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Boundary control of vibration of symmetric composite laminated plate

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Abstract: This paper presents a solution to the boundary stabilization of a symmetric composite laminated plate in free transverse vibration. The symmetric composite laminated plate dynamics is presented by a linear fourth order partial differential equation (PDE). A linear control law is constructed to stabilize the plate. The control force consists of feedback of the velocity at the boundary of the plate. The novelty of this article is that it is possible to stabilize asymptotically a free transversely vibrating symmetric composite laminated plate with simply supported boundary condition via boundary control without resorting to truncation of the model.

1. INTRODUCTION

The word composite in the term composite material signifies that two or more materials are combined on a macroscopic scale to form a useful third material. The advantage of composite materials is that, if well designed, they usually exhibit the best qualities of their components or constituents and often some qualities that neither constituent possesses. Some properties that can be improved by forming a composite material are: strength, stiffness, corrosion resistance, wear resistance, attractiveness, weight, fatigue life, temperature-dependent behavior, thermal-insulation, thermal conductivity, acoustical insulation, etc.

For a flexible system, its elastic effects are often modeled using a linear partial differential equation (PDE) and a set of boundary conditions. Since there has been little control synthesis work for PDE-based systems as compared to the abundance of control design techniques available for ordinary differential equations (ODEs), most of the proposed control approaches for elastic systems rely on discretizing the PDE model into a set of ODEs (Canbolat *et al.* 1998; Chrysafinos *et al.* 2006). FEM and finite assumed-modes are the common methods for approximating and descritizing a PDE into a set of ODEs. Since the actual number of modes in an elastic system is infinite (at least theoretically), it is often not clear how many modes should be included while constructing the discretized ODE model. In addition, if a large number of modes is utilized to approximate a PDE-based system, the order of the discretized linear ODE model is often relatively high. Hence, the resulting controller can be a complex high-order algorithm (Canbolat *et al.* 1998). Unfortunately, a stability result generated for a discretized ODE model under a proposed control cannot be generalized to the PDE model under the same controller. That is, the neglected higher order modes could possibly destabilize the flexible system under a discretized model based controller (i.e. spillover instability (Balas 1978, Meirovitch, Baruh 1983)). Also, some devices and instruments such as strain gages are needed to feedback the vibration information at different points of the object or an observer is required to estimate the vibration information. However, in many applications, using the measurement instruments at the interior points of the objects is impossible or at least very difficult.

In this paper, motion control of a symmetric composite laminated plate having a transverse vibration is considered. So far, in the literature the researchers have used different methods to control the transverse vibration by discretizing the governing PDE of the composite plates. To the best of our knowledge, boundary control method has not been used for this purpose (Birman, 2008; Zhang *et al.* 2007; Gaoa *et al.* 2003). To the best of our knowledge, boundary control method has not been used for this pupose. Boundary control (BC) is an efficient method to exclude the effects of both observation difficulty and control spillover problem. The boundary controllers designed for the nondiscretized PDE model are often simple compensators which ensure closed-loop stability for an infinite number of modes. A brief review of BC is given in reference (Shahruz *et al.* 1996). Several researchers have proposed boundary controllers for a variety of flexible systems such as strings, beams and plates. In reference (Shahruz *et al.* 1996), it is shown that feedback from the velocity at the boundary of a string can stabilize the vibration in the string. In the same reference, a boundary feedback state is used to control the vibration of an axially moving string. In both, (Littman *et al.* 1988; Shahruz *et al.* 1996), the control laws are implemented via a mass-damper-spring on the right-hand side of the string. The most important benefit of the boundary control is that it can stabilize the mechanical systems without using in-domain aligned distributed controllers and/or measurement instruments. This novelty is very important in the industry and aerospace problems.

Lagnese in 1989 suggested using the boundaries to stabilize a plate (Lagnese, 1989). Then, Rao showed that stabilization of the transverse vibration of the elastic plates by using boundary control is possible for special boundary conditions (Rao 1998, Littman *et al.* 1988). Then Liu, Jiang and Huang by using Rao's approach obtained the same results as

Lagnese's research, but they showed that only asymptotic and not exponential stability is obtainable (Liu *et al.* 2004).

The main goal of this article is to use boundary control to stabilize the vibration of elastic symmetric composite laminated plates with different boundary conditions. Our study is a novel extension of the work by Liu *et al.* and Rao for more boundary conditions and from isotropic to symmetric laminated composite plates (Liu *et al.*, 2004; Rao 1998). The stability will be proved using a Lyapunov functional.

2. GOVERNING EQUATIONS

Governing equations of vibrating plates are categorized as:

(1) Geometric equations

(2) Vibration equations

(3) Relations between stress and strain

The first two equations are the same for a variety of plates (isotropic, composite, FGM…) but the third equation depends on the type of the plate.

2.1 Geometric equations

For small deflection assumption (deflection in z direction (w) is small), w is a function of x and y and planar strains are expressed by the following equations (Timoshenko 1959):

$$\varepsilon_x = \frac{\partial u}{\partial x} - z\frac{\partial^2 w}{\partial x^2} \quad (1)$$

$$\varepsilon_y = \frac{\partial v}{\partial y} - z\frac{\partial^2 w}{\partial y^2} \quad (2)$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} - 2z\frac{\partial^2 w}{\partial x \partial y} \quad (3)$$

where u and v are displacements of middle plane in x direction and y directions, see figure 1.

2.2 Vibration equations

Assuming small deflection in z direction, the governing vibration equations for a transversely vibrating plate are obtained as follows (Timoshenko 1959):

$$\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = 0 \quad (4)$$

$$\frac{\partial N_y}{\partial y} + \frac{\partial N_{xy}}{\partial x} = 0 \quad (5)$$

$$\frac{\partial^2 M_x}{\partial x^2} + 2\frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} + \rho\frac{\partial^2 w}{\partial t^2} = 0 \quad (6)$$

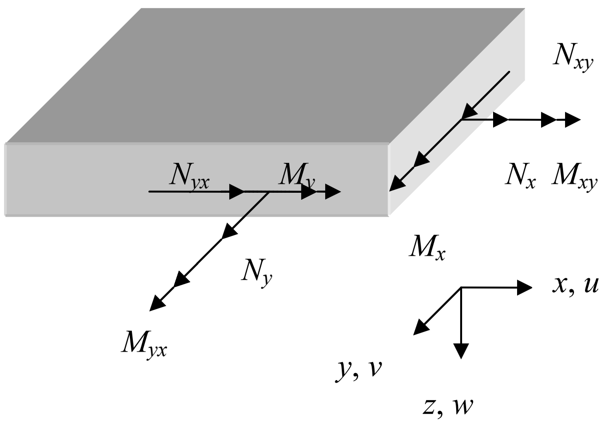

Fig. 1 Applied forces and moments and displacements along x, y and z axes

where N_x and N_y are in-plane forces in x and y directions, respectively N_{xy} and N_{yx} are shear forces in y and x directions and M_x and M_y and M_{xy} and M_{yx} are moments as shown in figure 1. Also, ρ is mass density of the plate.

2.3 Relation between stress and strain

Relation between stress and strain in plates depends on material symmetry. If we found this relation we could obtain the Forces and moments at each point of plate. Generally, for composite materials the forces and moments, at each point, is dependent planar strains and curvature of each point. This relation is expressed as follows, (Jones 1999):

$$\begin{bmatrix} N_x \\ N_y \\ N_{xy} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} \\ A_{12} & A_{22} & A_{26} \\ A_{16} & A_{26} & A_{66} \end{bmatrix} \begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{bmatrix} + \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix} \begin{bmatrix} -\frac{\partial^2 w}{\partial x^2} \\ -\frac{\partial^2 w}{\partial y^2} \\ -2\frac{\partial^2 w}{\partial x \partial y} \end{bmatrix} \quad (7)$$

$$\begin{bmatrix} M_x \\ M_y \\ M_{xy} \end{bmatrix} = \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix} \begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{bmatrix} + \begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{bmatrix} -\frac{\partial^2 w}{\partial x^2} \\ -\frac{\partial^2 w}{\partial y^2} \\ -2\frac{\partial^2 w}{\partial x \partial y} \end{bmatrix} \quad (8)$$

where A_{ij}, B_{ij} and D_{ij} are constant.

Also V_x (shear force in y direction) and V_y (shear force in x direction) expressed as follows:

$$V_x = \frac{\partial M_x}{\partial x} + 2\frac{\partial M_{xy}}{\partial y} = B_{11}\frac{\partial^2 u}{\partial x^2} + B_{12}\frac{\partial^2 v}{\partial x \partial y} + 3B_{16}\frac{\partial^2 u}{\partial y \partial x} + B_{16}\frac{\partial^2 v}{\partial x^2} + 2B_{26}\frac{\partial^2 v}{\partial y^2} + 2B_{66}(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 v}{\partial x \partial y}) - D_{11}\frac{\partial^3 w}{\partial x^3} - D_{12}\frac{\partial^3 w}{\partial x \partial y^2} - 4D_{16}\frac{\partial^3 w}{\partial y \partial x^2} - 2D_{26}\frac{\partial^3 w}{\partial y^3} - 4D_{66}\frac{\partial^3 w}{\partial x \partial y^2} \quad (9)$$

$$V_y = \frac{\partial M_y}{\partial y} + 2\frac{\partial M_{xy}}{\partial x} = B_{22}\frac{\partial^2 v}{\partial y^2} + B_{12}\frac{\partial^2 u}{\partial x \partial y} + 3B_{26}\frac{\partial^2 v}{\partial y \partial x} + B_{26}\frac{\partial^2 u}{\partial y^2} + 2B_{16}\frac{\partial^2 u}{\partial x^2} + 2B_{66}(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 u}{\partial x \partial y}) - D_{22}\frac{\partial^3 w}{\partial y^3} - D_{12}\frac{\partial^3 w}{\partial y \partial x^2} - 4D_{26}\frac{\partial^3 w}{\partial x \partial y^2} - 2D_{16}\frac{\partial^3 w}{\partial x^3} - 4D_{66}\frac{\partial^3 w}{\partial y \partial x^2} \quad (10)$$

If we replace equations (7) and (8) into equations (6), we obtain:

$$D_{11}\frac{\partial^4 w}{\partial x^4} + D_{22}\frac{\partial^4 w}{\partial y^4} + 2(D_{12} + 2D_{66})\frac{\partial^4 w}{\partial x^2 \partial y^2} + 4D_{16}\frac{\partial^4 w}{\partial x^3 \partial y} + 4D_{26}\frac{\partial^4 w}{\partial y^3 \partial x} - B_{11}\frac{\partial^3 u}{\partial x^3} - 3B_{16}\frac{\partial^3 u}{\partial x^2 \partial y} - B_{26}\frac{\partial^3 u}{\partial y^3} - (B_{12} + 2B_{66})\frac{\partial^3 u}{\partial y^2 \partial x} - B_{16}\frac{\partial^3 v}{\partial x^3} - (B_{12} + 2B_{66})\frac{\partial^3 v}{\partial x^2 \partial y} - 3B_{26}\frac{\partial^3 v}{\partial y^2 \partial x} - B_{22}\frac{\partial^3 v}{\partial y^3} + \rho\frac{\partial^2 w}{\partial t^2} = 0 \quad (11)$$

For laminated plates that are symmetric in both geometry and material properties about the middle surface, matrix B will be zero. For this case, the number of laminas should be odd. Also, laminas that are symmetric relative to the middle plane of middle lamina should have the same thickness.
For this case, the governing equation is expressed as:

$$D_{11}\frac{\partial^4 w}{\partial x^4} + 4D_{16}\frac{\partial^4 w}{\partial x^3 \partial y} + 2(D_{12} + 2D_{66})\frac{\partial^4 w}{\partial x^2 \partial y^2} + 4D_{26}\frac{\partial^4 w}{\partial y^3 \partial x} + D_{22}\frac{\partial^4 w}{\partial y^4} + \rho\frac{\partial^2 w}{\partial t^2} = 0 \quad (12)$$

2.2 Boundary conditions

In this paper we consider a plate with the following boundary conditions:

$$At \quad x = 0, \quad \begin{Bmatrix} w = 0 \\ M_x = 0 \end{Bmatrix} \quad (13)$$

$$At \quad x = a, \quad \begin{Bmatrix} w = 0 \\ M_x = 0 \end{Bmatrix} \quad (14)$$

$$At \quad y = 0, \quad \begin{Bmatrix} w = 0 \\ M_y = 0 \end{Bmatrix} \quad (15)$$

and

$$At \quad y = b, \quad \left\{ \begin{array}{l} M_y = 0 \\ V_y = (\frac{\partial M_y}{\partial y} + 2\frac{\partial M_{xy}}{\partial x}) = \\ \quad -(D_{12}\frac{\partial^3 w}{\partial x^2 \partial y} + D_{22}\frac{\partial^3 w}{\partial y^3} \\ \quad + 4D_{26}\frac{\partial^3 w}{\partial y^2 \partial x} + 4D_{66}\frac{\partial^3 w}{\partial x^2 \partial y} \\ \quad + 2D_{16}\frac{\partial^3 w}{\partial x^3}) = U \end{array} \right\} \quad (16)$$

3. BOUNDARY CONTROL

In order to apply Lyapunov's stability theorem to distributed parameter systems, it is necessary to introduce some definitions and lemmas. Furthermore, some basic theorems on which the stability proof is based will be presented.

Definition 1: An equilibrium state x_{eq} of a dynamic system is an element of the state-space Ξ such that $\eta(\phi(t,t_0)x_{eq}, x_{eq}) = 0$ for all $t \geq 0$ (the distance of its corresponding trajectory to that state is zero), where $\phi(t,t_0)$

is a continuous operator on Ξ and for any fixed $[t, t_0]$ it maps Ξ into itself. The set of all equilibrium states will be called the equilibrium set.

Definition 2: An invariant set M of a dynamic system is a subset Ξ of so that for any initial state $\mathbf{x}(t_0) \in M$, its corresponding trajectory will remain in Ξ for all $t \geq t_0$.

Definition 3: An asymptotically invariant set, M, of a distributed parameter dynamic system is uniformly asymptotically stable if

$$R(\phi(t,t_0)\mathbf{x_0}, M) \to 0 \text{ as } t - t_0 \to +\infty$$

Uniformly with respect to $t_0 \geq 0$, where δ_2 is sufficiently small and $\phi(t,t_0)\mathbf{x}(t_0)$ is the solution of the dynamic system at time t, starting at t_0.

Theorem 1 (Zubov, 1964): In order for an invariant set M of a dynamic system to be stable, it is necessary and sufficient that there exists a one-parameter family of functions $V(t)$, having the following properties:

1. On any element $\mathbf{x} \in S$ there is defined a function $V(\mathbf{x}, t)$ of the real argument t, defined for $t \geq t_0$, where $S = \{\mathbf{x} \in \Xi \mid 0 < \eta(\mathbf{x}, M) < r\}$

2. For any sufficiently small $\alpha_1 > 0$ it is possible to find a quantity $\alpha_2 > 0$ such that $V(\mathbf{x}, t) > \alpha_2$ for $\eta(\mathbf{x}(t_0), M) > \alpha_1$ and all $t \geq 0$

3. $V(\mathbf{x}, t) \to 0$ uniformly relative to $t \geq 0$ as $R(\mathbf{x}, M) \to 0$.

4. The functional $V(t)$ evaluated along the solution of the system does not increase for all $t \geq t_0$ for which it is defined, $\dot{V} \leq 0$.

5. Furthermore, if the functional $V(t)$ evaluated along the solution of dynamic system tends to zero as $t \to +\infty$ for all $t_0 \geq 0$ and $\eta(\mathbf{x}, M) < \delta_1$, where $\delta_1 > 0$ is sufficiently small, then the invariant set of the dynamic system will be asymptotically stable, and, conversely, if the invariant set is asymptotically stable, this holds, $\dot{V} \leq 0$.

Note that item 2 in the above theorem indicates positive definiteness of the function $V(\mathbf{x}, t)$. Item 3 requires that the function $V(\mathbf{x}, t)$ admit an infinitesimally upper limit. To prove stability of a distributed parameter system, one has to show that there exists a functional with the following properties:

1. The functional is positive definite with respect to a specified metric;

2. The functional admits an infinitesimally upper limit; and

3. The time derivative of the functional along the solutions of the underlying system is negative definite.

First we assume the following functional as a Lyapunov condidate and then we prove that the time derivative of this functional is negative definite.

$$V = \frac{1}{2}\int_0^b\int_0^a \{D_{11}\left(\frac{\partial^2 w}{\partial x^2}\right)^2 + 2D_{12}\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} + D_{22}\left(\frac{\partial^2 w}{\partial y^2}\right)^2 + 4D_{66}\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 + 4D_{16}\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial x \partial y} + 4D_{26}\frac{\partial^2 w}{\partial y^2}\frac{\partial^2 w}{\partial x \partial y} + \rho\left(\frac{\partial w}{\partial t}\right)^2\}dxdy \quad (17)$$

The above functional is the total energy (total kinetic energy and strain energy) of the vibrating composite plate thus it is a positive definite function. Its time derivative is:

$$\dot{V} = \frac{1}{2}\int_0^b\int_0^a \{2D_{11}\frac{\partial^2 w}{\partial x^2}\frac{\partial^3 w}{\partial x^2 \partial t} + 2D_{12}\left(\frac{\partial^2 w}{\partial x^2}\frac{\partial^3 w}{\partial y^2 \partial t} + \frac{\partial^2 w}{\partial y^2}\frac{\partial^3 w}{\partial x^2 \partial t}\right) + 2D_{22}\frac{\partial^2 w}{\partial y^2}\frac{\partial^3 w}{\partial y^2 \partial t} + 8D_{66}\frac{\partial^2 w}{\partial x \partial y}\frac{\partial^3 w}{\partial x \partial y \partial t} + 4D_{16}\left(\frac{\partial^2 w}{\partial x^2}\frac{\partial^3 w}{\partial x \partial y \partial t} + \frac{\partial^2 w}{\partial x \partial y}\frac{\partial^3 w}{\partial x^2 \partial t}\right) + 4D_{26}\left(\frac{\partial^2 w}{\partial y^2}\frac{\partial^3 w}{\partial x \partial y \partial t} + \frac{\partial^2 w}{\partial x \partial y}\frac{\partial^3 w}{\partial y^2 \partial t}\right) + 2\rho\frac{\partial w}{\partial t}\frac{\partial^2 w}{\partial t^2}\}dxdy \quad (18)$$

By replacing equation (12) in equation (18), we obtain:

$$\dot{V} = \frac{1}{2}\int_0^b\int_0^a \{2D_{11}\frac{\partial^2 w}{\partial x^2}\frac{\partial^3 w}{\partial x^2 \partial t} + 2D_{12}\left(\frac{\partial^2 w}{\partial x^2}\frac{\partial^3 w}{\partial y^2 \partial t} + \frac{\partial^2 w}{\partial y^2}\frac{\partial^3 w}{\partial x^2 \partial t}\right) + 2D_{22}\frac{\partial^2 w}{\partial y^2}\frac{\partial^3 w}{\partial y^2 \partial t} + 8D_{66}\frac{\partial^2 w}{\partial x \partial y}\frac{\partial^3 w}{\partial x \partial y \partial t} + 4D_{16}\left(\frac{\partial^2 w}{\partial x^2}\frac{\partial^3 w}{\partial x \partial y \partial t} + \frac{\partial^2 w}{\partial x \partial y}\frac{\partial^3 w}{\partial x^2 \partial t}\right) + 4D_{26}\left(\frac{\partial^2 w}{\partial y^2}\frac{\partial^3 w}{\partial x \partial y \partial t} + \frac{\partial^2 w}{\partial x \partial y}\frac{\partial^3 w}{\partial y^2 \partial t}\right) + 2\frac{\partial w}{\partial t}\left(-\left(D_{11}\frac{\partial^4 w}{\partial x^4} + 4D_{16}\frac{\partial^4 w}{\partial x^3 \partial y} + 2(D_{12} + 2D_{66})\frac{\partial^4 w}{\partial x^2 \partial y^2} + 4D_{26}\frac{\partial^4 w}{\partial y^3 \partial x} + D_{22}\frac{\partial^4 w}{\partial y^4}\right)\right)\}dxdy \quad (19)$$

Before we proceed, we introduce some basic lemmas.

Some required basic lemmas

By using integration by part we can verify each of the following integrals:

$$\int_0^b\int_0^a \frac{\partial^2 w}{\partial x^2}\frac{\partial^3 w}{\partial x^2 \partial t}dxdy = \int_0^b \left(\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial x \partial t} - \frac{\partial w}{\partial t}\frac{\partial^3 w}{\partial x^3}\right)\Big]_0^a dy + \int_0^b\int_0^a \frac{\partial^4 w}{\partial x^4}\frac{\partial w}{\partial t}dxdy \quad (20)$$

$$\int_0^b\int_0^a \frac{\partial^2 w}{\partial y^2}\frac{\partial^3 w}{\partial y^2 \partial t}dxdy = \int_0^a \left(\frac{\partial^2 w}{\partial y^2}\frac{\partial^2 w}{\partial y \partial t} - \frac{\partial w}{\partial t}\frac{\partial^3 w}{\partial y^3}\right)\Big]_0^b dx + \int_0^b\int_0^a \frac{\partial^4 w}{\partial y^4}\frac{\partial w}{\partial t}dxdy \quad (21)$$

$$\int_0^b\int_0^a \frac{\partial^2 w}{\partial y^2}\frac{\partial^3 w}{\partial x^2 \partial t}dxdy = \int_0^b \left(\frac{\partial^2 w}{\partial y^2}\frac{\partial^2 w}{\partial x \partial t} - \frac{\partial w}{\partial t}\frac{\partial^3 w}{\partial x \partial y^2}\right)\Big]_0^a dy + \int_0^b\int_0^a \frac{\partial^4 w}{\partial x^2 \partial y^2}\frac{\partial w}{\partial t}dxdy \quad (22)$$

$$\int_0^b\int_0^a \frac{\partial^2 w}{\partial x^2}\frac{\partial^3 w}{\partial y^2 \partial t}dxdy = \int_0^a \left(\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y \partial t} - \frac{\partial w}{\partial t}\frac{\partial^3 w}{\partial x^2 \partial y}\right)\Big]_0^b dx + \int_0^b\int_0^a \frac{\partial^4 w}{\partial x^2 \partial y^2}\frac{\partial w}{\partial t}dxdy \quad (23)$$

$$\int_0^b\int_0^a \frac{\partial^2 w}{\partial x^2}\frac{\partial^3 w}{\partial x \partial y \partial t}dxdy = \int_0^b \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y \partial t}\Big]_0^a dy - \int_0^a \frac{\partial^3 w}{\partial x^3}\frac{\partial w}{\partial t}\Big]_0^b dx + \int_0^b\int_0^a \frac{\partial^4 w}{\partial x^3 \partial y}\frac{\partial w}{\partial t}dxdy \quad (24\text{-a})$$

$$\int_0^b\int_0^a \frac{\partial^2 w}{\partial x^2}\frac{\partial^3 w}{\partial x \partial y \partial t}dxdy = \int_0^a \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial x \partial t}\Big]_0^b dx - \int_0^b \frac{\partial^3 w}{\partial x^2 \partial y}\frac{\partial w}{\partial t}\Big]_0^a dy + \int_0^b\int_0^a \frac{\partial^4 w}{\partial x^3 \partial y}\frac{\partial w}{\partial t}dxdy \quad (24\text{-b})$$

$$\int_0^b\int_0^a \frac{\partial^2 w}{\partial y^2}\frac{\partial^3 w}{\partial x \partial y \partial t}dxdy = \int_0^b \frac{\partial^2 w}{\partial y^2}\frac{\partial^2 w}{\partial y \partial t}\Big]_0^a dy - \int_0^a \frac{\partial^3 w}{\partial x \partial y^2}\frac{\partial w}{\partial t}\Big]_0^b dx + \int_0^b\int_0^a \frac{\partial^4 w}{\partial y^3 \partial x}\frac{\partial w}{\partial t}dxdy \quad (25\text{-a})$$

$$\int_0^b\int_0^a \frac{\partial^2 w}{\partial y^2}\frac{\partial^3 w}{\partial x \partial y \partial t}dxdy = -\int_0^b \frac{\partial^3 w}{\partial y^3}\frac{\partial w}{\partial t}\Big]_0^a dy + \int_0^a \frac{\partial^2 w}{\partial y^2}\frac{\partial^2 w}{\partial x \partial t}\Big]_0^b dx + \int_0^b\int_0^a \frac{\partial^4 w}{\partial y^3 \partial x}\frac{\partial w}{\partial t}dxdy \quad (25\text{-b})$$

$$\int_0^b\int_0^a \frac{\partial^2 w}{\partial x \partial y}\frac{\partial^3 w}{\partial x \partial y \partial t}dxdy = \int_0^b \frac{\partial^2 w}{\partial x \partial y}\frac{\partial^2 w}{\partial y \partial t}\Big]_0^a dy - \int_0^a \frac{\partial^3 w}{\partial y \partial x^2}\frac{\partial w}{\partial t}\Big]_0^b dx + \int_0^b\int_0^a \frac{\partial^4 w}{\partial x^2 \partial y^2}\frac{\partial w}{\partial t}dxdy \quad (26)$$

$$\int_0^b\int_0^a \frac{\partial^2 w}{\partial x \partial y}\frac{\partial^3 w}{\partial x^2 \partial t}dxdy = \int_0^b \left(\frac{\partial^2 w}{\partial x \partial y}\frac{\partial^2 w}{\partial x \partial t} - \frac{\partial w}{\partial t}\frac{\partial^3 w}{\partial x^2 \partial y}\right)\Big]_0^a dy + \int_0^b\int_0^a \frac{\partial^4 w}{\partial x^3 \partial y}\frac{\partial w}{\partial t}dxdy \quad (27)$$

$$\int_0^b\int_0^a \frac{\partial^2 w}{\partial x \partial y}\frac{\partial^3 w}{\partial y^2 \partial t}dxdy = \int_0^a \left(\frac{\partial^2 w}{\partial x \partial y}\frac{\partial^2 w}{\partial y \partial t} - \frac{\partial w}{\partial t}\frac{\partial^3 w}{\partial x \partial y^2}\right)\Big]_0^b dx + \int_0^b\int_0^a \frac{\partial^4 w}{\partial x \partial y^3}\frac{\partial w}{\partial t}dxdy \quad (28)$$

By using basic lemmas (20 – 28), we can convert integration on domain to integration on boundary; then:

$$\dot{V} = \frac{1}{2}\int_0^a \Big\{2D_{12}\Big(\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y \partial t} - \frac{\partial w}{\partial t}\frac{\partial^3 w}{\partial x^2 \partial y}\Big) + 2D_{22}\Big(\frac{\partial^2 w}{\partial y^2}\frac{\partial^2 w}{\partial y \partial t} - \frac{\partial^3 w}{\partial y^3}\frac{\partial w}{\partial t}\Big) - 8D_{66}\frac{\partial w}{\partial t}\frac{\partial^3 w}{\partial x^2 \partial y} - 4D_{16}\frac{\partial w}{\partial t}\frac{\partial^3 w}{\partial x^3} + 4D_{26}\Big(\frac{\partial^2 w}{\partial y \partial t}\frac{\partial^2 w}{\partial y \partial x} - 2\frac{\partial w}{\partial t}\frac{\partial^3 w}{\partial x \partial y^2}\Big)\Big\}\Big|_0^b dx$$

$$+\frac{1}{2}\int_0^b \Big\{2D_{12}\Big(\frac{\partial^2 w}{\partial y^2}\frac{\partial^2 w}{\partial x \partial t} - \frac{\partial w}{\partial t}\frac{\partial^3 w}{\partial x \partial y^2}\Big) + 2D_{11}\Big(\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial x \partial t} - \frac{\partial w}{\partial t}\frac{\partial^3 w}{\partial x^3}\Big) + 8D_{66}\frac{\partial^2 w}{\partial y \partial t}\frac{\partial^2 w}{\partial x \partial y} + 4D_{26}\frac{\partial^2 w}{\partial y^2}\frac{\partial^2 w}{\partial y \partial t} + 4D_{16}\Big(\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y \partial t} + \frac{\partial^2 w}{\partial x \partial y}\frac{\partial^2 w}{\partial x \partial t} - \frac{\partial w}{\partial t}\frac{\partial^3 w}{\partial x^2 \partial y}\Big)\Big\}\Big|_0^a dy \quad (29)$$

By Arrangement of $\dot{V}$, we can write

$$\dot{V} = \int_0^a \Big\{\frac{\partial w}{\partial t}V_y - \frac{\partial^2 w}{\partial y \partial t}M_y\Big\}\Big]_0^b dx + \int_0^b \Big\{\frac{\partial w}{\partial t}V_x - \frac{\partial^2 w}{\partial x \partial t}M_x\Big\}\Big]_0^a dy$$

$$+\frac{\partial w}{\partial t}\Big(2D_{16}\frac{\partial^2 w}{\partial x^2} + 2D_{26}\frac{\partial^2 w}{\partial y^2} + 4D_{66}\frac{\partial^2 w}{\partial x \partial y}\Big)\Big]_0^a \quad (30)$$

If boundary conditions (at x=0 and x=a), equations (13, 14) are applied to equation (30), $\dot{V}$ will be simplified to the following form:

At $x=0$ and $x=a$: $M_x = 0$ and $w = 0,\ \frac{\partial w}{\partial t} = 0$

So

$$\dot{V} = \frac{1}{2}\int_0^a (-M_y \frac{\partial^2 w}{\partial y \partial t} + V_y \frac{\partial w}{\partial t})\}\Big|_0^b dx \tag{31}$$

If boundary conditions at $y=0$ (equation 15)is applied to equation (31), $\dot{V}$ will be simplified to the following form:

At y=0: $w = 0, M_y = 0, \frac{\partial w}{\partial t} = 0, \frac{\partial^2 w}{\partial x \partial t} = 0$

Therefore

$$\dot{V} = \frac{1}{2}\int_0^a (-M_y \frac{\partial^2 w}{\partial y \partial t} + V_y \frac{\partial w}{\partial t})\Big|_{y=b} dx \tag{32}$$

Then, using (16), we have:
At y=b: $M_y = 0$ and $V_y = U$

$$\dot{V} = \frac{1}{2}\int_0^a (\frac{\partial w}{\partial t} U)\Big|_{y=b} dx \tag{33}$$

If we choose:

$U = -K\frac{\partial w}{\partial t}$ and $K > 0$

Then,

$$\dot{V} = -\frac{1}{2}\int_0^a \{K(\frac{\partial w}{\partial t})^2\}\Big|_{y=b} dx < 0 \tag{34}$$

4. CONCLUSION

According to the principle of conservation of energy, the work that is done by external forces and moments is equal to the total energy of the system. For our system, the external excitation (forces or moments) is applied on the boundaries of the rectangular composite plate. If we choose control forces/moments of the boundary controller to make the time derivative of the total energy of the system (which is positive definite function) be negative definite; then, the total energy of the plate can be used as the Lyapunov function of the system.

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