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On Dynamical Optimization of Conflict Hereditary Systems *

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Abstract: For dynamical systems with aftereffect, the problem of control under disturbances is considered within the game-theoretic approach of N.N. Krasovskii and A.I. Subbotin. The problem is posed in the class of strategies with memory (functions of the motion history). The value of optimal guaranteed result (OGR) depends here on an initial history. An appropriate functional equation of the Hamilton–Jacobi–Bellman–Isaacs (HJBI) type with co-invariant (*ci*-) derivatives is presented. It is shown that if the functional of OGR is *ci*-smooth then it is the classical solution of this equation, and the optimal strategy can be constructed by aiming in the direction of its *ci*-gradient. In the nonsmooth case, a generalization of the presented HJBI equation is obtained by using an appropriate directional derivatives. Here, for constructing the optimal control strategy, the method of aiming in the direction of *ci*-gradients of auxiliary Lyapunov type functionals is elaborated.

Keywords: Control theory; Differential games; Dynamic programming; Delay.

1. INTRODUCTION

In optimal control and differential games, the Hamilton–Jacobi (HJ) type nonlinear first order PDE is well known as the Bellman–Isaacs (BI) equation (see, e.g., Bellman [1957], Isaacs [1965], Krasovskii and Subbotin [1988]). It is a reflection of the optimality principle of dynamic programming in problems of feedback control of ordinary differential systems, where this principle is fulfilled in a finite-dimensional space of current states. If the value function (the function of optimal or, for problems with disturbances, optimal guaranteed result) is smooth (continuously differentiable) then it satisfies the BI equation in the classical sense. In this case, the optimal feedback strategy can be constructed by aiming in the direction of the gradient of the value function. However, generally the value function is not differentiable, and the corresponding BI equation does not have appropriate classical solutions. In such a situation, one can use generalized (minimax, viscosity) solutions (see, e.g., Crandall *et al.* [1987], Subbotin [1995]) and apply some appropriate nonsmooth constructions of extremal aiming. One can mention here such constructions as the aiming onto stable bridges (Krasovskii and Subbotin [1988]), the aiming onto accompany points (Krasovskii [1985]), the aiming in the direction of quasigradients of the value function (Garnysheva and Subbotin [1994]).

The present paper develops the similar technique for problems of control of conflict hereditary systems (Krasovskii and Lukoyanov [1996], Lukoyanov [2004], Osipov [1971]). The term “conflict hereditary systems” is used here to stress the following:

- the problem of control under disturbances is considered,
- the system under control is described by differential equations with aftereffect,
- the cost functional being optimized estimates both the terminal and past values of the system state,
- the control strategies with memory are admissible as functions of the motion history.

Unlike the ordinary case, in such a problem the optimality principle is not fulfilled if the control process is considered in a finite-dimensional space of current states. But the optimality principle is fulfilled in an appropriate infinite-dimensional space of the motion histories. This functional problem statement is closely related to the approach proposed by Krasovskii [1959] for problems of stability of time-delay systems. Thus, unlike the ordinary case, here strategies with memory give the opportunity to obtain an essentially better result. The paper deals with the case when both the concentrated and distributed delays act on the system (see assumption (5) below). The statement of the problem is given within the game-theoretic approach of Krasovskii and Subbotin [1988]. The optimal guaranteed result of the control is defined as a functional (the value functional) of an initial motion history.

For optimal control problems of systems with aftereffect, HJB equations and their generalizations are considered (see, e.g., Soner [1988], Wolenski [1994]) as classical partial or directional differential equations in appropriate Banach spaces. The present paper follows another approach relating to Krasovskii's method of Lyapunov functionals. An appropriate functional equation of the HJBI type is

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derived by using Kim's i-smooth calculus. If the value functional is ci-smooth then it satisfies this equation, and the optimal control strategy with memory is constructed by aiming in the direction of the ci-gradient of the value functional. In the general case, when the value functional is just only continuous, it is treated as the generalized Subbotin's minimax solution to the presented HJBI equation (see Krasovskii and Lukoyanov [2000], Lukoyanov [2003] for the related technique). Similarly as for systems described by ordinary differential equations (see Subbotin [1984]), the solution in this functional case can be also defined with the help of inequalities for derivatives in finite-dimensional directions. In order to construct the optimal strategy in the nonsmooth case, the method of aiming in the direction of ci-gradients of the Lyapunov functionals is elaborated. It is similar to the construction proposed in Garnysheva and Subbotin [1994] for control problems of systems described by ordinary differential equations.

2. AUXILIARY NOTATIONS

Let $t_*, T \in \mathbb{R}$ be such that $t_* < T$ and a function $x(\cdot) : [t_*, T] \mapsto \mathbb{R}^n$ be fixed. Symbols $x(t)$ and $x_t(\cdot)$ denote respectively the value of the function $x(\cdot)$ at a point $t \in [t_*, T]$ and the contraction of this function to $[t_*, t]$ (i.e., $x_t(\cdot) = \{x(\tau), t_* \leq \tau \leq t\}$). In particular, $x_T(\cdot) = x(\cdot)$, $x_t(\tau) = x(\tau)$ for $\tau \in [t_*, t]$. The space of continuous functions $x(\cdot) : [t_*, T] \mapsto \mathbb{R}^n$ is denoted by C. The notation C_t is used for the space of continuous functions $x_t(\cdot) : [t_*, t] \mapsto \mathbb{R}^n$. So, C_t is treated as a contraction of C to $[t_*, t]$. Denote by G the set of pairs $g = \{t, x_t(\cdot)\}$, where $t \in [t_*, T]$ and $x_t(\cdot) \in C_t$. Let

$$g_1 = \{t_1, x^{(1)}_{t_1}(\cdot)\} \in G, \ g_2 = \{t_2, x^{(2)}_{t_2}(\cdot)\} \in G.$$

A metric on G is defined as follows:

$$\rho(g_1, g_2) := \max_{i=0,1} \max_{t_* \leq \xi \leq t_{i+1}} \min_{t_* \leq \eta \leq t_{2-i}} (\dots)$$

$$(\dots) = \sqrt{(\xi - \eta)^2 + \|x^{(i+1)}(\xi) - x^{(2-i)}(\eta)\|^2}.$$

Here, $\|\cdot\|$ stands for the Euclidean norm of a vector.

3. STATEMENT OF THE PROBLEM

Consider a control process described by the dynamic equation with aftereffect:

$$\dot{x}(t) = f\big(t, x_t(\cdot), u(t), v(t)\big),$$
$$t_* \leq t^0 \leq t \leq T, \ x(t) \in \mathbb{R}^n, \ u(t) \in P, \ v(t) \in Q, \tag{1}$$

the initial condition:

$$x_{t^0}(\cdot) = x^0_{t^0}(\cdot) \in C_{t^0}, \tag{2}$$

and the cost functional:

$$\gamma\big(x(\cdot), u(\cdot), v(\cdot)\big) = \sigma\big(x(\cdot)\big) - \int_{t^0}^{T} h\big(t, x_t(\cdot), u(t), v(t)\big) \mathrm{d}t. \tag{3}$$

Here t is the time variable, $x(t)$ is the value of the state vector at the time t, $\dot{x}(t) := \mathrm{d}x(t)/\mathrm{d}t$ is the rate of its variation at this time, $x_t(\cdot)$ is the motion history that has realized up to the time t, $u(t)$ and $v(t)$ are respectively the current control and disturbance actions, t^0 is the initial time for the control process, and $[t_*, t^0]$ is treated as the interval of a priori accumulation of the initial history $x^0_{t^0}(\cdot)$. It is assumed that times t_*, t^0, T are given, and P, Q are known compact sets of finite-dimensional spaces. The process realization is the triple $\{x(\cdot), u(\cdot), v(\cdot)\}$, where $x(\cdot) : [t_*, T] \mapsto \mathbb{R}$ is the motion realization, $u(\cdot) : [t^0, T) \mapsto P$ is the control realization, and $v(\cdot) : [t^0, T) \mapsto Q$ is the disturbance realization.

Let us introduce the notation:

$$\overline{f}\big(t, x_t(\cdot), u, v\big) := \Big(f\big(t, x_t(\cdot), u, v\big), h\big(t, x_t(\cdot), u, v\big)\Big).$$

It is assumed that mappings $\overline{f} : G \times P \times Q \mapsto \mathbb{R}^n \times \mathbb{R}$ and $\sigma : C \mapsto \mathbb{R}$ are continuous, and the following estimate holds:

$$\big\|\overline{f}\big(t, x_t(\cdot), u, v\big)\big\| \leq L\big(t, x_t(\cdot)\big), \tag{4}$$

where

$$L\big(t, x_t(\cdot)\big) := \Big(1 + \max_{t_* \leq \tau \leq t} \|x(\tau)\|\Big) c, \quad c = \text{const} > 0.$$

It is assumed also that there exist a constant $\lambda > 1$, concentrated delays $\vartheta_k \in (0, t^0 - t_*]$, $k = \overline{1, K}$, and estimation times $T_m : t_* \leq T_1, T_m < T_{m+1}, m = \overline{1, M-1}$, $T_M = T$, such that, for any $t \in [t^0, T]$, $x(\cdot) \in C$, and $y(\cdot) \in C$, one has the following Lipschitz conditions:

$$\big\|\overline{f}\big(t, x_t(\cdot), u, v\big) - \overline{f}\big(t, y_t(\cdot), u, v\big)\big\|$$
$$\leq \lambda \left[\|w(t)\| + \sum_{k=1}^{K} \|w(t - \vartheta_k)\| + \sqrt{\int_{t_*}^{t} \|w(\tau)\|^2 \mathrm{d}\tau}\,\right],$$
$$\big|\sigma\big(x(\cdot)\big) - \sigma\big(y(\cdot)\big)\big| \tag{5}$$
$$\leq \lambda \left[\sum_{m=1}^{M} \|w(T_m)\| + \sqrt{\int_{t_*}^{T} \|w(\tau)\|^2 \mathrm{d}\tau}\,\right].$$

where $w(\cdot) = x(\cdot) - y(\cdot)$.

In particular, due to the given assumptions the initial value problem (1), (2) has the unique absolutely continuous on $[t^0, T]$ solution $x(\cdot) \in C$ for arbitrary initial history $x^0_{t^0}(\cdot) \in C_{t^0}$ and any Borel measurable realizations $u(\cdot) : [t^0, T) \mapsto P$ and $v(\cdot) : [t^0, T) \mapsto Q$.

The goal of the control is to minimize functional (3). Note that, since disturbance actions are unknown, the worst-case might occur when disturbances maximize (3).

The control strategy is treated as a function $U = U(g) \in P$, $g = \{t, x_t(\cdot)\} \in G$, $t \in [t^0, T)$. It acts on system (1) in the discrete time scheme on the basis of some partition

$$\Delta_\delta := \big\{t_i : \ t_1 = t^0, \ 0 < t_{i+1} - t_i \leq \delta, \ i = \overline{1, N}, \ t_{N+1} = T\big\}.$$

Let $g^0 = \{t^0, x^0_{t^0}(\cdot)\}$. Denote by $S(g^0, U, \Delta_\delta)$ the set of all triples $\{x(\cdot), u(\cdot), v(\cdot)\}$ such that $v(\cdot) : [t^0, T) \mapsto Q$ is a

Borel measurable function, $u(\cdot)$ is the piecewise constant control realization defined as follows:

$$u(t) = U\big(t_i, x_{t_i}(\cdot)\big), \quad t_i \le t < t_{i+1},\ i = \overline{1, N},$$

and $x(\cdot) \in C$ is absolutely continuous on $[t^0, T]$, satisfies (1) for almost all $t \in [t^0, T]$ and satisfies (2). So, $S(g^0, U, \Delta_\delta)$ is the set of all possible process realizations corresponding to the fixed initial position $g^0 = \{t^0, x^0_{t^0}(\cdot)\}$, control strategy U, and partition Δ_δ.

The guaranteed result for the strategy U is defined as follows:

$$\Gamma(g^0, U) := \lim_{\delta \downarrow 0} \sup_{\Delta_\delta} \sup \gamma\big(S(g^0, U, \Delta_\delta)\big).$$

The optimal guaranteed result is defined by the formula:

$$\Gamma^0(g^0) := \inf_U \Gamma(g^0, U).$$

It depends on the initial position $g^0 = \{t^0, x^0_{t^0}(\cdot)\}$. So, one can consider the functional of optimal guaranteed result:

$$G \ni g = \{t, x_t(\cdot)\} \mapsto \Gamma^0(g) = \Gamma^0\big(t, x_t(\cdot)\big) \in \mathbb{R}.$$

It is called the value functional of control problem (1)–(3).

The goal of the paper is to describe the infinitesimal properties of the value functional and to find the optimal control strategy. The results presented below are obtained as a development of Krasovskii and Lukoyanov [2000], and Lukoyanov [2003, 2004, 2006].

4. SMOOTH ESTIMATION OF THE GUARANTEED RESULT

For control problems of systems described by ordinary differential equations, the BI equation is derived within the dynamic programming method under the supposition that the value function is smooth. Similarly, to obtain an analogous equation for problems of the type (1)–(3), some smoothness of the value functional $\Gamma^0 : G \mapsto \mathbb{R}$ should be supposed. Note that, for a pair $\{t, x_t(\cdot)\} \in G$, the value of the time t determines the segment $[t_*, t]$, where the function $x_t(\cdot)$ has to be defined. Thus, the variables t and $x_t(\cdot)$ can not vary independently. Note also, by the substantive sense of the problem, the function $x_t(\cdot)$ is the motion history, and thus, can be varied only for the future instants of time. In other words, variation of $x_t(\cdot)$ consists in adding a "trunk" on $[t, t+\delta]$. So, it seems to be difficult to apply here the classical functional derivatives (see, e.g.,Wolenski [1994] for related technique). Hence, it is reasonable to use here some specific differentiability technique. The technique used below is based on the property of differentiability that was initially shown for Lyapunov functionals in problems of stabilization of hereditary dynamical systems.

Consider a functional $\varphi : G \mapsto \mathbb{R}$. Fix $g = \{t, x_t(\cdot)\} \in G$, $t < T$. Denote by $\mathrm{Lip}(g)$ the set of all functions $y(\cdot) \in C$ that coincide with $x_t(\cdot)$ on $[t_*, t]$ and are Lipschitz continuous on $[t, T]$. The functional φ is called the co-invariantly (*ci*-) differentiable at the point g if a number $\partial_t\varphi(g)$ and an n-vector $\nabla\varphi(g)$ exist such that, for any $y(\cdot) \in \mathrm{Lip}(g)$, the following relation is valid:

$$\varphi\big(t+\delta, y_{t+\delta}(\cdot)\big) - \varphi\big(t, x_t(\cdot)\big)$$
$$= \partial_t\varphi(g)\delta + \big\langle \nabla\varphi(g), y(t+\delta) - x(t)\big\rangle + o_{y(\cdot)}(\delta), \qquad (6)$$
$$\delta \in (0, T-t],$$

where $o_{y(\cdot)}$ depends on the choice of $y(\cdot) \in \mathrm{Lip}(g)$, $o_{y(\cdot)}(\delta)/\delta \to 0$ as $\delta \downarrow 0$. Here, $\langle \cdot, \cdot \rangle$ stands for the scalar product of vectors.

According to the terminology of Kim [1999], the quantities $\partial_t\varphi(g)$ and $\nabla\varphi(g)$ are the *ci*-derivative in t and the *ci*-gradient of the functional φ at the point g, respectively. In the terminology of Aubin and Haddad [2002], these quantities can be called Clio-derivatives. The functional φ is called *ci*-smooth if it is continuous, *ci*-differentiable at each point $g = \{t, x_t(\cdot)\} \in G$, $t < T$, and its *ci*-derivatives $\partial_t\varphi(g)$ and $\nabla\varphi(g)$ are continuous.

Denote

$$\chi(g, u, v, s) := \big\langle s, f(g, u, v)\big\rangle - h(g, u, v),$$
$$H(g, s) := \min_{u \in P} \max_{v \in Q} \chi(g, u, v, s).$$

If the functional φ is *ci*-smooth then the extremal strategy U^{e} can be defined by aiming in the direction of its *ci*-gradient:

$$U^{\mathrm{e}} = U^{\mathrm{e}}(g) := p\big(g, \nabla\varphi(g)\big),$$
$$p(g, s) \in \arg\min_{u \in P} \Big\{ \max_{v \in Q} \chi(g, u, v, s) \Big\}. \qquad (7)$$

Here the function $p : G \times \mathbb{R}^n \mapsto P$ is called the minimax pre-strategy.

The following results are proved on the basis of (6) similarly as for systems described by ordinary differential equations (see, e.g., Krasovskii and Subbotin [1988]).

Lemma 1. Let the *ci*-smooth functional φ satisfy the following differential inequality with *ci*-derivatives:

$$\partial_t\varphi(g) + H\big(g, \nabla\varphi(g)\big) \le 0,$$
$$g = \{t, x_t(\cdot)\} \in G,\ t^0 \le t < T. \qquad (8)$$

Then, for any $\eta > 0$ and any initial position $g^0 = \{t^0, x^0_{t^0}(\cdot)\} \in G$, a parameter $\delta > 0$ exists such that, for any partition Δ_δ of the time interval $[t^0, T]$ and any triple $\{x(\cdot), u(\cdot), v(\cdot)\} \in S(g^0, U^{\mathrm{e}}, \Delta_\delta)$, the following inequality is valid:

$$\varphi\big(T, x_T(\cdot)\big) - \int_{t^0}^{T} h\big(t, x_t(\cdot), u(t), v(t)\big)\mathrm{d}t \le \varphi(g^0) + \eta.$$

Theorem 1. Let the *ci*-smooth functional φ satisfy the HJ type equation with *ci*-derivatives

$$\partial_t\varphi(g) + H\big(g, \nabla\varphi(g)\big) = 0,$$
$$g = \{t, x_t(\cdot)\} \in G,\ t^0 \le t < T, \qquad (9)$$

and the condition at the right end point

$$\varphi\big(T, x_T(\cdot)\big) = \sigma\big(x(\cdot)\big),\ x_T(\cdot) = x(\cdot) \in C. \qquad (10)$$

Then functional φ is the value functional of the problem (1)–(3), i.e. $\Gamma^0 = \varphi$; and the extremal strategy (7) is optimal, i.e. $\Gamma(g^0, U^{\mathrm{e}}) = \Gamma^0(g^0)$, $g^0 \in G$.

5. NONSMOOTH CASE

Theorem 1 gives a solution of the considered problem if the value functional Γ^0 is *ci*-smooth. However, as a rule, this functional does not possess sufficient properties of smoothness, the problem (9), (10) has no solutions in the classical sense, and smooth estimations on the basis of (7) do not give satisfactory results. So, it is reasonable to apply here the nonsmooth technique of generalized (minimax, viscosity) solutions to the problem (9), (10), and nonsmooth constructions of extremal aiming.

Let the functional $\varphi : G \mapsto \mathbb{R}$ be continuous. Fix $g = \{t, x_t(\cdot)\} \in G$, $t < T$ and $l \in \mathbb{R}^n$. Denote

$$\partial^-\{\varphi(g)|l\} = \liminf_{\delta\downarrow 0} \left[\varphi\big(t+\delta, y^l_{t+\delta}(\cdot)\big) - \varphi(g)\right]\delta^{-1},$$
$$\partial^+\{\varphi(g)|l\} = \limsup_{\delta\downarrow 0} \left[\varphi\big(t+\delta, y^l_{t+\delta}(\cdot)\big) - \varphi(g)\right]\delta^{-1},$$

where

$$y^l(\tau) = \begin{cases} x_t(\tau) & \text{for } \tau \in [t_*, t), \\ x_t(t) + (\tau - t)l & \text{for } \tau \in [t, T]. \end{cases}$$

The values $\partial^-\{\varphi(g)|l\}$ and $\partial^+\{\varphi(g)|l\}$ are called the lower and upper semi-derivatives of the functional φ at the point g in the finite-dimensional direction l.

Take λ from condition (5). Fix a number ε such that $0 < \varepsilon < \varepsilon^0 := \exp\{(K+3)\lambda(t^0 - T)\}$. Let $t \in [t^0, T]$ and $w(\cdot) \in C$. Denote

$$\alpha^\varepsilon(t) := \exp\{(K+3)\lambda(t^0 - t)\}/\varepsilon,$$

$$\beta^\varepsilon_1\big(t, w_t(\cdot)\big) := \sqrt{\varepsilon^4 + 2\lambda \sum_{k=1}^{K} \int_{t-\vartheta_k}^{t} \|w(\tau)\|^2 \mathrm{d}\tau + \|w(t)\|^2},$$

$$\beta^\varepsilon_2\big(t, w_t(\cdot)\big) := \sqrt{\varepsilon^4 + 2\lambda \int_{t_*}^{t} \|w(\tau)\|^2 \mathrm{d}\tau + \|w(t)\|^2}.$$

Consider the auxiliary Lyapunov type functional

$$\nu_\varepsilon\big(t, w_t(\cdot)\big) = \alpha^\varepsilon(t)\beta^\varepsilon_1\big(t, w_t(\cdot)\big) + \big(\alpha^\varepsilon(t) - 1\big)\beta^\varepsilon_2\big(t, w_t(\cdot)\big). \tag{11}$$

It is *ci*-smooth, and

$$\begin{aligned}\partial_t\nu_\varepsilon\big(t, w_t(\cdot)\big) = \\ &- (K+3)\lambda\alpha^\varepsilon(t)\Big(\beta^\varepsilon_1\big(t, w_t(\cdot)\big) + \beta^\varepsilon_2\big(t, w_t(\cdot)\big)\Big) \\ &+ \frac{\lambda\alpha^\varepsilon(t)}{\beta^\varepsilon_1\big(t, w_t(\cdot)\big)} \sum_{k=1}^{K} \Big(\|w(t)\|^2 - \|w(t-\vartheta_k)\|^2\Big) \\ &+ \frac{\lambda\big(\alpha^\varepsilon(t) - 1\big)}{\beta^\varepsilon_2\big(t, w_t(\cdot)\big)} \|w(t)\|^2,\end{aligned}$$

$$\nabla\nu_\varepsilon\big(t, w_t(\cdot)\big) = \left(\frac{\lambda\alpha^\varepsilon(t)}{\beta^\varepsilon_1\big(t, w_t(\cdot)\big)} + \frac{\lambda\big(\alpha^\varepsilon(t) - 1\big)}{\beta^\varepsilon_2\big(t, w_t(\cdot)\big)}\right) w(t).$$

By (5), for any $x(\cdot), y(\cdot) \in C$ and $t \in [t^0, T]$, it follows

$$\begin{aligned}\partial_t\nu_\varepsilon\big(t, w_t(\cdot)\big) + H\Big(t, x_t(\cdot), \nabla\nu_\varepsilon\big(t, w_t(\cdot)\big)\Big) \\ - H\Big(t, y_t(\cdot), \nabla\nu_\varepsilon\big(t, w_t(\cdot)\big)\Big) \le 0,\end{aligned} \tag{12}$$

where $w(\cdot) = x(\cdot) - y(\cdot)$.

Let the initial position $g^0 = \{t^0, x^0_{t^0}(\cdot)\} \in G$ be fixed. Define the set

$$X^0 := \Big\{x(\cdot) \in \mathrm{Lip}(g^0) : \|\dot{x}(t)\| \le L\big(t, x_t(\cdot)\big) \text{ for almost all } t \in [t^0, T]\Big\}. \tag{13}$$

It is a compact set in C. By (4), it contains all possible realizations of the motion.

Consider now the following transformation of the functional φ:

$$\varphi_\varepsilon(g) := \min_{y(\cdot)\in X^0} \Big[\varphi\big(t, y_t(\cdot)\big) + \nu_\varepsilon\big(t, w_t(\cdot)\big)\Big], \tag{14}$$

where $g = \{t, x_t(\cdot)\} \in G$, and $w_t(\cdot) = x_t(\cdot) - y_t(\cdot)$.

Let $y^\varepsilon(\cdot \mid g)$ be the function giving the minimum in (14), and $w^\varepsilon_t(\cdot \mid g) = x_t(\cdot) - y^\varepsilon_t(\cdot \mid g)$. Define the so-called extremal ε-strategy $U^{\mathrm{e}}_\varepsilon$ as follows:

$$U^{\mathrm{e}}_\varepsilon = U^{\mathrm{e}}_\varepsilon(g) = p\Big(g, \nabla\nu_\varepsilon\big(t, w^\varepsilon_t(\cdot \mid g)\big)\Big), \tag{15}$$

where $p(g, s)$ is the minimax pre-strategy defined by (7).

Lemma 2. Let φ be a lower semicontinuous functional that satisfies the inequality

$$\begin{aligned}\min_{l\in B(g)} \Big[\partial^-\{\varphi(g)|l\} - \langle s, l\rangle\Big] + H(g, s) \le 0, \\ g = \{t, x_t(\cdot)\} \in G,\ t^0 \le t < T,\ s \in \mathbb{R}^n,\end{aligned} \tag{16}$$

where $B(g) := \{l \in \mathbb{R}^n : \|l\| \le L(g)\}$ ($L(g)$ is defined as in (4)). Then, for any $\eta > 0$ and any initial position $g^0 = \{t^0, x^0_{t^0}(\cdot)\} \in G$, parameters $\varepsilon > 0$ and $\delta > 0$ exist such that, for any partition Δ_δ of the time interval $[t^0, T]$ and any triple $\{x(\cdot), u(\cdot), v(\cdot)\} \in S(g^0, U^{\mathrm{e}}_\varepsilon, \Delta_\delta)$, the following inequality is valid:

$$\varphi\big(T, x_T(\cdot)\big) - \int_{t^0}^{T} h\big(t, x_t(\cdot), u(t), v(t)\big)\mathrm{d}t \le \varphi(g^0) + \eta.$$

Theorem 2. The functional $\varphi : G \mapsto \mathbb{R}$ is the value functional of control problem (1)–(3), i.e. $\varphi = \Gamma^0$, if and only if it is continuous and satisfies condition (10), differential inequality (16), and the following inequality:

$$\begin{aligned}\max_{l\in B(g)} \Big[\partial^+\{\varphi(g)|l\} - \langle s, l\rangle\Big] + H(g, s) \ge 0, \\ g = \{t, x_t(\cdot)\} \in G,\ t^0 \le t < T,\ s \in \mathbb{R}^n.\end{aligned} \tag{17}$$

In this case, the extremal ε-strategy (15) is optimal, i.e.

$$\limsup_{\varepsilon\downarrow 0} \Gamma(g^0, U^{\mathrm{e}}_\varepsilon) = \Gamma^0(g^0),\ g^0 \in G.$$

Theorem 2 demonstrates the attainability of the optimal guaranteed result in the class of feedback controls with memory.

Remark 1. For *ci*-differentiable functionals φ, inequality (16) is equivalent to inequality (8). Respectively, the pair of inequalities (16), (17) is equivalent to equality (9). So, differential inequalities (16), (17) are a natural generalization of equation (9) for the nonsmooth case.

Remark 2. To construct the optimal ε-strategy on the basis of (14), (15), one can use, instead of functional (11) and set (13), any other functional ν_ε and set $X \supset X^0$ satisfying the following conditions:

(a) The functional ν_ε is nonnegative, *ci*-smooth, and satisfies the inequality
$$\nu_\varepsilon\big(t, w_t(\cdot) \equiv 0\big) \le \mu(\varepsilon),$$
where $\mu(\varepsilon) \to 0$ as $\varepsilon \downarrow 0$.
(b) For any $x(\cdot), y(\cdot) \in X$, inequality (12) is valid.
(c) The minimum in (14) exists for $X^0 = X$.
(d) The following estimate holds:
$$\sup_{y(\cdot)\in X} \big|\sigma\big(y(\cdot)\big)\big| < \infty.$$
(e) For any $\zeta > 0$ and $\eta > 0$, a number $\varepsilon > 0$ exists such that, for any $x(\cdot), y(\cdot) \in X$, the inequality
$$\nu_\varepsilon\big(T, w_T(\cdot)\big) < \zeta,$$
where $w_T(\cdot) = w(\cdot) = x(\cdot) - y(\cdot)$, implies
$$\big|\sigma\big(x(\cdot)\big) - \sigma\big(y(\cdot)\big)\big| < \eta.$$

As a rule, the appropriate choice of the auxiliary functional ν_ε and the set X allows to simplify calculations essentially.

6. EXAMPLE

Consider a simple example. Let $\vartheta > 0$ and $t_* = -\vartheta$. Denote
$$f\big(t, x_t(\cdot)\big) := A(t)x(t) + A_\vartheta(t)x(t-\vartheta) + \int_{-\vartheta}^{t} F(t,\xi)x(\xi)\mathrm{d}\xi, \tag{18}$$
where $A(t)$, $A_\vartheta(t)$ and $F(t,\xi)$ are continuous $n \times n$-matrix-functions. Let the control process be described by the dynamic equation:
$$\dot{x}(t) = f\big(t, x_t(\cdot)\big) + u(t) + v(t),$$
$$0 \le t^0 \le t \le T, \quad x(t) \in \mathbb{R}^n, \tag{19}$$
$$u(t) \in \mathbb{R}^n: \ \|u(t)\| \le 1, \quad v(t) \in \mathbb{R}^n: \ \|v(t)\| \le 1,$$
the initial condition:
$$x_{t^0}(\cdot) = x^0_{t^0}(\cdot) \in C_{t^0}, \tag{20}$$
and the cost functional:
$$\gamma = \sigma\big(x(\cdot)\big) - \frac{1}{2}\int_{t^0}^{T} \Big(\|v(t)\|^2 - \|u(t)\|^2\Big)\mathrm{d}t. \tag{21}$$
Then
$$H\big(t, x_t(\cdot), s\big) = \Big\langle s, f\big(t, x_t(\cdot)\big)\Big\rangle, \tag{22}$$
and the problem (9), (10) takes the form
$$\begin{cases} \partial_t\varphi\big(t, x_t(\cdot)\big) + \Big\langle \nabla\varphi\big(t, x_t(\cdot)\big),\, f\big(t, x_t(\cdot)\big)\Big\rangle = 0, \\ \qquad \{t, x_t(\cdot)\} \in G, \quad 0 \le t < T; \\ \varphi\big(T, x_T(\cdot)\big) = \sigma\big(x(\cdot)\big),\ x_T(\cdot) = x(\cdot) \in C. \end{cases} \tag{23}$$

According to Krasovskii and Lukoyanov [2000], the minimax solution to (23) is described by the formula:
$$\varphi\big(t, x_t(\cdot)\big) = \sigma\Big(z\big(\,\cdot\,|t, x_t(\cdot)\big)\Big), \tag{24}$$
where $z\big(\,\cdot\,|t, x_t(\cdot)\big)$ is the solution of the following differential equation with aftereffect:
$$\dot{z}(\tau) = f\big(\tau, z_\tau(\cdot)\big), \quad t \le \tau \le T, \tag{25}$$
under the initial condition:
$$z_t(\cdot) = x_t(\cdot). \tag{26}$$
Note that, due to (25), (26), the following equality is valid:
$$\partial^{\mp}\Big\{\sigma\Big(z\big(\,\cdot\,|t, x_t(\cdot)\big)\Big)\,\Big|\, f\big(t, x_t(\cdot)\big)\Big\} = 0$$
These relations and (22) imply that functional (24) satisfies inequalities (16) and (17). Thus, by Theorem 2, it is the value functional of the control problem (18)–(21).

Denote by $\Psi(\tau, t)$ the $n \times n$-matrix-function such that $\Psi(\tau, t) = 0$ for $t > \tau$, $\Psi(\tau, \tau)$ is the identity matrix, and, for $t < \tau$,
$$\begin{aligned}\frac{\mathrm{d}\Psi(\tau,t)}{\mathrm{d}t} &= -\Psi(\tau,t)A(t) - \Psi(\tau, t+\vartheta)A_\vartheta(t+\vartheta) \\ &\quad - \int_t^\tau \Psi(\tau,\xi)F(\xi,t)\mathrm{d}\xi.\end{aligned} \tag{27}$$
By virtue of (18), for $\tau \in [t, T]$, the solution to (25), (26) can be represented in the following form:
$$\begin{aligned} z\big(\tau|t, x_t(\cdot)\big) &= \Psi(\tau,t)x_t(t) + \int_t^{t+\vartheta} \Psi(\tau,\xi)A_\vartheta(\xi)x(\xi-\vartheta)\mathrm{d}\xi \\ &\quad + \int_{-\vartheta}^{t}\int_t^\tau \Psi(\tau,\xi)F(\xi,\eta)x_t(\eta)\mathrm{d}\xi\mathrm{d}\eta. \end{aligned} \tag{28}$$
Let us take, for example, $\sigma\big(x(\cdot)\big) = \|x(T)\|^2$. In this case, (24) implies
$$\varphi\big(t, x_t(\cdot)\big) = \big\|z\big(T|t, x_t(\cdot)\big)\big\|^2. \tag{29}$$
By virtue of (27) and (28), functional (29) is *ci*-smooth, and
$$\partial_t\varphi = -2\Big\langle z\big(T|t, x_t(\cdot)\big), \Psi(T,t)f\big(t, x_t(\cdot)\big)\Big\rangle,$$
$$\nabla\varphi = 2\Psi^\top(T,t)z\big(T|t, x_t(\cdot)\big), \tag{30}$$
where $\Psi^\top$ is the transpose to Ψ. By substituting (29), (30) into (23), it can be checked directly that functional (29) satisfies (23). So, by Theorem 1, the optimal strategy U^0 exists in this case. It can be constructed by (7):
$$U^0\big(t, x_t(\cdot)\big) = p(\nabla\varphi),$$
where $\nabla\varphi$ is taken from (30), and p is the minimax prestrategy of the control problem (18)–(21):
$$p(s) = \begin{cases} -s & \text{if } \|s\| \le 1, \\ -s/\|s\| & \text{otherwise.} \end{cases} \tag{31}$$

On the other hand, suppose $\sigma\big(x(\cdot)\big) = \|x(T)\|$. Then

$$\varphi\big(t, x_t(\cdot)\big) = \big\|z\big(T|t, x_t(\cdot)\big)\big\|. \tag{32}$$

Functional (32) is not *ci*-differentiable at points $\{t, x_t(\cdot)\} \in G$ where $z\big(T|t, x_t(\cdot)\big) = 0$. So, the constructions of Section 4 do not work here. But the optimal ε-strategy $U^0_\varepsilon(\cdot)$ can be constructed on the basis of (14), (15). It is convenient to choose here

$$\nu_\varepsilon\big(t, w_t(\cdot)\big) = \frac{\big\|z\big(T|t, w_t(\cdot)\big)\big\|^2}{2\varepsilon}.$$

The set $X = X^0$ can be defined accordingly to (13). One can check directly that in the considered case the functional ν_ε satisfies conditions (a)–(e) of Remark 1. The corresponding calculations provide

$$\nabla\nu_\varepsilon\big(t, w^\varepsilon_t(\cdot)\big) = \begin{cases} \Psi^\top(T,t)z/\|z\| & \text{if } \|z\| \geq \varepsilon, \\ \Psi^\top(T,t)z/\varepsilon & \text{otherwise,} \end{cases}$$

$$U^0_\varepsilon\big(t, x_t(\cdot)\big) = p\Big(\nabla\nu_\varepsilon\big(t, w^\varepsilon_t(\cdot)\big)\Big).$$

Here $z = z\big(T|t, x_t(\cdot)\big)$, and p is pre-strategy (31).

7. CONCLUSION

The presented results show that, for control problems of hereditary systems, equation (9) is a functional analogue of the standard HJBI equation. In particular, it gives an effective criterion to check the optimality of *ci*-smooth functionals. Differential inequalities (16), (17) provide effective criteria to check the optimality of piecewise *ci*-smooth functionals and of envelopes of *ci*-smooth functionals (see, e.g., Lukoyanov [2006] for related technique). These properties are typical for the value functional.

According to the terminology of the differential game theory (see, e.g., Krasovskii [1985], Subbotin [1984], Osipov [1971]), inequality (16) expresses in infinitesimal form the property of u-stability of the value functional. Respectively, inequality (17) expresses the property of v-stability.

In the terminology of the theory of generalized solutions of HJ equations, inequalities (16) and (17) define respectively the upper and lover generalized solutions of equation (9) (see Subbotin [1995], and also Lukoyanov [2003, 2006]). Thus, by Theorem 2, the value functional of the control problem (1)–(3) is the unique minimax solution of the problem (9), (10).

Transformation (14) is similar to the transformation used in Garnysheva and Subbotin [1994] to define quasigradients of nonsmooth functions. In nonsmooth analysis (see, e.g., Clarke *et al.* [1998]), proximal gradients are defined with the help of similar transformations. In this paper, functional (11) is an appropriate analogue of auxiliary functions used in the mentioned constructions.

Conditions of the type (12) play an important role in the theory of viscosity solutions of the first order PDEs (see, e.g., condition A4 in Crandall *et al.* [1987]).

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