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A Eulerian approach to the analysis of rendez-vous algorithms

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Abstract: In this paper we analyze rendez-vous algorithms in the situation when agents can only exchange information below a given distance threshold R. We study the system under an Eulerian point of view considering (possibly continuous) probability distributions of agents and we present convergence results both in discrete and in continuous time. The limit distribution is always necessarily a convex combination of delta functions at least R far apart from each other: in other terms these algorithms are locally aggregating. Numerical simulations seem to show that starting from continuous distributions, in general these algorithms do not converge to a unique delta (rendez-vous) in agreement with previous literature on this subject.

Keywords: multi-agent systems; rendez-vous; distributed averaging algorithms

1. INTRODUCTION

Consider N agents moving in $\mathbb{R}^q$ according to the following rule: at every time t (discrete or continuous), each agent i, which is in position $x_i(t)$, computes the barycenter of the positions of those agents which are within (Euclidean) distance R from him and it moves towards it with a certain velocity. This is a well known rendez-vous algorithm: if one can prove that the corresponding geometric graph (connecting agents which are below distance R) does not change, it easily follows that all agents converge to the same position (rendez-vous). In general however the graph changes (and could disconnect). Consequently, the overall dynamical system is non-linear (possibly discontinuous) and hard to analyze. While more robust rendez-vous algorithms assuring that the communication graph never looses links, have been considered in **?**), the behavior of these more basic algorithms is largely unexplored. Literature on this argument is quite vast: recent contributions are **?**) and for applications in opinion dynamical models **?**). In **?**) a convergence result for the discrete time case and $q = 1$ (agents on a line) is presented: in the limit configuration, for any two agents i and j, $x_i(\infty)$ and $x_j(\infty)$ are either equal or their distance is larger than R. If we assume that the initial positions $x_i(0)$ all stay in a hypercube $K = [-L, L]^q$, the set of limit positions $\Omega = \{x_i(\infty)\}$ will be a subset of K: in particular $|\Omega| = O(2L/R)^q$ (independent on the number of agents). The exact cardinality of Ω of course depends on the initial positions. Simulations (in the case $q = 1$) presented in **?**) seem to indicate that for a large number of agents uniformly distributed on the interval, $|\Omega| \approx L/R$ (the limit positions are around $2R$ far apart), but there is no analytical proof of this result.

The main goal of our paper is to study the behavior of these algorithms for large number of agents in any dimension, in discrete and continuous time. In order to achieve this, we believe that a fundamental step is to study these dynamical models for continuous distributions of agents. Some considerations in this sense are already in **?**), our point of view is however different: we undertake in fact an "Eulerian" point of view, substituting labelled agents, with probability measures of agents. Our results (Theorems 1 and 3, Corollary 2) show that, for a special important case of rendez-vous algorithms preserving global average, the algorithms, both in discrete than in continuous time and in any dimension q, converge to a limit configuration which is always a convex combinations of deltas with reciprocal distance at least R. At our knowledge these results are new (for $q > 1$). At the moment we do not have (except in very special symmetric case Corollary 6) any theoretical tool to predict the final number of deltas. As in the cited literature, numerical simulations seem to show that only local aggregation takes place and that the limit configuration exhibit more than one delta even if we start from initial distributions with connected support. The exact number of these deltas seems to depend not only on R and on the initial distribution, but also on the specific averaging scheme used.

2. FORMULATION OF THE PROBLEM

2.1 The dynamical model

Assume we have a family V of N agents in the space $\mathbb{R}^q$ (typically with $q = 1, 2, 3$). The position of agent $j \in V$ at time t will be denoted by $x_j(t)$ and we will assume that they will all undergo the same dynamics governed by a linear input/output law which can be in either discrete or continuous time.

The discrete-in-time model In this setting we assume the evolution of all systems to take place over the lattice $0, \tau, 2\tau, \dots$ where $\tau > 0$ is a fixed time. For the sake of notational simplicity, we will write $x_j(t)$ for $x_j(\tau t)$ with $t \in \mathbb{N}$. In this case we assume the evolution law of every agent to be of the following type:

$$x_j(t+1) = x_j(t) + \tau u_j(t) \tag{1}$$

The vector $u_j(t) \in \mathbb{R}^q$ plays the role of a control velocity function that each agent can autonomously choose on the basis of the information available at time t: it will be a function of the position $x_j(t)$ as well of the information transmitted by its neighbors. More precisely, we fix a function $\xi : \mathbb{R}^q \to \mathbb{R}^+$ such that $\xi(0) > 0$ and we define the averages

$$m_j(t) = \frac{\sum_{k \in V} \xi(x_k(t) - x_j(t)) x_k(t)}{\sum_{k \in V} \xi(x_k(t) - x_j(t))} \,. \tag{2}$$

The control functions which we want to consider are of type

$$\tau u_j(t) = p_j[m_j(t) - x_j(t)]$$

where $p_j \in]0,1[$ are scalar (which can also be function of the position of the agents). We thus obtain

$$x_j(t+1) = p_j m_j(t) + (1 - p_j) x_j(t) \,. \tag{3}$$

The new position is a convex combination of the average of the other agents weighted according to their distance from $x_j(t)$ and $x_j(t)$ itself. The N coupled evolution equations (3) can be compactly written as follows. We consider $X(t) \in \mathbb{R}^{N \times q}$ where $X(t)_{js}$ is the s-th component of $x_j(t)$. For any $X \in \mathbb{R}^{N \times q}$ we define $A(X) \in \mathbb{R}^{N \times N}$ by

$$A(X)_{jk} = p_j(X) \frac{\xi(X_k - X_j)}{\sum_{h \in V} \xi(X_h - X_j)} + (1 - p_j(X)) \delta_{i,k} \tag{4}$$

where X_h denotes the h-th row of X and where $\delta_{i,k}$ is equal to 1 if $i = k$ and 0, otherwise. Then, the global dynamical system becomes

$$X(t+1) = A(X(t)) X(t) \tag{5}$$

It is immediate to check that matrices $A(X)$ are always stochastic.

We say that the $A(X)$ satisfies the *rendez-vous problem* if for every initial condition $X(0)$, there exists $\alpha \in \mathbb{R}^q$ such that

$$\lim_{t \to +\infty} X(t)_{js} = \alpha_s \quad \forall j \in V \quad \forall s = 1, \dots, q \,.$$

Moreover, we say that it satisfies the *barycentral rendez-vous problem* if $\alpha = N^{-1} \sum_j x_j(0)$.

The case typically treated in the literature is when $\xi = \mathbf{1}_{B(0,R)}$ the indicator function of the ball $B(0,R)$, for some $R > 0$, and $p_j(t) = 1$ for every j and t. In this paper we will instead focus on weight profiles ξ which are continuous: the basic case we have in mind is when ξ is a continuous radial decreasing function, supported inside $B(0,R)$. On one side, this continuity assumption seems to be necessary for our type of results. On the other hand, simulation seem to show that the behavior of this algorithm, when ξ is close to the indicator function, is quite close to the one with exactly $\xi = \mathbf{1}_{B(0,R)}$. Our main theoretical results are for the case when, for some fixed constant λ,

$$p_j(X) = \lambda \sum_{k \in V} \xi(x_k(t) - x_j(t)) \,,$$

$$A(X)_{jk} = \lambda \xi(x_k(t) - x_j(t)) + (1 - \lambda \sum_{k \in V} \xi(x_k(t) - x_j(t))) \delta_{jk} \tag{6}$$

The peculiarity of this case consists in the fact that $A(X)$ is symmetric. Simple consequence of this is that the global average of the system is preserved: $\sum_k x_k(t)$ is a constant.

The continuous-in-time model In this case we consider instead the law of agents to be

$$\dot{x}_j(t) = u_j(t) \tag{7}$$

where now $t \in \mathbb{R}^+$. We consider the same choice for the control signals $u_j(t)$. This leads, in the same compact notation, to

$$\dot{X}(t) = (A(X(t)) - I) X(t) \tag{8}$$

One difficulty we have to face with the continuous model is that if $A(X)$ is not continuous, the above closed loop equation cannot be interpreted in the classical sense. Notice that this happens when ξ is discontinuous, as for instance an indicator function.

3. A CONTINUOUS-IN-SPACE MODEL

When the number of agents N is very large, one can identify the set of agents with a mass distribution μ_t in $\mathbb{R}^q$, which varies in time [1] according to a suitable strategy based on some communication model, as in the previous section. In this identification, borrowing the terminology from Fluid Dynamics, we abandon the "Lagrangean" point of view used above (in which the independent variable j labels individual agents followed in their evolution) in favor of an "Eulerian" point of view, in which the independent variable x denotes a point in space occupied at each time by an infinitesimal mass of agents. Since agents are neither created nor destroyed, the total mass of μ_t is preserved, hence we may assume that μ_t is a probability measure in $\mathbb{R}^q$. As before, the time variable t can be discrete as well as continuous, whereas the model is continuous in space by construction. This means that the mass distribution μ_t can be in principle any Borel probability measure in $\mathbb{R}^q$: for instance, a large number of agents uniformly distributed in an interval $[a,b]$ can be well represented by normalized Lebesgue measure on $[a,b]$. Another example is the discrete-in-space model of the previous section, which reduces to a particular case of the continuous-in-space model (see the end of the next section).

The initial condition is therefore a probability measure μ_0 in $\mathbb{R}^q$ which is assigned, and represents the initial spatial configuration of agents. Then, we have to distinguish between the discrete-time and the continuous-time models.

3.1 Discrete time

If times t takes discrete values, we are led to consider a dynamical system of the kind

$$\mu_{t+1} = T(\mu_t) \mu_t, \quad t = 0, 1, 2, \dots \,, \tag{9}$$

where, for each time t, $T(\mu_t)$ is some linear operator which maps the space of probability measures in $\mathbb{R}^q$ to itself.

In order to derive a model which generalizes (and is consistent with) the discrete case, we have to further specialize the structure of the operator $T(\mu_t)$. Suppose that, at time t, we have a mass distribution μ_t, and let $x \in \mathbb{R}^q$ be a point in the support of μ_t (i.e. the smallest

[1] Throughout this section, we use the subscript t to denote dependence on time, and not differentiation with respect to t. Similarly, any object $O(t,x)$ depending on both the time variable t and the space variable x will be denoted by $O_t(x)$.

closed set which carries all the mass). We suppose that the agents at x (or, more precisely, the infinitesimal mass of agents $\mu_t(dx)$ at x), move from position x at time t to position $\tilde{x}$ at time $t+1$, according to some strategy which we will specify later: it is convenient to write the displacement $\tilde{x} - x$ as the product of a fixed time step $\tau > 0$ times a velocity vector $V_t(x)$, i.e.,

$$\tilde{x} - x = \tau V_t(x). \tag{10}$$

Introducing the time step τ, as already done in Sect. 2.1.1, allows us to consider arbitrary clock times τ, 2τ, 3τ etc. at which the agents actually move, whereas the variable t takes integer values only. With this notation, determining a strategy reduces to specifying how the velocity field $V_t(x)$ is computed, at time t, by the agent(s) at position x.

Before further specifying the choice of $V_t(x)$, we point out that (10) can be written as

$$\mu_{t+1} = \gamma_t \# \mu_t, \quad t = 0, 1, 2, \ldots, \tag{11}$$

where $\gamma_t : \operatorname{supp} \mu_t \subseteq \mathbb{R}^q \to \mathbb{R}^q$ is the mapping $\gamma_t(x) = x + \tau V_t(x)$ while the symbol $\gamma_t\#$ denotes the corresponding action on measures (called the *push forward* of a measure, see e.g. **?**)), formally defined by

$$\gamma_t \# \mu_t(E) = \mu_t(\gamma_t^{-1}(E)) \quad \text{for every Borel set } E.$$

Note that $\gamma_t \# \mu_t$ is unaffected by values of γ_t possibly taken outside the support of μ_t. With this definition, (11) uniquely defines the mass distribution μ_{t+1} by the identity

$$\int_{\mathbb{R}^q} f(x)\, d\mu_{t+1} = \int_{\mathbb{R}^q} f(x + \tau V_t(x))\, d\mu_t \tag{12}$$

for every (bounded and Borel) function f. Choosing f to be the characteristic function of a set E, for instance, makes this definition consistent with the intuitive idea that a point x in the support of μ_t moves (at time $t+1$) to the point $x + \tau V_t(x)$ in the support of μ_{t+1}.

The choice of the velocity field $V_t(x)$ in (11) is the counterpart of the choice of the velocity controls $u_j(t)$ in previous formulation.

In the spirit of the communication constraint considered above, we require that an agent at x should compute its velocity $V_t(x)$ by looking only at those other agents which are a distance less than some given communication radius $R > 0$ apart from x. Mathematically, this means that the vector $V_t(x)$ should depend only on the *restriction* of the measure μ_t to the ball of radius R centered at x. A possible choice for $V_t(x)$ is the following. As in Section 2, suppose we have fixed a positive number R (the communication radius), a non negative, continuous function $\xi(x) : \mathbb{R}^q \to \mathbb{R}$ which is supported and positive in the ball $B(0, R)$, and, for every probability μ, a continuous function $p(\mu)(x) : \mathbb{R}^q \to]0, 1[$ (also continuous with respect to μ). For any $x \in \operatorname{supp}(\mu_t)$, we put

$$V_t(x) = p(\mu_t)(x) \left[\frac{\displaystyle\int_{\mathbb{R}^q} y \xi(y - x)\, d\mu_t(y)}{\displaystyle\int_{\mathbb{R}^q} \xi(y - x)\, d\mu_t(y)} - x \right] \tag{13}$$

Note that the ratio of the two integrals is a sort of center of mass (weighted by ξ) of the neighbouring agents that the agent at x can see, within radius R. This is precisely the continuous analog of (2).

Note that, when $x \in \operatorname{supp}(\mu_t)$, the integral at the denominator is always positive, thus (13) does define a velocity vector at every point $x \in \operatorname{supp}(\mu_t)$. The case which we will be able to analyze theoretically is when

$$p(\mu)(x) = \int_{\mathbb{R}^q} \xi(y - x)\, d\mu(y) \tag{14}$$

In this case, $V_t(x)$ can be rewritten in the form

$$V_t(x) = \int_{\mathbb{R}^q} (y - x) \xi(y - x)\, d\mu_t(y), \quad \forall x \in \operatorname{supp}(\mu_t)\,. \tag{15}$$

Note that, in presence of the same mass distribution μ_t, a change in $p(\mu_t)(x)$ only changes the velocity field in (13) by scalar multiples: this however affects the resulting new mass distribution μ_{t+1}, and hence the resulting dynamics is a different one. However, numerical simulations (see Sect. **??**) show the same qualitative behaviour for the dynamics when $p(x)$ is given by our choice (14) and $p(x) = 1$. Our choice, however, leads to a system which is much simpler to handle mathematically, and lends itself to a much more detailed analysis for what concerns the asymptotic behaviour for large times.

This is the main result of this section, for a proof see **?**).

Theorem 1. Assume that ξ is as above with the additional assumption that $\xi(x) = \xi(-x)$. Consider the dynamical system (11) relative to velocity field (15) and any time step $\tau > 0$ such that $\tau ||\xi||_\infty < 1$, and let μ_0 be a probability measure on $\mathbb{R}^q$ with compact support. Then, the sequence of probability measures μ_t converges, as $t \to \infty$, to a limit probability measure μ_∞, which is a purely atomic measure, whose atoms are a distance at least R apart from one another.

In the previous theorem, convergence of measures is meant in the weak-star topology, i.e.

$$\lim_{t \to \infty} \int_{\mathbb{R}^q} \eta(x)\, d\mu_t(x) = \int_{\mathbb{R}^q} \eta(x)\, d\mu_\infty(x)$$

for every bounded and continuous test function η. Note that, no matter what the initial measure is, Dirac masses arise in the limit, hence the weak-star topology is natural in this context.

We now discuss the precise relation of these models with the discrete-in-space models considered above. Straightforward verification shows that if we consider sequences of atomic probability measures

$$\mu_t = \frac{1}{N} \sum_{j \in V} \delta_{x_j(t)}.$$

(with $|V| = N$) solving (11), then, the trajectories $x_j(t)$ solve (1) with $u_j(t) = V_t(x_j(t))$. Using the compact notation $X(t)$ introduced above, we have that $X(t)$ solves (5) with $A(X)$ as in (4). In particular, if V_t is given by (15), we obtain that $A(X)$ is equal to the choice (6). We have thus the following obvious consequence of previous result.

Corollary 2. Assume that ξ is as above with the additional assumption that $\xi(x) = \xi(-x)$. Consider the dynamical system (5) with $A(X)$ given by (6). For any initial condition $X(0)$, the sequence matrix $X(t)$ converges to a matrix $X(\infty)$ such that

$$X_h(\infty) \neq X_k(\infty) \;\Rightarrow\; ||X_h(\infty) - X_k(\infty)|| \geq R\,,$$

(where, we recall, X_h denotes the h-th row of X).

3.2 Continuous time

If the time variable t varies continuously, we are naturally lead to a partial differential equation which governs the evolution in time of the mass distributions μ_t. At every time $t \geq 0$, the velocity field is again given by (13), whereas the fact that the movement of the mass distribution is dictated by $V_t(x)$, is expressed by the continuity equation

$$\frac{\partial}{\partial t}\mu_t + \operatorname{div} V_t \mu_t = 0, \tag{16}$$

which being homogeneous also carries the information that mass is neither created nor destroyed. If every μ_t is a priori known to be smooth (i.e., an absolutely continuous measure with smooth density), then (16), coupled with (13), can be considered in principle as a nonlinear partial differential equation, and classical solutions can be sought. However, this is not our case: indeed, since the equation is nonlinear, even if the initial mass distribution μ_0 is smooth, it is not clear whether smoothness is preserved for all times. Moreover, we are interested in the large-time behaviour, hence in global solutions which shall possibly show local concentration phenomena, and we are also interested in nonsmooth initial measures (in fact, we seek a formulation which contains the discrete-in-space case as a particular case, when the initial mass distribution is purely atomic).

Therefore, (16) is to be meant in the sense of measures. Formally, if we multiply (16) by a smooth test function $\eta(x)$ with compact support, and we integrate in space (by part in the last term), we obtain

$$\frac{d}{dt}\int_{\mathbb{R}^q} \eta(x)\, d\mu_t(x) = \int_{\mathbb{R}^q} \nabla\eta(x)\cdot V_t(x)\, d\mu_t(x). \tag{17}$$

This suggests the following notion of solution. We say that a family of probability measures μ_t, $t \geq 0$, is a solution of (16), if for every test function $\eta(x)$, continuous with compact support in $\mathbb{R}^q$, the function

$$t \mapsto \int_{\mathbb{R}^q} \eta(x)\, d\mu_t(x), \quad t \geq 0,$$

is absolutely continuous in $[0,\infty)$, and satisfies (17) for almost every $t > 0$.

Note that this definition makes sense for every initial probability measure μ_0. Moreover, this notion of solution allows one to see the discrete-in-time case as an explicit Euler approximation to the continuous case. Indeed, let μ_n, $n = 0, 1, 2, \ldots$ be a solution, with time step $\tau > 0$, of the problem with discrete time, i.e. rewriting (12) with n in place of t,

$$\int_{\mathbb{R}^q} \eta(x)\, d\mu_{n+1} = \int_{\mathbb{R}^q} \eta(x + \tau V_n(x))\, d\mu_n \tag{18}$$

If we expand

$$\eta(x + \tau V_n(x)) = \eta(x) + \tau\nabla\eta(x)\cdot V_n(x) + O(\tau^2)$$

and we assume, as it is the case when $V_t(x)$ is given by (13), that $V_t(x)$ is uniformly bounded, plugging into (18) and neglecting the term $O(\tau^2)$ we obtain

$$\int_{\mathbb{R}^q} \eta(x)\, d\mu_{n+1} = \int_{\mathbb{R}^q} \eta(x)\, d\mu_n + \tau\int_{\mathbb{R}^q} \nabla\eta(x)\cdot V_n(x)\, d\mu_n,$$

an explicit time-discretization of (17).

For a generic velocity field given by (13) when the initial datum has compact support, we are able to prove existence and uniqueness of a solution, in the previous sense. However, with the special velocity field (15), we are also able to study the asymptotic behaviour of solutions for large times, as in the discrete-in-time model.

Theorem 3. Assume that ξ is as above with the additional assumption that $\xi(x) = \xi(-x)$. Let μ_0 be any probability measure on $\mathbb{R}^q$. Then there exists a unique solution μ_t of (16) with the velocity field given by (15). Moreover, as time t tends to infinity, the probability measures μ_t converge to a limit measure μ_∞, which is a purely atomic measure, whose atoms are a distance at least R apart from one another.

3.3 Symmetries

Both in the continuous-in-time and in the discrete-in-time models, we are not in general able to predict the number of deltas which will appear in the limit configuration starting from a given initial measure μ_0 (even when μ_0 is the uniform distribution). However some further information on μ_∞ can be obtained in the presence of symmetries.

Suppose for the moment we are back to the discrete-in-time model. We have the following result.

Proposition 4. Let $U : \mathbb{R}^q \to \mathbb{R}^q$ be a linear isometry. Let ξ be a radial function and let $p(\mu)(x)$ be such that for any probability measure μ, it holds $p(U\#\mu)(Ux) = p(\mu)(x)$ for all x. Consider now μ_t the solution of (11) with respect to the initial condition μ_0. Then $U\#\mu_t$ is the solution of (11) with respect to the initial condition $U\#\mu_0$.

Proof. Denote by $W_t(x)$ the velocity field $V_t(x)$ when we consider the measure $U\#\mu_t$. It is immediate to check that $UV_t(x) = W_t(Ux)$ for any $x \in \mathbb{R}^q$. Notice now that

$$\begin{aligned}
&\int_{\mathbb{R}^q} f(x)\, d(U\#\mu_{t+1})(x) = \int_{\mathbb{R}^q} f(Ux)\, d\mu_{t+1}(x)\\
&= \int_{\mathbb{R}^q} f(U(x + V_t(x)))\, d\mu_t(x) = \int_{\mathbb{R}^q} f(Ux + W_t(Ux))\, d\mu_t(x)\\
&= \int_{\mathbb{R}^q} f(x + W_t(x))\, d(U\#\mu_t(x)\,.
\end{aligned}$$

This proves the result. ■

This has some interesting consequences.

Corollary 5. Let $U : \mathbb{R}^q \to \mathbb{R}^q$ be a linear isometry. Let ξ be a radial function and let $V_t(x)$ be given by (15). Then,

$$U\#\mu_0 = \mu_0 \;\Rightarrow U\#\mu_\infty = \mu_\infty$$

A probability measure μ is said to have a radial symmetry with respect to $x_0 \in \mathbb{R}^q$ if for any rotation U centered in x_0 we have that $U\#\mu = \mu$. We can now present our last result.

Corollary 6. Let ξ be a radial function and let $V_t(x)$ be given by (15). If μ_0 has radial symmetry with respect to x_0, then $\mu_\infty = \delta_{x_0}$.

Proof. By Corollary 5, we have that μ_∞ also has radial symmetry with respect to x_0, and since it must be a finite combination of deltas, it has to coincide with δ_{x_0}. ■

Similar considerations also hold in the continuous-time case. Notice that in this context, a key point underlying the proof of Proposition 4 is the uniqueness result for the corresponding partial differential equation.

4. NUMERICAL SIMULATIONS

At last, we provide some insight about the behavior of the continuous-in-time model introduced in Sect. 3.2, through numerical simulations, mainly in 1 dimension. We assume here that the probability measures μ_t are absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$, i.e., there exists a density function $\rho(t,x) \geq 0$, which $\forall t$ is compactly supported in x and satisfies $\int_{\mathbb{R}} \rho(t,x)\,dx = 1$, such that $d\mu_t = \rho(t,x)\,dx$. The initial density $\rho(0,x) = \rho_0(x)$ is assigned. The continuity equation (16) becomes

$$\frac{\partial \rho}{\partial t} + \frac{\partial F}{\partial x} = 0\,, \tag{19}$$

where the flux is given by

$$F(t,x) = V_t(x)\rho(t,x) \tag{20}$$

and the velocity V is defined as in (13). Note that, unlike in classical conservation laws, our flux $F(t,x)$ depends on ρ in a non-local manner. We discretize (**??**) by a standard finite-volume approach (see, e.g., **?**)). Let $\Delta x > 0$ be a constant spatial discretization step. For $j \in \mathbb{Z}$, the nodes $x_{j+1/2} = (j+\frac{1}{2})\Delta x$ define a sequence of cells $\mathcal{V}_j = [x_{j-1/2}, x_{j+1/2}]$. Let $\Delta t_n > 0$ be a sequence of time steps, which define time instants $t_{n+1} = t_n + \Delta t_n$, $n \geq 0$, from $t_0 = 0$; set $\lambda_n = \Delta t_n/\Delta x$. The numerical method is obtained by integrating (**??**) over the volume $\mathcal{V}_j \times [t_n, t_{n+1}]$ and applying the divergence theorem, after introducing approximate cell-averages

$$\rho_{n,j} \simeq \frac{1}{\Delta x}\int_{x_{j-1/2}}^{x_{j+1/2}} \rho(t_n,x)\,dx$$

and approximate fluxes (termed numerical fluxes)

$$F_{n,j+1/2} \simeq \frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} F(t, x_{j+1/2})\,dt\,.$$

We obtain the scheme

$$\rho_{n+1,j} = \rho_{n,j} - \lambda_n(F_{n,j+1/2} - F_{n,j-1/2})\,, \qquad j \in \mathbb{Z}, \tag{21}$$

which updates the cell-averages at time t_{n+1} from their values at time t_n, provided we specify how the numerical fluxes depend on the cell averages $\rho_{n,k}$. In all subsequent simulations, we will adopt the popular choice of the *upwind* fluxes, in which information is taken from the side where particles arrive; precisely, assuming that an approximate velocity $V_{n,j+1/2}$ is defined at time t_n and at the interface point $x_{j+1/2}$, we set

$$F_{n,j+1/2} = \begin{cases} V_{n,j+1/2}\rho_{n,j} & \text{if } V_{n,j+1/2} \geq 0\,, \\ V_{n,j+1/2}\rho_{n,j+1} & \text{if } V_{n,j+1/2} < 0\,. \end{cases}$$

The definition of the approximate velocity $V_{n,j+1/2}$ is based on (13) with $p(\mu) = 1$ or on (15); in both cases, we use as weight ξ an even nonnegative function whose support is $[-R, R]$. Precisely, we set

$$V_{n,j+1/2} = \frac{\displaystyle\int_{-R}^{R} z\,\xi(z)\rho_n(z + x_{j+1/2})\,dz}{\displaystyle\int_{-R}^{R} \xi(z)\rho_n(z + x_{j+1/2})\,dz}\,, \tag{22}$$

or, in the second case,

$$V_{n,j+1/2} = \int_{-R}^{R} z\,\xi(z)\rho_n(z + x_{j+1/2})\,dz\,, \tag{23}$$

where ρ_n is the piecewise constant function which equals $\rho_{n,k}$ on the cell $\mathcal{V}_k$, $k \in \mathbb{Z}$.

Uniform (in Δt and n) L^1-stability is assured by the CFL (Courant-Friedrichs-Lewy) condition

$$C_n := \lambda_n \max_{j\in\mathbb{Z}} |V_{n,j+1/2}| \leq 1\,;$$

hereafter, we will invariably choose Δt_n so that $C_n = 1$.

The first set of simulations is relative to an initial density ρ_0 which is constant $= 1$ on the interval $[0.5, 1.5]$ and zero elsewhere. We report the results for two choices of the weight function ξ: i) a C^∞ function ξ_1 obtained by slightly smoothing the characteristic function of the interval $[-R, R]$ near $\pm R$ to ensure continuity; ii) the piecewise linear hat function $\xi_2(z) = 1 - |z|/R$. In both cases, the observed asymptotic dynamics is consistent with the theoretical prediction of Theorem 3, i.e., a finite number of deltas is created whose minimal distance is invariably larger than R. Figure **??**, obtained with $\xi = \xi_1$ and $\Delta x = 1/1600$, provides a sort of "bifurcation diagram", in which the position of the discrete deltas in the interval $[0.5, 1.5]$ (represented on the horizontal axis) is plotted against $\log(1/R)$ (represented on the vertical axis). This scenario appears to be quite robust with respect to numerical discretization errors; besides, the alternative choices (**??**) or (**??**) of velocity have little influence on the limit pattern, the only significant difference being in the speed of the evolution. Similar diagrams have been reported in the recent literature (see **?**) and the references therein), stemming from related but different models.

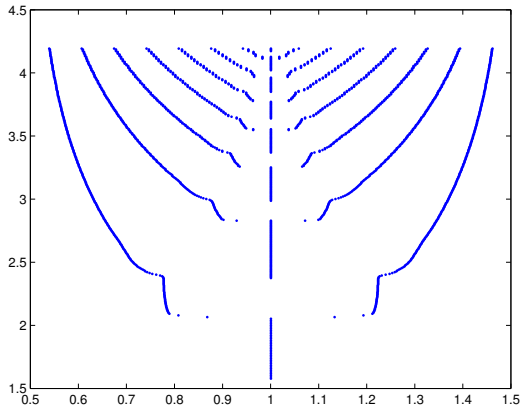

Fig. 1. Position of discrete deltas vs $\log(1/R)$ (with $\Delta x = 1/1600$) for the piecewise constant initial density

Next, we investigate the influence of the weight ξ on the asymptotic patterns. For both choices $\xi = \xi_1$ and $\xi = \xi_2$ and several values of R, we have monitored the number $\#\delta$ of deltas, as well as the inter-delta distance, expressed by the ratios $\sigma_{\min} = d_{\min}/R$ and $\sigma_{\max} = d_{\max}/R$, where $d_{\min}$ and $d_{\max}$ are the minimal and maximal distance between two consecutive deltas. Indeed, it is conjectured in the literature mentioned above that this ratio should be around 2. Our results, given in Table **??**, do suggest the existence of a limit intra-delta distance significantly larger that R; yet they also indicate a clear dependence on the particular weight assigned to the neighboring agents in the communication graph.

Finally, we consider two non-constant initial distributions, a linear and a parabolic one given by the left-hand plots

Table 1. Number of deltas and intra-delta distance as a function of R

	$\xi = \xi_1$			$\xi = \xi_2$		
R	$\#\delta$	$\sigma_{\min}$	$\sigma_{\max}$	$\#\delta$	$\sigma_{\min}$	$\sigma_{\max}$
0.10	4	2.367	2.450	5	1.800	2.175
0.08	5	2.250	2.635	7	1.646	1.802
0.06	7	2.250	2.472	9	1.792	1.930
0.04	11	2.167	2.250	14	1.583	1.812
0.02	22	2.208	2.333	28	1.583	1.792

of Fig. **??**. The resulting patterns of discrete deltas, obtained at convergence for $R = 0.04$ with $\xi = \xi_1$ and $\Delta x = 1/800$, are shown in the corresponding right-hand plots. The figures show a characteristic feature of the communication model under investigation, namely, the mass transportation occurs only locally (on a scale proportional to R), but not globally (on the scale of the support of the initial density). Indeed, the strengths of the limit deltas retain the linear or parabolic behavior of the initial data. The "bifurcation" diagram for such cases (see Fig.**??**)

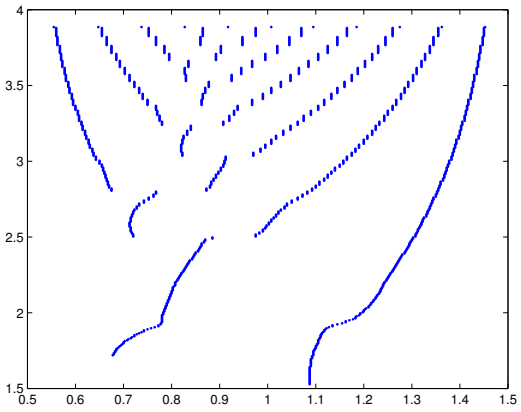

Fig. 2. Position of discrete deltas vs $\log(1/R)$ (with $\Delta x = 1/1600$) for the piecewise linear initial density

indicates that the loss of symmetry does not destroy the mechanism of one-by-one increment of the number of deltas as R decreases, observed above. The ratios $\sigma_{\min}$ and $\sigma_{\max}$ are 2.125 and 2.292 (linear case) and 2.066 and 2.312 (parabolic case).

At last, we show that the 1D scenario illustrated so far is somehow representative of the "structurally stable" behavior of our model in higher dimensions. Consider the two-dimensional situation of an initial constant density, whose support is a circle centered at the origin. According to Corollary 6, the limit measure is a single Dirac mass centered at the origin. However, the discretization errors generated by the use of a Cartesian grid of rectangular cells in the plane immediately destroy the rotational symmetry; no matter how refined is the grid, mass concentrates locally as in the 1D case, giving rise to a pattern of discrete deltas whose reciprocal distance is related to the size of R. Fig. **??** provides an example of such limit pattern, for $R = 0.08$ and $\Delta x = \Delta y = 1/32$.

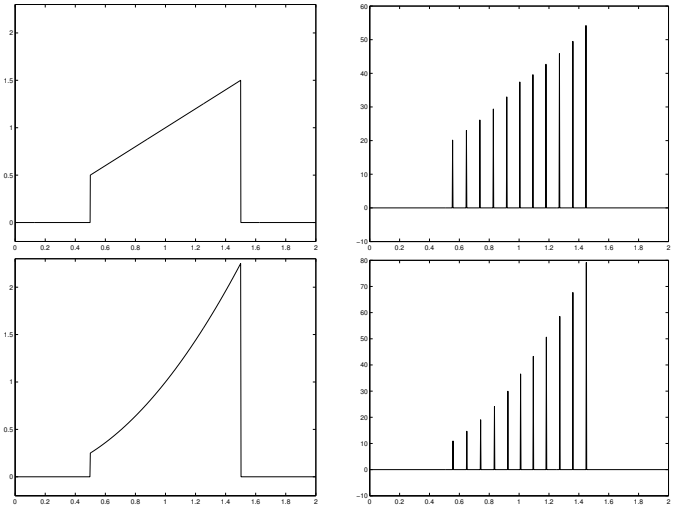

Fig. 3. Initial piecewise linear density (top left); discrete deltas at convergence (top right); initial piecewise parabolic density (bottom left); discrete deltas at convergence (bottom right)

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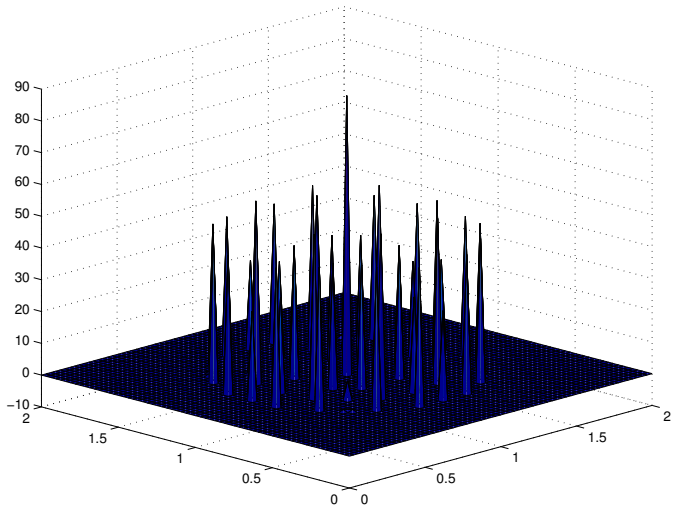

Fig. 4. Limit pattern of discrete deltas for $R = 0.08$, with $\Delta x = \Delta y = 1/32$, starting from an initial piecewise constant density supported in a circle around the origin.