.

In order to simplify the notations, we will write ϕ_k, λ_k, $\mathscr{C}_k$ instead of $\phi_{k,0}$, $\lambda_{k,0}$, $\mathscr{C}_{k,0}$. We have

$$\lambda_k = \frac{k^2 \pi^2}{2}, \quad \phi_k = \begin{cases} \sqrt{2}\cos(k\pi x), \text{ when } k \text{ is odd}, \\ \sqrt{2}\sin(k\pi x), \text{ when } k \text{ is even}. \end{cases} \quad (4)$$

When $k \in \mathbb{N}$, $\sigma \in \mathbb{R}$, $\Psi_0 = \phi_{k,\sigma}$ and $u \equiv \sigma$, the solution of (1)-(2)-(3) is given by $\Psi_{k,\sigma}(t,x) = \phi_{k,\sigma}(x)e^{-i\lambda_{k,\sigma}t}$.

Finally, for $s, \sigma \in \mathbb{R}$, we define

$$H_{(\sigma)}^s(I, \mathbb{C}) = D(A_\sigma^{s/2}),$$

equipped with the norm

$$\|\varphi\|_{H_{(\sigma)}^s} = \left(\sum_{k=1}^{\infty} \lambda_{k,\sigma}^s |\langle \varphi, \phi_{k,\sigma}\rangle|^2\right)^{1/2}.$$

The goal of this paper is the study of the stabilization of the system (Σ) around $\mathscr{C}_{k,\sigma}$. More precisely, for $k \in \mathbb{N}^*$ and $\sigma \in \mathbb{R}$ small, we state feedback laws $u = u_{k,\sigma}(\Psi)$ for which the solution of (1)-(2)-(3) with $u(t) = u_{k,\sigma}(\Psi(t))$ is such that

$$\limsup_{t \to +\infty} \text{dist}_{L^2(I,\mathbb{C})}(\Psi(t), \mathscr{C}_{k,\sigma})$$

is arbitrarily small. We consider the convergence toward the circle $\mathscr{C}_{k,\sigma}$ because the wave function Ψ is defined up to a phase factor. For simplicity sakes, we will only work with the ground state $\Psi_{1,\sigma}$. However, the whole arguments remain valid for the general case.

Note that, even though the feedback stabilization of a quantum system necessitates more complicated models taking into account the measurement backaction on the system (see e.g. Handel et al. [2005], Mirrahimi and Handel [2007]), the kind of strategy considered in this paper can be helpful for the open-loop control of closed quantum systems. Indeed, one can apply the stabilization techniques for the Schrödinger equation in simulation and retrieve the control signal that will be then applied in open-loop on the real physical system. As it will be detailed below, in the bibliographic overview, such kind of strategy has been widely used in the context of finite dimensional quantum systems.

Before going through the statement of the main result and the technicalities of a proof, let us give a brief bibliography on the context.

The controllability of a finite dimensional quantum system, $\iota\frac{d}{dt}\Psi = (H_0 + u(t) H_1)\Psi$ where $\Psi \in \mathbb{C}^N$ and H_0 and H_1 are $N \times N$ Hermitian matrices with coefficients in $\mathbb{C}$ has been very well explored Ramakrishna et al. [1995], Albertini and D'Alessandro [2003]. However, this does not guarantee the simplicity of the trajectory generation. Very often the chemists formulate the task of the open-loop control as a cost functional to be minimized. Optimal control techniques (see e.g., Shi et al. [1988]) and iterative stochastic techniques (e.g, genetic algorithms Li et al. [2002]) are then two classes of approaches which are most commonly used for this task.

When some non-degeneracy assumptions concerning the linearized system are satisfied, Mirrahimi et al. [2005a] provides another method based on Lyapunov techniques for generating trajectories. The relevance of such a method for the control of chemical models has been studied in Mirrahimi et al. [2005b]. As mentioned above, the closed-loop system is simulated and the retrieved control signal is applied in open-loop. Such kind of strategy has already been applied widely in this framework Chen et al. [1995], Sugawara [2003].

The situation is much more difficult when we consider an infinite dimensional configuration and less results are available. However, the controllability of the system (1)-(2)-(3) is now well understood. In Turinici [2000], the author states some non-controllability results for general Schrödinger systems. However, this negative result is due to the choice of the functional spaces that does not allow the controllability. Indeed, if we consider different functional spaces, one can get positive controllability results. In Beauchard [2005], the local controllability of the system (1)-(2)-(3) around the ground state $\Psi_{1,\sigma}$, for σ small is proved. The main tools in this aim are the Nash-Moser implicit function theorem, the return method and the quantum adiabatic theory. Furthermore, in Beauchard and Coron [2006], the steady-state controllability of this nonlinear system is proved (i.e. the particle can be moved in finite time from an eigenstate Ψ_k to another one Ψ_j).

Concerning the trajectory generation problem for infinite dimensional systems still much less results are available. The simplicity of the feedback law found by the Lyapunov techniques in Mirrahimi et al. [2005a], Beauchard et al. [2007] suggests the use of the same approach for infinite dimensional configurations. However, an extension of the convergence analysis to the PDE configuration is not at all a trivial problem. Indeed, it requires the pre-compactness of the closed-loop trajectories, a property that is difficult to prove in infinite dimension.

In Mirrahimi [2006], one of the authors proposes a Lyapunov-based method to approximately stabilize a particle in a 3D finite potential well under some restrictive assumptions. The author assumes that the system is initialized in the finite dimensional discrete part of the spectrum. Then, the idea consists in proposing a Lyapunov function which encodes both the distance with respect to the target state and the necessity of remaining in the discrete part of the spectrum. In this way, he prevents the possibility of the "mass lost phenomenon" at infinity. Finally, applying some dispersive estimates of Strichartz type, he ensures the approximate stabilization of an arbitrary eigenstate in the discrete part of the spectrum.

In this paper, we study the stabilization of the ground state $\Psi_{1,\sigma}$ for σ in a neighborhood of 0. Adapting the techniques proposed in Mirrahimi [2006], we ensure the approximate stabilization

of the system around arbitrary eigenstates and under general settings. The main idea consists in proposing a Lyapunov function which encodes the distance with respect to the target state and at the same time prevents the "mass lost phenomenon" in high-energy eigenstates.

The case $\sigma = 0$ represent a degenerate case (the linearized system is not controllable; to be compared with the finite dimensional problem Mirrahimi et al. [2005a]). Choosing $\sigma \neq 0$ sufficiently small, we can remove this degeneracy. Therefore, the problem is a little bit simpler in this case and will be considered first.

In the next section, we consider the non-degenerate case $\sigma \neq 0$. We provide the design technique and we check out the performance of the control law on a simulation. Next, we state the main result and give the main steps of the proof without going through the detailed technicalities. A more complete version of the proof will be soon submitted as a journal paper.

In Section 3, we consider the case $\sigma = 0$. Applying an implicit Lyapunov theory we will remove the degeneracy problem. The performance of the implicit design will be checked out on a simulation. The proof of the provided result is quite similar to the finite dimensional case Beauchard et al. [2007] and can be adapted accordingly.

Finally, in Conclusion, we will address the possibility of applying the provided techniques in order to get a global controllability result.

2. STABILIZATION OF $\mathscr{C}_{1,\sigma}$ WITH $\sigma \neq 0$

2.1 Control design

In this section, we consider the non-degenerate case of $\sigma \neq 0$ for σ sufficiently small. We assume the initial state Ψ_0 to be in L^2 and therefore we may consider its expansion over the orthonormal eigenbasis, $\{\phi_{k,\sigma}\}_{k=1}^{\infty}$ of A_σ:

$$\Psi_0 = \sum_{k=1}^{\infty} \left\langle \phi_{k,\sigma} \mid \Psi_0 \right\rangle \phi_{k,\sigma}.$$

Here $\langle . \mid . \rangle$ denotes the Hermitian product of $L^2(I, \mathbb{C})$. While trying to stabilize the ground state $\phi_{1,\sigma}$, the first approach would be to consider the simple Lyapunov function

$$\widetilde{\mathscr{V}}(\Psi) = 1 - |\langle \Psi \mid \phi_{1,\sigma} \rangle|^2.$$

Just as in the finite the dimensional case Beauchard et al. [2007], the feedback law

$$\tilde{u}(\Psi) = \Im(\langle x\Psi \mid \phi_{1,\sigma} \rangle \langle \phi_{1,\sigma} \mid \Psi \rangle)$$

where $\Im$ denotes the imaginary part of a complex, ensures the decrease of the Lyapunov function. However, trying to adapt the convergence analysis, based on the use of the LaSalle invariance principle, the pre-compactness of the trajectories in L^2 constitutes a major obstacle. Note that, in order to be able to apply the LaSalle principle for an infinite dimensional system, one certainly needs to prove such a pre-compactness result. In the particular case of the infinite potential well, it even seems that, one can not hope such a result. Indeed, phenomenons such as the L^2-mass lost in the high energy levels do not allow this property to hold true.

Similarly to Mirrahimi [2006], the approach of this paper is to avoid the population to go through the very high energy levels,

while trying to stabilize the system around $\phi_{1,\sigma}$. We, therefore, consider the Lyapunov function

$$\mathscr{V}_\varepsilon(\Psi) = 1 - (1-\varepsilon)\sum_{k=1}^{N} |\langle \Psi \mid \phi_{k,\sigma}\rangle|^2 - \varepsilon|\langle \Psi \mid \phi_{1,\sigma}\rangle|^2. \quad (5)$$

Here, the cut-off dimension N is fixed so that the initial state Ψ_0 is approximately spanned by the first N eigenstates; i.e.

$$1 - \sum_{k=1}^{N} |\langle \Psi \mid \phi_{k,\sigma}\rangle|^2$$

is sufficiently small (this will be detailed in the main theorem below). Note that this is always possible (as Ψ_0 is in L^2) and makes the Lyapunov function depend on the initial state. However, as it will be detailed in the Remark 2, taking Ψ_0 in a Sobolev space H^s with $s > 0$, we can choose this cut-off dimension as a function of the H^s-norm.

Deriving the Lyapunov function $\mathscr{V}_\varepsilon$ with respect to time and inserting the dynamics (1) with the control $u = \sigma + v$, we have

$$\frac{d\mathscr{V}_\varepsilon}{dt} = -v(t)\Im\left((1-\varepsilon)\sum_{k=1}^{N} \langle x\Psi \mid \phi_{k,\sigma}\rangle\langle \phi_{k,\sigma} \mid \Psi\rangle + \varepsilon\langle x\Psi \mid \phi_{1,\sigma}\rangle\langle \phi_{1,\sigma} \mid \Psi\rangle\right).$$
$$(6)$$

Thus, the feedback law

$$v = v_\varepsilon(\Psi) = c\Im\left((1-\varepsilon)\sum_{k=1}^{N} \langle x\Psi \mid \phi_{k,\sigma}\rangle\langle \phi_{k,\sigma} \mid \Psi\rangle + \varepsilon\langle x\Psi \mid \phi_{1,\sigma}\rangle\langle \phi_{1,\sigma} \mid \Psi\rangle\right)$$
$$(7)$$

with $c > 0$, trivially ensures the decrease of the Lyapunov function $\mathscr{V}_\varepsilon$. We claim that, the solution of (1)-(2)-(3) with the initial condition Ψ_0 (approximately spanned by the first N eigenstates) and the control $u = \sigma + v_\varepsilon(\Psi)$ satisfies

$$\limsup_{t\to+\infty} \text{dist}_{L^2}(\Psi(t), \mathscr{C}_{1,\sigma}) \leq \varepsilon. \quad (8)$$

Before stating the main result and going through the proof, let us check this on a simulation.

As mentioned above the constant σ needs to be small. In fact, one should choose σ, such that the perturbation σx is small compared to the the operator $-\frac{1}{2}\frac{\partial^2}{\partial x^2}$. We choose it here to be $\sigma = 2e+1$. We consider the initial state Ψ_0 to be spanned by the 3 first eigenstates of A_σ. More precisely, we take: $\Psi_0 = \frac{1}{\sqrt{2}}(\phi_{1,\sigma} + \phi_{3,\sigma})$. As it will be seen in Section 3, such an initial state is particularly hard to stabilize in a near degenerate situation. The Figure 1 illustrates the simulation of the closed-loop system when $u = \sigma + v_\varepsilon$ with $c = 1e+3$ and $\varepsilon = 5e-2$. The simulations have been done applying a third order split-operator method, where instead of computing $\exp(-i\,dt\,(A_\sigma + v_\varepsilon x))$ at each time step, we compute

$$\exp(-i\,dt\,A_\sigma/2)\exp(-i\,dt\,v_\varepsilon x)\exp(-i\,dt\,A_\sigma/2).$$

Moreover, we consider a Galerkin discretization over the first 20 modes of the system (it turns out, by considering higher modal approximations, that 20 modes are completely sufficient to get a trustable result).

2.2 Main result and convergence analysis

The main result of Section 2 is the following

Theorem 1. Let $N \in \mathbb{N}^*$. There exists $\sigma^\sharp = \sigma^\sharp(N) > 0$ such that, for every $\sigma \in (-\sigma^\sharp, \sigma^\sharp) - \{0\}$, $\gamma \in (0,1)$, $\varepsilon > 0$, and $\Psi_0 \in \mathbb{S} \cap H^2 \cap H_0^1$ verifying

$$\sum_{k=N+1}^{\infty} |\langle \Psi_0, \phi_{k,\sigma}\rangle|^2 < \frac{\varepsilon\gamma^2}{1-\varepsilon} \quad \text{and} \quad |\langle \Psi_0, \phi_{1,\sigma}\rangle| \geq \gamma, \quad (9)$$
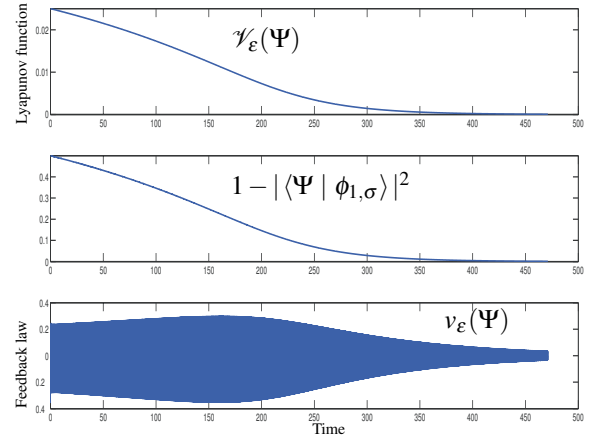

Fig. 1. The approximate stabilization of $\Psi_{1,\sigma}$, where $\Psi_0 = \frac{1}{\sqrt{2}}(\phi_{1,\sigma} + \phi_{3,\sigma})$ and therefore the cut-off dimension is 3; as it can be seen the closed-loop system reaches the .05-neighborhood of $\phi_{1,\sigma}$ in a time $T = 150\pi$ corresponding to about 200 periods of the longest natural period corresponding to the ground to the first excited state.

the Cauchy problem (1)-(2)-(3) with $u(t) = \sigma + v_{\sigma,N,\varepsilon}(\Psi(t))$, where $v_{\sigma,N,\varepsilon}$ is given by (7), has a unique strong solution Ψ, moreover, this solution satisfies

$$\liminf_{t\to+\infty} |\langle \Psi(t), \phi_{1,\sigma}\rangle|^2 \geq 1 - \varepsilon. \quad (10)$$

Remark 2. The cut-off constant N can be uniformly chosen when the initial state lies in a bounded subset of H_0^1. Indeed, when the H^s-norm (for some $s > 0$) of Ψ_0 is less than some constant Γ, one can easily find a cut-off constant $N(s,\Gamma)$ only depending on s and Γ, such that the first part of the assumption (9) is satisfied.

Remark 3. The second part of the assumption (9) does not play a crucial role in practice. Actually, even for an initial state such that $\langle \Psi_0 \mid \phi_{1,\sigma}\rangle = 0$, applying a resonant control field including the transition frequencies between the other N eigenstates and the ground state, one can always make sure to have instantaneously a non-zero population in $\phi_{1,\sigma}$.

Before going through the proof of the approximate stabilization (10), we need to establish the well-posed ness of the closed-loop system.

Lemma 4. Let $\sigma \in \mathbb{R}$, $N \in \mathbb{N}^*$, $\varepsilon > 0$ and $\Psi_0 \in \mathbb{S}$. There exists a unique weak solution Ψ of (1)-(2)-(3) with $u = \sigma + v_{\sigma,N,\varepsilon}(\Psi)$, i.e. $\Psi \in C^0(\mathbb{R}_+, \mathbb{S}) \cap C^1(\mathbb{R}_+, H_{(0)}^{-2}(I,\mathbb{C}))$, the equality (1) holds in $H_{(0)}^{-2}(I,\mathbb{C})$.

Proof of Lemma 4: Consider $T > 0$ such that

$$TNe^{NT} < 1. \quad (11)$$

In order to build solutions on $[0,T]$, we apply the Banach fixed point theorem to the map

$$\Theta : C^0([0,T],\mathbb{S}) \to C^0([0,T],\mathbb{S})$$
$$\xi \mapsto \Psi$$

where Ψ is the solution of (1)-(2)-(3) with $u = \sigma + v_{\sigma,N,\varepsilon}(\xi(t))$.

The map Θ is well defined and maps $C^0([0,T],\mathbb{S})$ into itself. Indeed, when $\xi \in C^0([0,T],\mathbb{S})$, then $u : t \mapsto \sigma + v_{\sigma,N,\varepsilon}(\xi(t))$ is continuous and thus the existence of a unique weak solution Ψ

is a classical consequence. Notice that the map Θ takes values in $C^0([0,T],\mathbb{S})) \cap C^1([0,T],H^{-2}_{(0)})$.

Let us prove that Θ is a contraction of $C^0([0,T],\mathbb{S})$. Let $\xi_j \in C^0([0,T],\mathbb{S})$, $v_j = v_{\sigma,N,\varepsilon}(\xi_j)$, $\Psi_j = \Theta(\xi_j)$, for $j = 1,2$ and $\Delta = \Psi_1 - \Psi_2$. We have

$$\Delta(t) = i \int_0^t e^{-iA_\sigma(t-s)}[v_1 x \Delta(s) + (v_1 - v_2)x\Psi_2(s)]ds.$$

We easily have $\|v_j\|_{L^\infty(0,T)} \leq N$ for $j = 1,2$ and $\|v_1 - v_2\|_{L^\infty(0,T)} \leq 2N\|\xi_1 - \xi_2\|_{C^0([0,T],L^2)}$ thus

$$\|\Delta(t)\|_{L^2} \leq \int_0^t N\|\Delta(s)\|_{L^2} + N\|\xi_1 - \xi_2\|_{C^0([0,T],L^2)}ds. \quad (12)$$

Therefore, Gronwall Lemma provides

$$\|\Delta(t)\|_{C^0([0,T],L^2)} \leq \|\xi_1 - \xi_2\|_{C^0([0,T],L^2)}NTe^{NT},$$

and (11) justifies that Θ is a contraction of the Banach space $C^0([0,T],\mathbb{S})$. Therefore, there exists a fixed point $\Psi \in C^0([0,T],\mathbb{S})$ such that $\Theta(\Psi) = \Psi$. Since Θ takes values in $C^0([0,T],\mathbb{S}) \cap C^1([0,T],H^{-2}_{(0)})$, necessarily Ψ belongs to this space, thus, it is a weak solution of (1)-(2)-(3) on $[0,T]$.

We have introduced a time $T > 0$ and, for every $\Psi_0 \in \mathbb{S}$, we have built a weak solution $\Psi \in C^0([0,T],\mathbb{S})$ of (1)-(2)-(3) on $[0,T]$. Thus, for a given initial condition $\Psi_0 \in \mathbb{S}$, we can apply this result on $[0,T]$, $[T,2T]$, $[2T,3T]$ etc. This proves the existence and uniqueness of global weak solutions for the closed-loop system. $\square$

Applying Lemma 4, it is a classical consequence that, for an initial state $\Psi_0 \in H^2 \cap H_0^1$ the unique weak solution is actually a strong solution (see e.g. Pazy [1983]).

In order to prove the approximate stabilization, we need to state a few Lemmas. After some simple but tedious computations, based on the Rayleigh-Schrödinger perturbation theory, we have the following Lemma:

Lemma 5. Let $N \in \mathbb{N}^*$. There exists $\sigma^\sharp(N) > 0$ such that, for every $\sigma \in (-\sigma^\sharp,\sigma^\sharp) - \{0\}$, $j_2,k_2 \in \mathbb{N}^*$, and $j_1,k_1 \in \{1,\cdots,N\}$, verifying $j_1 \neq j_2$ and $k_1 \neq k_2$,

- $\langle x\phi_{j_2,\sigma} \mid \phi_{1,\sigma} \rangle \neq 0$;
- $\lambda_{k_1,\sigma} - \lambda_{k_2,\sigma} = \lambda_{j_1,\sigma} - \lambda_{j_2,\sigma}$ implies $(j_1,j_2) = (k_1,k_2)$.

For a proof of the first part, we refer to Beauchard [2005]. The second part of the of the Lemma can be proven considering the second order expansion of the eigenvalues with respect to σ. This necessitates simple but tedious computations and we leave it to the reader.

Lemma 6. Let $\sigma > 0$, $N \in \mathbb{N}$, $\varepsilon > 0$ and $(\Psi_0^n)_{n \in \mathbb{N}}$ be a sequence of $\mathbb{S}$ and $\Psi_0^\infty \in L^2$ with $\|\Psi_0^\infty\|_{L^2} \leq 1$ be such that

$$\lim_{n \to +\infty} \Psi_0^n = \Psi_0^\infty \text{ strongly in } H^{-1}(I,\mathbb{C}).$$

Let Ψ^n (resp. Ψ^∞) be the weak solution of (1)-(2)-(3) with $u(t) = \sigma + v_{\sigma,N,\varepsilon}(\Psi^n(t))$ (resp. with $u(t) = \sigma + v_{\sigma,N,\varepsilon}(\Psi^\infty(t))$). Then, for every $\tau > 0$,

$$\lim_{n \to +\infty} \Psi^n(\tau) = \Psi^\infty(\tau) \text{ strongly in } H^{-1}(I,\mathbb{C}).$$

Proof of Lemma 6: We introduce $\mathscr{C} > 0$ such that,

$$\|x\varphi\|_{H^{-1}} \leq \mathscr{C}\|\varphi\|_{H^{-1}}, \forall \varphi \in H^{-1}(I,\mathbb{C}). \quad (13)$$

Such a constant does exist. Indeed, for every $\xi \in H_0^1(I,\mathbb{C})$, $x\xi \in H_0^1(I,\mathbb{C})$ and

$$\|x\xi\|_{H_0^1} = \left(\int_I |x\xi' + \xi|^2 dx\right)^{1/2} \leq \|\xi'\|_{L^2}(1 + C_P)$$

where C_P is the Poincaré constant on I. Thus, for $\varphi \in H^{-1}(I,\mathbb{C})$, we have

$$\begin{aligned}
\|x\varphi\|_{H^{-1}(I,\mathbb{C})} &= \sup\left\{\langle x\varphi,\xi\rangle; \xi \in H_0^1(I,\mathbb{C}), \|\xi\|_{H_0^1} = 1\right\} \\
&\leq \sup\left\{\|\varphi\|_{H^{-1}}\|x\xi\|_{H_0^1}; \xi \in H_0^1(I,\mathbb{C}), \|\xi\|_{H_0^1} = 1\right\} \\
&\leq (1 + C_P)\|\varphi\|_{H^{-1}}.
\end{aligned}$$

In order to simplify the notations, in this proof, we write $v(\Psi)$ instead of $v_{\sigma,N,\varepsilon}(\Psi)$. We have

$$\begin{aligned}
(\Psi^n - \Psi^\infty)(t) &= e^{-iAt}(\Psi_0^n - \Psi_0^\infty) + i\int_0^t e^{-iA(t-s)}\sigma x(\Psi^n - \Psi^\infty)(s)ds \\
&+ i\int_0^t e^{-iA(t-s)}[v(\Psi^n(s)) - v(\Psi^\infty(s))]x\Psi^n(s)ds \\
&+ i\int_0^t e^{-iA(t-s)}v(\Psi^\infty(s))x[\Psi^n(s) - \Psi^\infty(s)]ds.
\end{aligned}$$

Using (7), $\|\Psi^n(s)\|_{L^2} = 1$, $\|\Psi^\infty(s)\|_{L^2} \leq 1$ and the fact that $\phi_{k,\sigma}, x\phi_{k,\sigma} \in H_0^1(I,\mathbb{C})$ for $k = 1,...,N$, we get

$$|v(\Psi^n(s)) - v(\Psi^\infty(s))| \leq 2N\mathscr{C}C_\sigma(N)\|(\Psi^n - \Psi^\infty)(s)\|_{H^{-1}}, \quad (14)$$

where $C_\sigma(N) = \sup\{\|\phi_{k,\sigma}\|_{H_0^1(I,\mathbb{C})}; k \in \{1,...,N\}\}$. The semigroup e^{-iAt} preserves the H^{-1}-norm thus, using $|v(\Psi^\infty(s))| \leq N$ and (14), we get

$$\begin{aligned}
\|(\Psi^n - \Psi^\infty)(t)\|_{H^{-1}} &\leq \|\Psi_0^n - \Psi_0^\infty\|_{H^{-1}} \\
&+ \mathscr{C}\int_0^t [|\sigma| + 2NC_\sigma(N) + N]\|\Psi^n(s) - \Psi^\infty(s)\|_{H^{-1}}ds.
\end{aligned}$$

We conclude thanks to the Gronwall Lemma. $\square$

Proof of the approximate stabilization (10): Applying (6) and (7), $\mathscr{V}_\varepsilon$ defined in (5) is a non-increasing positive function. There exists $\alpha \in [0,\mathscr{V}_\varepsilon(\Psi_0)]$ such that $\mathscr{V}_\varepsilon[\Psi(t)] \to \alpha$ when $t \to +\infty$. Since $\Psi_0 \in \mathbb{S}$ and (9) holds we have

$$\begin{aligned}
\mathscr{V}_\varepsilon(\Psi_0) &= 1 - (1-\varepsilon)\sum_{k=1}^N |\langle\Psi,\phi_{k,\sigma}\rangle|^2 - \varepsilon|\langle\Psi,\phi_{1,\sigma}\rangle|^2 \\
&< 1 - (1-\varepsilon)\left(1 - \frac{\varepsilon\gamma^2}{1-\varepsilon}\right) - \varepsilon\gamma^2 < \varepsilon,
\end{aligned}$$

thus $\alpha \in [0,\varepsilon)$.

Since $\|\Psi(t)\|_{L^2} = 1$ for $t \geq 0$, there exists $\Psi_\infty \in L^2(I,\mathbb{C})$, and an increasing sequence of times $\{t_n\}_{n=1}^\infty \nearrow \infty$, such that

$$\Psi(t_n) \to \Psi_\infty \text{ weakly in } L^2(I,\mathbb{C}) \text{ and strongly in } H^{-1}(I,\mathbb{C}).$$

Let ξ be the solution of

$$\begin{cases}
i\frac{\partial\xi}{\partial t} = A_\sigma\xi - v_{\sigma,N,\varepsilon}(\xi(t))x\xi, x \in I, t \in (0,+\infty), \\
\xi(t,\pm 1/2) = 0, \\
\xi(0) = \Psi_\infty.
\end{cases}$$

Thanks to the Lemma 6, for every $\tau > 0$, $\Psi(t_n + \tau) \to \xi(\tau)$ strongly in $H^{-1}(I,\mathbb{C})$ when $n \to +\infty$. Thus $\mathscr{V}_\varepsilon(\Psi(t_n + \tau)) \to \mathscr{V}_\varepsilon(\xi(\tau))$ when $n \to +\infty$, as $\mathscr{V}_\varepsilon(.)$ is continuous for the L^2-weak topology. Therefore $\mathscr{V}_\varepsilon(\xi(\tau)) \equiv \alpha$. As $\mathscr{V}_\varepsilon(\xi(t))$ is a non-increasing function of time and as $\mathscr{V}_{\sigma,N,\varepsilon}(\xi(0)) = \alpha$, (6) and (7) imply $v_{\sigma,N,\varepsilon}(\xi(\tau)) \equiv 0$. Thus

$$\xi(\tau) = \sum_{k=1}^\infty \langle\Psi_\infty,\phi_{k,\sigma}\rangle\phi_{k,\sigma}e^{-i\lambda_{k,\sigma}\tau}$$

where

$$\Im\left(\sum_{k=1}^N \sum_{j\in\mathbb{N}^*, j\neq k} a_k\langle\Psi_\infty,\phi_{j,\sigma}\rangle\langle x\phi_{j,\sigma},\phi_{k,\sigma}\rangle\overline{\langle\Psi_\infty,\phi_{k,\sigma}\rangle}e^{i(\lambda_{k,\sigma}-\lambda_{j,\sigma})\tau}\right) \equiv 0 \quad (15)$$

with $a_1 = 1$ and $a_k = 1 - \varepsilon$ for $k > 1$. Applying Lemma 5 and the Ingham inequality (see e.g. Krabs [1992]), we get

$$\langle \Psi_\infty, \phi_{j,\sigma} \rangle \overline{\langle \Psi_\infty, \phi_{1,\sigma} \rangle} = 0, \forall j \geq 2. \qquad (16)$$

Let us prove that $\langle \Psi_\infty, \phi_{1,\sigma} \rangle \neq 0$. Since $\|\Psi^\infty\|_{L^2} \leq 1$, we easily have

$$\mathscr{V}_\varepsilon(\Psi_\infty) \geq \varepsilon - \varepsilon |\langle \Psi^\infty, \phi_{1,\sigma} \rangle|^2.$$

Moreover, $\mathscr{V}_\varepsilon(\Psi_\infty) = \alpha < \varepsilon$, thus $\varepsilon > \varepsilon - \varepsilon |\langle \Psi_\infty, \phi_{1,\sigma} |^2$. This ensures $\langle \Psi_\infty, \phi_{1,\sigma} \rangle \neq 0$. Therefore (16) justifies the existence of $\beta \in \mathbb{C}$ with $|\beta| \leq 1$ such that $\Psi_\infty = \beta \phi_{1,\sigma}$. Then, $\varepsilon > \alpha = \mathscr{V}_\varepsilon(\Psi_\infty) = 1 - |\beta|^2$, thus $|\beta|^2 > 1 - \varepsilon$. Finally, we have

$$\lim_{n \to +\infty} |\langle \Psi(t_n), \phi_{1,\sigma} \rangle|^2 = |\langle \Psi_\infty, \phi_{1,\sigma} \rangle|^2 = |\beta|^2 > 1 - \varepsilon.$$

This holds for every sequence $(t_n)_{n \in \mathbb{N}}$ thus (10) is proved. $\square$

3. STABILIZATION OF $\mathscr{C}_1$

3.1 Control design

In this section, we consider the degenerate case of $\sigma = 0$. This degeneracy (non-controllability of the linearized system around ϕ_1) comes from the fact that the result of Lemma 5 does not hold true for $\sigma = 0$. This particularly makes the control task much harder. In order to see this, let us reconsider the example of Figure 1. Considering the initial state $\Psi_0 = \frac{1}{\sqrt{2}}(\phi_1 + \phi_3)$ and calculating $v_\varepsilon(\Psi)$ given by (7), one can easily see that for the symmetry reasons $v_\varepsilon(\Psi) \equiv 0$. In fact, ϕ_1 and ϕ_3 have the same parity and therefore $\langle x\phi_1 \mid \phi_3 \rangle = 0$.

In order to overcome this degeneracy, we apply an implicit Lyapunov technique, also considered in Beauchard et al. [2007] for the finite dimensional case.

In this aim, we consider the Lyapunov function

$$\mathscr{V}_\varepsilon(\Psi) = 1 - (1-\varepsilon) \sum_{k=1}^N |\langle \Psi \mid \phi_{k,\sigma(\Psi)} \rangle|^2 - \varepsilon |\langle \Psi \mid \phi_{1,\sigma(\Psi)} \rangle|^2, \qquad (17)$$

where the function $\Psi \mapsto \sigma(\Psi)$ is implicitly defined as below

$$\sigma(\Psi) = \theta(\mathscr{V}_\varepsilon(\Psi)), \qquad (18)$$

for a slowly varying real function θ. We claim that such a function $\sigma(\Psi)$ exists. When Ψ solves (Σ), we have

$$\begin{aligned}
\frac{d\mathscr{V}_\varepsilon}{dt} = &-v(\Psi)\Im\Big((1-\varepsilon) \sum_{k=1}^N \langle x\Psi \mid \phi_{k,\sigma(\Psi)} \rangle \langle \phi_{k,\sigma(\Psi)} \mid \Psi \rangle \\
&+ \varepsilon \langle x\Psi \mid \phi_{1,\sigma(\Psi)} \rangle \langle \phi_{1,\sigma(\Psi)} \mid \Psi \rangle \Big) \\
&- \frac{d\sigma(\Psi)}{dt} 2\Re\Big((1-\varepsilon) \sum_{k=1}^N \langle \Psi, \phi_{k,\sigma(\Psi)} \rangle \langle \frac{d\phi_{k,\sigma(\Psi)}}{d\sigma}, \Psi \rangle \\
&+ \varepsilon \langle \Psi, \phi_{1,\sigma(\Psi)} \rangle \langle \frac{d\phi_{1,\sigma(\Psi)}}{d\sigma}, \Psi \rangle \Big).
\end{aligned}$$

where the notation $\frac{d\phi_{k,\sigma(\Psi)}}{d\sigma}$ means the derivative of the map $\sigma \mapsto \phi_{k,\sigma}$ taken at the point $\sigma = \sigma(\Psi)$. By definition of $\sigma(\Psi)$, we have

$$\frac{d\sigma(\Psi)}{dt} = \theta'(\mathscr{V}_\varepsilon(\Psi)) \frac{d\mathscr{V}_\varepsilon}{dt}.$$

Thus, the feedback law $u(\Psi) := \sigma(\Psi) + v_\varepsilon(\Psi)$ where

$$\begin{aligned}
v_\varepsilon(\Psi) = \\
c\Im\Big((1-\varepsilon) \sum_{k=1}^N \langle x\Psi \mid \phi_{k,\sigma(\Psi)} \rangle \langle \phi_{k,\sigma(\Psi)} \mid \Psi \rangle + \varepsilon \langle x\Psi \mid \phi_{1,\sigma(\Psi)} \rangle \langle \phi_{1,\sigma(\Psi)} \mid \Psi \rangle \Big)
\end{aligned}$$

with $c > 0$ ensures

$$\frac{d\mathscr{V}_\varepsilon}{dt} = -2c\mu v_\varepsilon(\Psi)^2.$$

Here

$$\frac{1}{\mu} = 1 + 2\theta'(\mathscr{V}(\Psi)) \times$$

$$\Re\Big((1-\varepsilon) \sum_{k=1}^N \langle \Psi, \phi_{k,\sigma(\Psi)} \rangle \langle \frac{d\phi_{k,\sigma(\Psi)}}{d\sigma}, \Psi \rangle + \varepsilon \langle \Psi, \phi_{1,\sigma(\Psi)} \rangle \langle \frac{d\phi_{1,\sigma(\Psi)}}{d\sigma}, \Psi \rangle \Big)$$

is a positive constant, when $\|\theta'\|_{L^\infty}$ is small enough. Thus $t \mapsto \mathscr{V}_\varepsilon(\Psi(t))$ is not increasing.

We claim that, the solution of (1)-(2)-(3) with initial condition Ψ_0 and the control $u = \sigma(\Psi) + v_\varepsilon(\Psi)$ satisfies

$$\limsup_{t \to +\infty} \text{dist}_{L^2}(\Psi(t), \mathscr{C}_1) \leq \varepsilon.$$

Before stating the main result of this section, let us check this claim on a simulation. We consider the exact same example as in Figure 1. As stated at the beginning of the section, the initial state $\Psi_0 = \frac{1}{\sqrt{2}}(\phi_1 + \phi_3)$ is particularly a hard state to stabilize around ϕ_1. In the simulations of Figure 2, we consider the function $\theta(r) = \eta r$ with $\eta = 7e + 2$. Furthermore, in the feedback design v_ε, we consider $c = 1e + 3$ and $\varepsilon = 5e - 2$. The numerical scheme is similar to the simulations of Figure 1. In order to calculate the implicit part of the feedback design $\sigma(\Psi)$, we apply a fixed point algorithm.
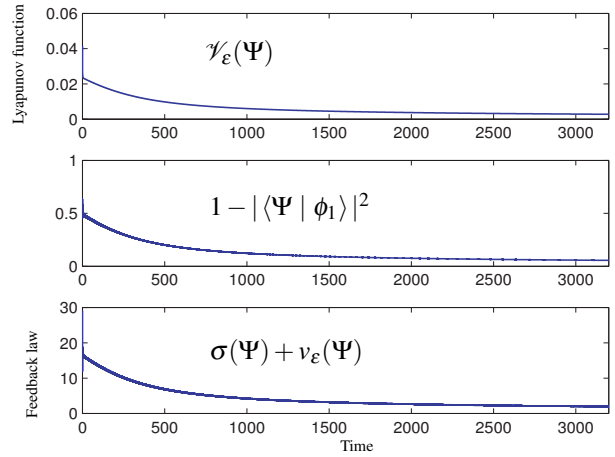

Fig. 2. The approximate stabilization of Ψ_1, where $\Psi_0 = \frac{1}{\sqrt{2}}(\phi_1 + \phi_3)$ and therefore the cut-off dimension is 3; as it can be seen, the closed-loop system reaches the .05-neighborhood of ϕ_1 in a time $T = 1000\pi$ corresponding to about 1300 periods of the longest natural period corresponding to the ground to the first excited state.

3.2 Main result

First, let us state the existence of the implicit function $\sigma(\Psi)$ that will be used in the feedback law. When X is a normed space, $a \in X$ and $r > 0$, we use the notation $B_X(a, r) := \{y \in X; \|y - a\|_X < r\}$.

Lemma 7. Let $N \in \mathbb{N}^*$, $\varepsilon > 0$, and $\theta \in C^\infty(\mathbb{R}_+, [0, \sigma^*])$ be such that

$$\theta(0) = 0, \; \theta(s) > 0, \forall s > 0 \; \|\theta'\|_{L^\infty} < c^*, \qquad (19)$$

for a sufficiently small c^*. There exists a unique $\sigma \in C^\infty(B_{L^2}(0,2), [0, \|\theta\|_{L^\infty}])$ such that

$$\sigma(\psi) = \theta(\mathscr{V}_\varepsilon(\psi)), \forall \psi \in B_{L^2}(0,2),$$

where $\mathscr{V}_\varepsilon$ is defined by (17).

The proof of Lemma 7 is completely similar to the one given in Beauchard et al. [2007] and is left to the author.

Here is the main result of this section:

Theorem 8. Let $N \in \mathbb{N}^*$, $\gamma \in (0,1)$, $\varepsilon > 0$, $\theta \in C^\infty(\mathbb{R}_+, [0, \sigma^*])$ verifying (19) and $\|\theta\|_{L^\infty}$ sufficiently small. Let $\sigma \in C^\infty(B_{L^2}(0,2), [0, \|\theta\|_{L^\infty}])$ be as in Lemma 7. For every $\Psi_0 \in \mathbb{S} \cap (H^2 \cap H_0^1)(I, \mathbb{C})$ with

$$\sum_{k=N+1}^\infty |\langle \Psi_0, \phi_k \rangle|^2 < \frac{\varepsilon \gamma^2}{32(1 - \varepsilon/2)} \quad \text{and} \quad |\langle \Psi_0, \phi_1 \rangle| \geq \gamma, \quad (20)$$

the Cauchy problem (1)-(2)-(3) with $u = \sigma(\Psi) + v_{\sigma(\Psi), N, \varepsilon}(\Psi)$ has a unique strong solution ψ, moreover this solution satisfies

$$\liminf_{t \to +\infty} |\langle \Psi(t), \phi_1 \rangle|^2 \geq 1 - \varepsilon. \quad (21)$$

Remark 9. Similarly to the Remark 2, the cut-off dimension N can be uniformly chosen as a function of the H_0^1-norm of Ψ_0. Moreover, similarly to the Remark 3, the second part of the assumption (20) is not restrictive in practice.

The proof of the Theorem 8 can be adapted following the same steps as in Beauchard et al. [2007] and the proof of the Theorem 1. It is therefore left to the reader. A complete version of this paper including a detailed proof of this theorem will be soon submitted as a journal paper.

4. CONCLUSION

In this paper, we considered the approximate stabilization of an infinite dimensional quantum system given by (Σ) and corresponding to a non-relativistic charged particle in an infinite potential well. Applying an implicit Lyapunov technique, we were able to stabilize any ε-neighborhood of an eigenstate ϕ_k of the system. Note that, this neighborhood is defined in the L^2-norm.

In a previous article Beauchard [2005], one of the authors proved the local controllability of the same system around the ground state in an H^7 sense. If, we were able to adapt the result of this paper to get an H^7-approximate stabilization of the ground state, together with the result of Beauchard [2005], we would be able to show the global controllability of the system in this functional space. Such a statement is quite a strong controllability result for this system. It, therefore, constitutes an interesting direction to be explored.

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