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A Class of Nonlinear RLC Circuits Globally Stabilizable by Proportional plus Integral Controllers

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Abstract: In this note we identify graph-theoretic conditions which allow to write an RLC circuit as port-Hamiltonian with constant input matrices. We show that under additional monotonicity conditions of the network's components, the circuit enjoys the property of relative passivity, an extended notion of classical passivity. The property of relative passivity is then used to build simple, yet robust and globally stable, Proportional plus Integral controllers. *Copyright © 2008 IFAC.*

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1. INTRODUCTION

In this note, we look at the problem of global output regulation of nonlinear RLC networks using Proportional plus Integral (PI) controllers. Besides their simplicity and widespread popularity, it is well known that PI control is robust *vis–à–vis* parameter uncertainty—due to the integral action that is necessary to reject constant disturbances, even in the nonlinear context (Byrnes et al., 1997). The main contribution of this note is to show that for a large class of nonlinear RLC circuits with regulated voltage and current sources we can exploit the property of passivity to ensure that the problem in question can be solved with a simple PI controller around the sources port variables. This brief is a sequel of Jayawardhana et al. (2006), where we investigated the important property of *relative passivity* of general nonlinear systems and applied it for PI stabilization of a restricted class of RLC circuits.

We show that a very large class of RLC circuits enjoys the relative passivity property and can, therefore, be stabilized via PI control. Instrumental for this work is the use of port–Hamiltonian models to describe the RLC networks. This allows us to establish in a straightforward manner the property of relative passivity and identify some additional assumptions on the characteristic functions of the circuit elements that are sufficient to make the stability result global.

The search for energy (and power) based Hamiltonian models for physical systems with external ports and (possibly nonlinear) dissipation is interesting in its own right and has attracted considerable attention in the last decades. One way to obtain such models is to start from a Brayton–

Moser (Brayton and Moser, 1964a,b; Weiss et al., 1998) or a Lagrangian formulation and then perform the necessary transformations to arrive at a Hamiltonian description (Chua and McPherson, 1974; Blankenstein, 2005). A critical assumption in this procedure is that the characteristics of the network components are bijective. To avoid this limitation we prefer, in the spirit of network modeling, to proceed from the port–Hamiltonian lossless models of Maschke et al. (1995), see also Bernstein and Liberman (1989), and add the required sources and dissipation terminations.

2. PORT-HAMILTONIAN FORMULATION OF NONLINEAR RLC NETWORKS

The purpose of this section is to set the energy-based models that will be central in the subsequent analysis. A direct constructive method for obtaining Hamiltonian models for LC circuits has been proposed by Bernstein and Liberman (1989). Following the suggestion of Maschke et al. (1995), in this section we extend this method by adding ports to account for voltage and currents sources and resistive elements, as shown in Figure 1.[1]

We consider RLC networks satisfying the following assumptions.

Assumption 1. Capacitors are charge controlled and inductors are flux controlled with characteristics given by

$$v_{\mathrm{q}} = \hat{v}_{\mathrm{q}}(q) \quad \text{and} \quad i_{\phi} = \hat{i}_{\phi}(\phi) \,, \qquad (1)$$

where $q, v_{\mathrm{q}} \in \mathbb{R}^{n_{\mathrm{q}}}$ are the capacitors charges and voltages, and $\phi, i_{\phi} \in \mathbb{R}^{n_{\phi}}$ are the inductors fluxes and currents.

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[1] The inclusion of resistive ports was independently proposed by B. Maschke and published in Maschke (1998) without a proof.
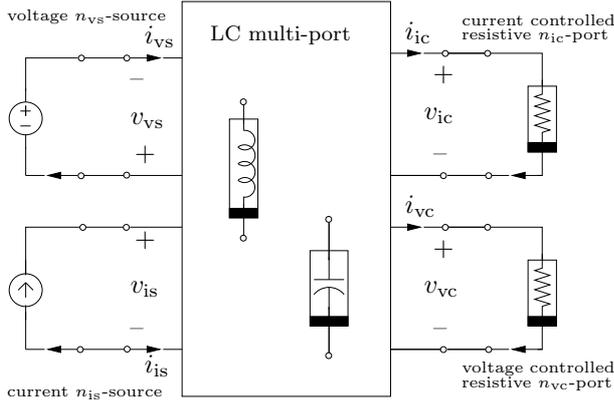
Fig. 1. An RLC network with sources and dissipation elements viewed as ports.

Furthermore, $\hat{v}_q(q)$ and $\hat{i}_\phi(\phi)$ have symmetric Jacobians—that is, they are gradients of scalar functions.

The first condition of this assumption allows us to write the electric and magnetic energies as functions of q and ϕ, i.e.,

$$\mathcal{E}_q(q) = \int_0^q \hat{v}_q^\top(\xi)d\xi + \mathcal{E}_q(0)$$

and

$$\mathcal{E}_\phi(\phi) = \int_0^\phi \hat{i}_\phi^\top(\xi)d\xi + \mathcal{E}_\phi(0) \, ,$$

while the second one guarantees that these functions do not depend on the integration path. This is of course automatically satisfied if the energy-storing elements are all single-port. To define the rest of the network's components we denote by

$$v_{\rm ic} = \hat{v}_{\rm ic}(i_{\rm ic}), \quad i_{\rm ic}, v_{\rm ic} \in \mathbb{R}^{n_{\rm ic}}$$

and

$$i_{\rm vc} = \hat{i}_{\rm vc}(v_{\rm vc}), \quad v_{\rm vc}, i_{\rm vc} \in \mathbb{R}^{n_{\rm vc}}$$

the current and voltage controlled resistors. Time-varying voltage and current sources are represented by $v_{\rm vs} \in \mathbb{R}^{n_{\rm vs}}$ and $i_{\rm is} \in \mathbb{R}^{n_{\rm is}}$ respectively.

Assumption 2. The graph $\mathcal{G}$ associated to the network has a tree $\mathcal{T}$ containing all capacitors, voltage sources and current controlled resistors. For future reference, denote by $\mathcal{L}$ the set of links corresponding to $\mathcal{T}$.

This assumption excludes loops formed exclusively by capacitors and/or voltage sources as well as cut sets formed exclusively by inductors and/or current sources. This in turn means that we can choose $\phi, q, v_{\rm vs}$ and $i_{\rm is}$ independently without violating Kirchhoff's laws.

It is possible to write Kirchhoff's voltage law in compact form as $Bv = 0$, where B is the fundamental loop matrix and v is the vector of branch voltages (Desoer and Kuh, 1969). Moreover, if v is partitioned as $v = \mathrm{col}(v_\mathcal{L}, v_\mathcal{T})$, where $v_\mathcal{L}$ and $v_\mathcal{T}$ are the branch voltages of $\mathcal{L}$ and $\mathcal{T}$ respectively, then B takes the form $B = [I \ F]$ (with I and F of appropriate dimensions). If we further partition $v_\mathcal{L}, v_\mathcal{T}$ and F as [2]

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[2] The negative sign of the voltage-sources indicates that voltage drops across these branches are opposite to current flow.

$$v_\mathcal{L} = \begin{bmatrix} v_\phi \\ v_{\rm is} \\ v_{\rm vc} \end{bmatrix}, \quad v_\mathcal{T} = \begin{bmatrix} \hat{v}_q(q) \\ -v_{\rm vs} \\ \hat{v}_{\rm ic}(i_{\rm ic}) \end{bmatrix}$$

and

$$F = \begin{bmatrix} F_{\phi-q} & F_{\phi-{\rm vs}} & F_{\phi-{\rm ic}} \\ F_{{\rm is}-q} & F_{{\rm is}-{\rm vs}} & F_{{\rm is}-{\rm ic}} \\ F_{{\rm vc}-q} & F_{{\rm vc}-{\rm vs}} & F_{{\rm vc}-{\rm ic}} \end{bmatrix} \, ,$$

then, because of Faraday's law ($v_\phi = \dot{\phi}$), we can write

$$\dot{\phi} = -F_{\phi-q}\hat{v}_q(q) + F_{\phi-{\rm vs}}v_{\rm vs} - F_{\phi-{\rm ic}}\hat{v}_{\rm ic}(i_{\rm ic}) \quad (2a)$$

$$v_{\rm is} = -F_{{\rm is}-q}\hat{v}_q(q) + F_{{\rm is}-{\rm vs}}v_{\rm vs} - F_{{\rm is}-{\rm ic}}\hat{v}_{\rm ic}(i_{\rm ic}) \quad (2b)$$

$$v_{\rm vc} = -F_{{\rm vc}-q}\hat{v}_q(q) + F_{{\rm vc}-{\rm vs}}v_{\rm vs} - F_{{\rm vc}-{\rm ic}}\hat{v}_{\rm ic}(i_{\rm ic}) \quad (2c)$$

Similarly, we can partition the currents as $i = \mathrm{col}(i_\mathcal{L}, i_\mathcal{T})$ with

$$i_\mathcal{L} = \begin{bmatrix} \hat{i}_\phi(\phi) \\ -i_{\rm is} \\ \hat{i}_{\rm vc}(v_{\rm vc}) \end{bmatrix}, \quad i_\mathcal{T} = \begin{bmatrix} i_q \\ i_{\rm vs} \\ i_{\rm ic} \end{bmatrix}$$

and write Kirchhoff's current law as $i = B^\top i_\mathcal{L}$. Simple matrix bookkeeping, together with the Charge Conservation principle ($i_q = \dot{q}$) shows that

$$\dot{q} = F_{\phi-q}^\top \hat{i}_\phi(\phi) - F_{{\rm is}-q}^\top i_{\rm is} + F_{{\rm vc}-q}^\top \hat{i}_{\rm vc}(v_{\rm vc}) \quad (3a)$$

$$i_{\rm vs} = F_{\phi-{\rm vs}}^\top \hat{i}_\phi(\phi) - F_{{\rm is}-{\rm vs}}^\top i_{\rm is} + F_{{\rm vc}-{\rm vs}}^\top \hat{i}_{\rm vc}(v_{\rm vc}) \quad (3b)$$

$$i_{\rm ic} = F_{\phi-{\rm ic}}^\top \hat{i}_\phi(\phi) - F_{{\rm is}-{\rm ic}}^\top i_{\rm is} + F_{{\rm vc}-{\rm ic}}^\top \hat{i}_{\rm vc}(v_{\rm vc}) \, . \quad (3c)$$

Setting $\mathcal{E} = \mathcal{E}_\phi + \mathcal{E}_q$ and noting that $\hat{v}_q(q) = \nabla_q\mathcal{E}$ and $\hat{i}_\phi(\phi) = \nabla_\phi\mathcal{E}$, we can write (2) and (3) as the port-Hamiltonian system (van der Schaft, 2000)

$$\dot{x} = J\nabla_x\mathcal{E} + g_1 u_1 + g_2 u_2(y_2) \quad (4a)$$

$$y_1 = g_1^\top \nabla_x\mathcal{E} + h_{11}u_1 + h_{12}u_2(y_2) \quad (4b)$$

$$y_2 = -g_2^\top \nabla_x\mathcal{E} + h_{12}^\top u_1 + h_{22}u_2(y_2) \, , \quad (4c)$$

with state $x = \mathrm{col}(\phi, q)$, inputs $u_1 = \mathrm{col}(v_{\rm vs}, i_{\rm is})$ and $u_2 = \mathrm{col}(\hat{v}_{\rm ic}(i_{\rm ic}), \hat{i}_{\rm vc}(v_{\rm vc}))$, outputs $y_1 = \mathrm{col}(i_{\rm vs}, i_{\rm is})$ and $y_2 = \mathrm{col}(i_{\rm ic}, v_{\rm vc})$. The system parameters are

$$J = \begin{bmatrix} 0 & -F_{\phi-q} \\ F_{\phi-q}^\top & 0 \end{bmatrix} \quad g_1 = \begin{bmatrix} F_{\phi-{\rm vs}} & 0 \\ 0 & -F_{{\rm is}-q}^\top \end{bmatrix}$$

$$g_2 = \begin{bmatrix} -F_{\phi-{\rm ic}} & 0 \\ 0 & F_{{\rm vc}-q}^\top \end{bmatrix} \quad h_{11} = \begin{bmatrix} 0 & -F_{{\rm is}-{\rm vs}}^\top \\ F_{{\rm is}-{\rm vs}} & 0 \end{bmatrix}$$

$$h_{12} = \begin{bmatrix} 0 & F_{{\rm vc}-{\rm vs}}^\top \\ -F_{{\rm is}-{\rm ic}} & 0 \end{bmatrix} \quad h_{22} = \begin{bmatrix} 0 & F_{{\rm vc}-{\rm ic}}^\top \\ -F_{{\rm vc}-{\rm ic}} & 0 \end{bmatrix} \, .$$

A feature of the network representation (4) is that the power-flow relationship is clearly revealed. Indeed, the time derivative of the energy is [3]

$$\dot{\mathcal{E}} = \nabla\mathcal{E}^\top(J\nabla\mathcal{E} + g_1 u_1 + g_2 u_2)$$
$$= \nabla\mathcal{E}^\top J\nabla\mathcal{E} + (y_1 - h_{11}u_1 - h_{12}u_2)^\top u_1 -$$
$$- (y_2 - h_{12}^\top u_1 - h_{22}u_2)^\top u_2 \, .$$

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[3] For ease of notation, whenever clear from the context, we drop the sub-index from the gradient operator, e.g., we write $\nabla\mathcal{E}$ instead of $\nabla_x\mathcal{E}$.
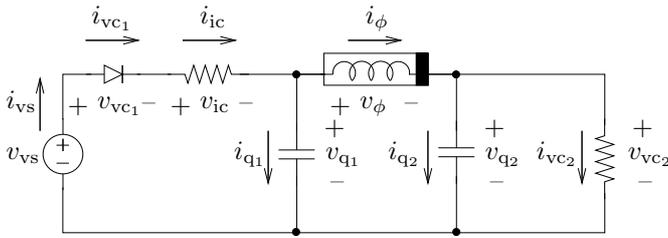
Fig. 2. An RLC circuit.

By noting that J, h_{11} and h_{22} are skew symmetric, we verify that

$$\dot{\mathcal{E}} = y_1^\top u_1 - y_2^\top u_2 , \qquad (5)$$

which shows that the rate at which the stored energy increases equals the difference between the power delivered by the sources and the power dissipated by the resistors.

Remark 1. Notice that for system (4) to be well defined, it is necessary that a unique y_2 —solution of (4c)— exists. In this brief we assume that such a y_2 exists. The interested reader is referred to Stern (1966); Roska (1981), where sufficient conditions for existence and uniqueness can be found.

Remark 2. If the characteristics of inductors and capacitors (1) are bijective it is possible to relax Assumption 2 by finding a reduced equivalent network containing no inductor cut sets or capacitor loops. A precise notion of equivalence, as well as the explicit procedure to carry out the transformation can be found in Sangiovanni-Vicentelli and Wang (1978), see also Cahill (1969). Note, however, that in the general nonlinear case, practical use of these procedures is impeded by the requirement of an explicit solution of (4). An alternative way to relax Assumption 2 is to enforce Kirchhoff's laws using the notion of port-Hamiltonian models with constraints (Jeltsema and Scherpen, 2003).

2.1 Example

As an example, consider the circuit shown in Figure 2. We model the diode as a nonlinear, voltage controlled resistor characterized by

$$i_{vc_1} = I_{s_1} \left(\exp \left(\frac{v_{vc_1}}{V_T} \right) - 1 \right) ,$$

where I_{s_1} is the saturation current and $V_T = 25\text{mV}$. The other dissipative elements are, a linear conductance governed by $i_{vc_2} = G v_{vc_2}$ and the linear resistor $v_{ic} = R i_{ic}$. The nonlinear inductor, described by

$$i_\phi = I_{s_\phi} \tanh \left(\frac{\phi}{\delta_\phi} \right) ,$$

saturates at a current I_{s_ϕ} and has, at the origin, an incremental inductance of δ_ϕ / I_{s_ϕ}. For simplicity, we consider linear capacitors of the form $v_{q_1} = q_1/C_1$ and $v_{q_2} = q_2/C_2$.

The directed graph corresponding to the circuit is given in Figure 3. The tree that satisfies Assumption 2 has been highlighted with thick lines.

The energy of the circuit is given by

$$\mathcal{E} = I_{s_\phi} \delta_\phi \ln \left(\left| \cosh \left(\frac{\phi}{\delta_\phi} \right) \right| \right) + \frac{q_1^2}{2C_1} + \frac{q_2^2}{2C_2} + \mathcal{E}(0) \quad (6)$$
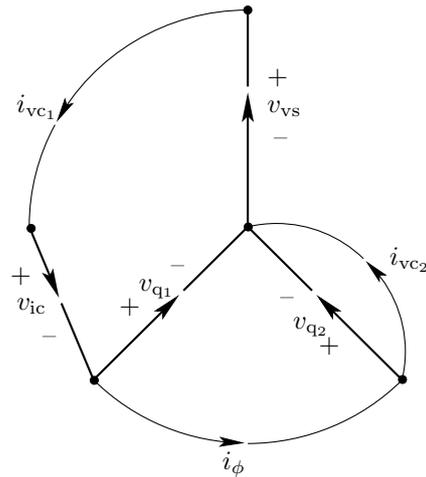

Fig. 3. Directed graph corresponding to the circuit of Figure 2. The thick straight lines represent the branches of $\mathcal{T}$. The thin arcs represent the links that belong to $\mathcal{L}$.

and its gradient by

$$\nabla \mathcal{E} = \begin{bmatrix} I_{s_\phi} \tanh(\phi/\delta_\phi) \\ q_1/C_1 \\ q_2/C_2 \end{bmatrix} .$$

Recall that $F = \{F_{\lambda-\tau}\}$, $\lambda \in \mathcal{L}$, $\tau \in \mathcal{T}$ is constructed according to the following rule:

$$F_{\lambda-\tau} = \begin{cases} 1 & \text{if } \tau \text{ is in the loop formed by } \lambda \text{ and} \\ & \text{their directions are equal.} \\ 0 & \text{if } \tau \text{ is not in the loop formed by } \lambda. \\ -1 & \text{if } \tau \text{ is in the loop formed by } \lambda \text{ but} \\ & \text{their directions are opposite.} \end{cases}$$

Since there are no current sources, the matrix F reduces to

$$F = \left[\begin{array}{c|c|c} F_{\phi-q} & F_{\phi-vs} & F_{\phi-ic} \\ \hline F_{vc-q} & F_{vc-vs} & F_{vc-ic} \end{array} \right] = \left[\begin{array}{cc|c|c} -1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & -1 & 0 & 0 \end{array} \right] .$$

Finally, equation (4) becomes

$$\begin{bmatrix} \dot{\phi} \\ \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \nabla \mathcal{E} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} u_2 \qquad (7a)$$

$$i_{vs} = [0 \ 1 \ 0] u_2 \qquad (7b)$$

$$\begin{bmatrix} i_{ic} \\ v_{vc_1} \\ v_{vc_2} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \nabla \mathcal{E} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} v_{vs} + \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} u_2 , (7c)$$

where

$$u_2 = \hat{u}_2(y_2) = \begin{bmatrix} R i_{ic} \\ I_{s_1} \left(\exp \left(v_{vc_1}/V_T \right) - 1 \right) \\ G v_{vc_2} \end{bmatrix} .$$

3. RELATIVE PASSIVITY AND GLOBAL OUTPUT REGULATION VIA PI CONTROL

In this section, we study the problem of global output regulation of RLC networks described by (4). We will exploit their port–Hamiltonian structure to show that, under some suitable additional conditions on their characteristic

functions, the problem can be solved with a simple PI controller.

It is well known (van der Schaft, 2000) that output regulation to *zero* of systems of the form

$$\dot{x} = f(x) + g(x)u, \quad y = h(x), \tag{8}$$

is relatively simple if the map $u \mapsto y$ is passive with a non–negative storage function. That is, if there exists a scalar function $V(x) \geq 0$ such that $\dot{V} \leq u^\top y$. In this case, a simple proportional controller, $u = -K_P y$ with $K_P = K_P^\top > 0$, ensures that (along all bounded trajectories) $y(t) \to 0$. Regulation will be global if the storage function is radially unbounded, which ensures boundedness of all trajectories.

It is widely known that RLC networks consisting of passive inductors, capacitors and resistors are passive (Desoer and Kuh, 1969). This can be readily seen for circuits described by (4), using (5), $y_2^\top u_2 \geq 0$, and the fact that $\mathcal{E}$ is non–negative for passive inductors and capacitors. In most practical applications of RLC circuits the control objective is *not* to drive the output to zero but to a desired value $y_1^\star \neq 0$. In this case it is natural to look for passivity *relative* to $y_1^\star$ and it's corresponding input (Desoer and Vidyasagar, 1975). More precisely, instead of looking for passivity of the map $u_1 \mapsto y_1$, we look for passivity of the map $\tilde{u}_1 \mapsto \tilde{y}_1$, where $\tilde{u}_1 \triangleq u_1 - u_1^\star$, $\tilde{y}_1 \triangleq y_1 - y_1^\star$ and $u_1^\star, x^\star$ satisfy

$$0 = J\nabla\mathcal{E}^\star + g_1 u_1^\star + g_2 u_2(y_2^\star) \tag{9a}$$

$$y_1^\star = g_1 \nabla\mathcal{E}^\star + h_{11} u_1^\star + h_{12} u_2(y_2^\star) \tag{9b}$$

$$y_2^\star = -g_2 \nabla\mathcal{E}^\star + h_{12}^\top u_1^\star + h_{22} u_2(y_2^\star) \tag{9c}$$

with $\nabla\mathcal{E}^\star \triangleq \nabla\mathcal{E}\big|_{x=x^\star}$ for some $y_2^\star$.

Passive linear systems are passive relative to any equilibrium input–output pair $(u_1^\star, y_1^\star)$, a property that is simply revealed by shifting the origin in the state space model (8) and noting that the storage function for the map $u \mapsto y$ will also qualify as storage function for $\tilde{u} \mapsto \tilde{y}$. Although this is in general not true for nonlinear systems (Jayawardhana et al., 2006), it follows from Assumption 2 that the voltage of each element of $v_\mathcal{L}$ (in particular those of $\dot{\phi}$) is a *linear* function of the voltages of $v_\mathcal{T}$ (cf. (2a)). Likewise, the current of each element of $i_\mathcal{T}$ (in particular those of $\dot{q}$) is a linear function of the currents of $i_\mathcal{L}$ (cf. (3a)). This known fact (Chua and Green, 1976) is nicely captured in (4a) and suggests that for this class of RLC networks, relative passivity holds. To prove it we need one last assumption.

Assumption 3. The characteristics of the inductors and capacitors are strictly increasing and continuously differentiable. Those of the resistors are monotone non-decreasing. [4]

Lemma 1. Under Assumption 3 the map $\tilde{u}_1 \mapsto \tilde{y}_1$ is passive with the positive definite storage function

$$H(x) \triangleq \mathcal{E}(x) - (x - x^\star)^\top \nabla\mathcal{E}^\star - \mathcal{E}(x^\star),$$

which is, furthermore, strictly convex.

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[4] Recall that $f : \mathbb{R}^n \to \mathbb{R}^n$ is monotone non-decreasing if $(a - b)^\top(f(a) - f(b)) \geq 0, \forall a, b \in \mathbb{R}^n$. It is strictly increasing if the inequality is strict whenever $a \neq b$.

Proof. Since $\nabla\mathcal{E} = \text{col}(\hat{i}_\phi(\phi), \hat{v}_q(q))$ and the inductors and capacitors characteristics are strictly increasing we get that $\mathcal{E}$ is strictly convex (Hiriart-Urruty and Lemaréchal, 1993, p.185). It is immediately verified that

$$H(a\lambda + (1 - \lambda)b) - [\lambda H(a) + (1 - \lambda)H(b)] =$$
$$\mathcal{E}(a\lambda + (1 - \lambda)b) - [\lambda\mathcal{E}(a) + (1 - \lambda)\mathcal{E}(b)] ,$$

which proves that strict convexity of $\mathcal{E}$ is equivalent to strict convexity of H. Moreover, since $\nabla H(x^\star) = 0$, $H(x^\star) = 0$ is a unique global minimum and therefore H is positive definite (with respect to $x^\star$).

The network (4) can be written in terms of $\tilde{u}$ and $\tilde{y}$ as follows

$$\dot{x} = J\nabla\mathcal{E} + g_1 u_1^\star + g_1\tilde{u}_1 + g_2 u_2^\star + g_2\tilde{u}_2$$
$$= J(\nabla\mathcal{E} - \nabla\mathcal{E}^\star) + g_1\tilde{u}_1 + g_2\tilde{u}_2$$
$$\tilde{y}_1 = g_1^\top(\nabla\mathcal{E} - \nabla\mathcal{E}^\star) + h_{11}\tilde{u}_1 + h_{12}\tilde{u}_2$$
$$\tilde{y}_2 = -g_2^\top(\nabla\mathcal{E} - \nabla\mathcal{E}^\star) + h_{12}^\top\tilde{u}_1 + h_{22}\tilde{u}_2 ,$$

where the second line is due to (9a). The derivative of H is then obtained as before:

$$\dot{H} = (\nabla\mathcal{E} - \nabla\mathcal{E}^\star)^\top(J(\nabla\mathcal{E} - \nabla\mathcal{E}^\star) + g_1\tilde{u}_1 + g_2\tilde{u}_2)$$
$$= \tilde{y}_1^\top\tilde{u}_1 - \tilde{y}_2^\top\tilde{u}_2 . \tag{10}$$

Finally, the monotonicity of the resistors leads to $\dot{H} \leq \tilde{y}_1^\top\tilde{u}_1$.

Remark 3. Strict convexity of $\mathcal{E}$ might seem too strong if all that we need is an H bounded from below. The reason for imposing strict convexity is that, together with the existence of a minimum, it implies radial unboundedness (Jayawardhana et al., 2006). This will prove useful for global stabilization later on.

As indicated above, the output of a passive system can be regulated to zero with a proportional feedback. In the present context this evokes a control law of the form $\tilde{u}_1 = -K_P\tilde{y}_1$, whose implementation $u_1 = -K_P\tilde{y} + u_1^\star$ clearly requires the exact value of the feed-through term $u_1^\star$. The latter —obtained from the solution of (9)— needs the precise knowledge of the system's parameters rendering the controller highly non–robust. This problem can be circumvented by the use of an integral action, as shown in the following theorem.

Theorem 2. Consider network (4). Under Assumption 3 the PI controller

$$\dot{\xi} = -\tilde{y}_1 \tag{11a}$$

$$u_1 = K_I\xi - K_P\tilde{y}_1 , \tag{11b}$$

with $K_I = K_I^\top > 0$ and $K_P = K_P^\top > 0$, ensures that for all initial conditions, $y_1(t)$ converges asymptotically to $y_1^\star$, that is,

$$\lim_{t\to\infty} y_1(t) = y_1^\star .$$

Moreover, it globally stabilizes the equilibrium point $(x^\star, \xi^\star \triangleq K_I^{-1}u^\star)$. [5] If in addition, the closed-loop system (4), (11) satisfies the detectability condition

$$\tilde{y}_1(t) \equiv 0 \implies \lim_{t\to\infty}(x(t), \xi(t)) = (x^\star, \xi^\star), \tag{12}$$

then the equilibrium is globally asymptotically stable.

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[5] By global stability we mean stability in the sense of Lyapunov plus boundedness of the solutions for every initial condition.

Proof. It is straightforward to see that the (shifted) candidate Lyapunov function

$$V(x, \xi) = H(x) + \frac{1}{2}(\xi - \xi^\star)^\top K_I(\xi - \xi^\star)$$

is indeed a Lyapunov function:

$$
\begin{aligned}
\dot{V} &= \dot{H} + \dot{\xi}^\top K_I(\xi - \xi^\star) \\
&= \tilde{y}_1^\top \tilde{u}_1 - \tilde{y}_2^\top \tilde{u}_2 - \tilde{y}_1^\top K_I(\xi - \xi^\star) \\
&= \tilde{y}_1^\top \tilde{u}_1 - \tilde{y}_2^\top \tilde{u}_2 - \tilde{y}_1^\top (\tilde{u}_1 + K_P \tilde{y}_1) \\
&= -\tilde{y}_2^\top \tilde{u}_2 - \tilde{y}_1^\top K_I \tilde{y}_1 \leq 0 .
\end{aligned}
$$

The second equation is due to (10) and (11a), while the third is due to (11b) and the definition of $\xi^\star$. Non positivity is due to the monotonicity of the resistors and the positive definiteness of K_I. It follows then from standard Lyapunov theory that $(x^\star, \xi^\star)$ is stable. Since V is also radially unbounded (see Remark 3) the solutions remain bounded for any initial condition (Khalil, 1996, p. 124). From LaSalle's invariance principle (Salle and Lefschetz, 1961, p. 66) we conclude that $y_1(t)$ converges to $y_1^\star$ and that the detectability condition (12) leads to asymptotic stability.

3.1 Example (cont.)

The derivatives of the inductor's and capacitors' characteristics are, respectively,

$$\frac{I_{s_\phi}}{\delta_\phi} \operatorname{sech}^2\left(\frac{\phi}{\delta_\phi}\right), \quad \frac{1}{C_1} \quad \text{and} \quad \frac{1}{C_2} .$$

Those of the resistive elements are

$$R, \quad \frac{I_{s_1}}{V_T} \exp\left(\frac{v_{vc_1}}{V_T}\right) \quad \text{and} \quad G .$$

Assumption 3 is verified, since strict positivity of the derivatives implies strict monotonicity.

In order to obtain the equilibrium input-output pairs, it is necessary to solve (9):

$$y_2^\star = \begin{bmatrix} i_{ic}^\star \\ v_{vc_1}^\star \\ v_{vc_2}^\star \end{bmatrix} = \begin{bmatrix} y_1^\star \\ V_T \ln(y_1^\star/I_{s_1} + 1) \\ y_1^\star/G \end{bmatrix}$$

$$x^\star = \begin{bmatrix} \phi^\star \\ q_1^\star \\ q_2^\star \end{bmatrix} = \begin{bmatrix} \delta_\phi \operatorname{artanh}(y_1^\star/I_{s_\phi}) \\ y_1^\star C_1/G \\ y_2^\star C_2/G \end{bmatrix}$$

$$u_1^\star = v_{vs} = v_{vc_1}^\star + y_1^\star\left(\frac{1}{G} + R\right)$$

Suppose that the system parameters are

$$I_{s_\phi} = 30\,\text{mA}, \quad \delta_\phi = 250\,\mu\text{Wb}, \quad C_1 = C_2 = 2\,\mu\text{F}$$

and

$$R = 100\,\Omega, \quad I_{s_1} = 0.1\,\mu\text{A}, \quad G = 1\,\text{mA/V} .$$

Suppose further that we want to stabilize the diode's current at $y_1^\star = 10\text{mA}$. For an initial condition

$$x(0) = \begin{bmatrix} 1\,\text{mWb} \\ 1\,\mu\text{C} \\ 0.5\,\mu\text{C} \end{bmatrix},$$

the PI controller (11), with $K_I = 500\,\text{V/mA} \cdot \text{s}$ and $K_P = 100\,\Omega$, produces the set of currents and voltages shown in Figure 4. It can be seen that both, y_1 and u_1 converge to their preset values of $10\,\text{mA}$ and $11.3\,\text{V}$ respectively.
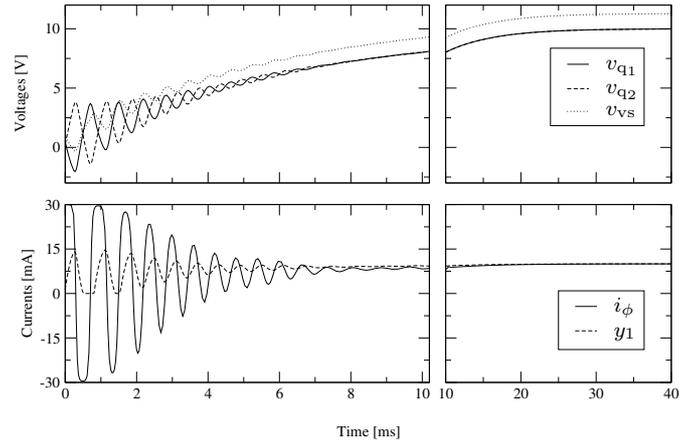

Fig. 4. Simulated response of the regulated circuit.

4. CONCLUSIONS

We have identified in this paper a large class of nonlinear RLC circuits that can be (globally) stabilized with simple PI controllers. Instrumental for our proof was the establishment of the property of relative passivity, which is satisfied by RLC circuits with monotonic characteristic functions. It has been shown that the incorporation of the integral action robustifies the controller, obviating the need for the exact knowledge of the circuit parameters. This well-known property of PI controllers —that underlies its huge popularity in applications— is particularly critical here, where the implementation of the control law without the integral action requires the computation of the constant input associated to desired equilibria. Current research is under way to explore the use of relative passivity to induce oscillations on the circuit, a behavior that is desired in many practical problems.

REFERENCES

G. M. Bernstein and M. A. Liberman. A method for obtaining a canonical Hamiltonian for nonlinear LC circuits. *IEEE Trans. Circuits Syst.*, 36:411 – 420, March 1989.

G. Blankenstein. Geometric modeling of nonlinear RLC circuits. *IEEE Trans. Circuits Syst. I*, 52:396 – 404, February 2005.

R. K. Brayton and J. K. Moser. A theory of nonlinear networks, I. *Quart. Appl. Math.*, 12:1 – 33, April 1964a.

R. K. Brayton and J. K. Moser. A theory of nonlinear networks, II. *Quart. Appl. Math.*, 12:81 – 104, July 1964b.

C. I. Byrnes, F. Delli Priscoli, and A. Isidori. *Output Regulation of Nonlinear Systems.* Birkhäuser, Boston, 1997.

L. W. Cahill. On the selection of state variables for nonlinear RLC networks. *IEEE Trans. Circuit Theory*, 16:553 – 555, November 1969.

L. O. Chua and D. N. Green. Graph-theoretic properties of dynamic nonlinear networks. *IEEE Trans. Circuits Syst.*, 23:292 – 312, May 1976.

L. O. Chua and J. D. McPherson. Explicit topological formulation of Lagrangian and Hamiltonian equations for nonlinear networks. *IEEE Trans. Circuits Syst.*, 21: 277 – 286, March 1974.

C. A. Desoer and E. S. Kuh. *Basic Circuit Theory.* McGraw-Hill Kogausha, Ltd., Tokyo, 1969.

C. A. Desoer and M. Vidyasagar. *Feedback Systems: Input-Output Properties.* Academic Press, Inc., New York, 1975.

J. Hiriart-Urruty and C. Lemaréchal. *Convex Analysis and Minimization Algorithms I.* Springer-Verlag, New York, 1993.

B. Jayawardhana, R. Ortega, E. García-Canseco, and F. Castaños. Passivity of nonlinear incremental systems: Application to PI stabilization of nonlinear RLC circuits. In *Proc. Conference on Decision and Control*, page ThIP2.17, San Diego, December 2006.

D. Jeltsema and J. Scherpen. A dual relation between port-hamiltonian systems and the Brayton-Moser equations for nonlinear switched RLC circuits. *Automatica*, 39:969 – 979, 2003.

H. K. Khalil. *Nonlinear Systems.* Prentice-Hall, Upper Saddle River, New Jersey, 1996.

B. Maschke. Interconnexion et structure des systèmes hamiltoniens commandés : Une approche réseau. Technical report, Université Paris-Sud XI, Orsay, France, August 1998. Mémoire présenté pour obtenir l'Habilitation á diriger les recherches.

B. Maschke, A. J. van der Schaft, and P. C. Breedveld. An intrinsic Hamiltonian formulation of the dynamics of LC-circuits. *IEEE Trans. Circuits Syst. I*, 42:73 – 82, February 1995.

T. Roska. The limits of modeling of nonlinear circuits. *IEEE Trans. Circuits Syst.*, 28:212 – 216, March 1981.

J. La Salle and S. Lefschetz. *Stability by Liapunov's Direct Method With Applications.* Academic Press, Inc., New York, 1961.

A. L. Sangiovanni-Vicentelli and Y. T. Wang. On equivalent dynamic networks: Elimination of capacitor loops and inductor cutsets. *IEEE Trans. Circuits Syst.*, 25: 174 – 177, March 1978.

T. E. Stern. On the equations of nonlinear networks. *IEEE Trans. Circuit Theory*, 13:74 – 81, March 1966.

A. J. van der Schaft. $\mathcal{L}_2$-*Gain and Passivity Techniques in Nonlinear Control.* Springer-Verlag, London, 2000.

L. Weiss, W. Mathis, and L. Trajković. A generalization of Brayton-Moser's mixed potential function. *IEEE Trans. Circuits Syst. I*, 45:423 – 427, April 1998.