Proceedings of the 17th World Congress
The International Federation of Automatic Control
Seoul, Korea, July 6-11, 2008

Explicit Formulas for ISS Stabilization of Nonlinear Systems Subject to Bounded Inputs and Disturbances

Hisakazu Nakamura * **Nami Nakamura** * **Hirokazu Nishitani** *

* *Nara Institute of Science and Technology, Takayama 8916-5, Ikoma, Nara, 631-0192 Japan (e-mail:hisaka-n@is.naist.jp)*

Abstract: Control Lyapunov functions (CLFs), CLF based controller designs and disturbance attenuation have attracted much attention in nonlinear control theory.
However, little research exists that considers both the input constraint and the disturbance attenuation problems. For input constrained systems, we cannot stabilize under unbounded disturbances in general. Therefore, we propose an input-to-state stabilizable robust control Lyapunov function (ISS-RCLF) and an asymptotically stabilizable robust control Lyapunov function (AS-RCLF) for an input-restricted nonlinear system. In this paper, we propose a stabilizing controller for input and disturbance constrained nonlinear systems using ISS-RCLF, which becomes continuous if an ISS-RCLF has an ISS-CLF small control property. Moreover, we clarify the condition to be satisfied for an AS-RCLF and an ISS-RCLF, and when a proper function becomes an ISS-RCLF. Finally, we show the effectiveness of the proposed method by computer simulation.

1. INTRODUCTION

Control Lyapunov functions (CLF) and CLF-based controller designs have attracted much attention in nonlinear control theory. Control of nonlinear systems with input constraints is also considered to be an important problem, and there exists some research on stabilization using CLF [10, 11, 12]. Recently, we proposed generalized controllers for input constrained nonlinear systems using CLF [14, 15].

On the other hand, disturbance attenuation is an important problem in control theory. Input-to-stability (ISS) plays quite an important role in the problem [3, 4, 17]. Sontag and Wang [9] proposed a controller that achieves ISS, and Liberzon et al. [13] proposed a controller that guarantees integral input-to-state stability (iISS). Moreover, Krstić and Li [8] proposed an inverse optimal controller that achieves input-to-state stability.

However, there exist few studies that consider both the input constraint and the disturbance attenuation problems [6] and there is no research based on CLFs. In this paper, we consider the disturbance attenuation problem for input constrained nonlinear systems. For input constrained systems, we cannot stabilize the origin with unbounded disturbances. Hence, we may not apply previously proposed methods using an input-to-state stabilizable control Lyapunov function (ISS-CLF) or an integral input-to-state stabilizable control Lyapunov function (iISS-CLF) directly to input constrained nonlinear systems. Therefore, we propose an input-to-state stabilizable robust control Lyapunov function (ISS-RCLF) and an asymptotically stabilizable robust control Lyapunov function (AS-RCLF) for an input-restricted nonlinear system.

In this paper, we propose a stabilizing controller for input and disturbance constrained nonlinear systems using ISS-RCLF, which becomes continuous if an ISS-RCLF has an ISS-CLF small control property. Moreover, we clarify the condition to be satisfied for an AS-RCLF and an ISS-RCLF, and when a proper function becomes an ISS-RCLF. Finally, we show the effectiveness of the proposed method by computer simulation.

2. PRELIMINARIES

In this paper, we consider a disturbance attenuation problem for input constrained nonlinear systems. We introduce our previous results on nonlinear control of bounded-input nonlinear systems. We use the following notations: $\mathbb{R}_{>0} := (0, \infty)$ and $\mathbb{R}_{\geq 0} := [0, \infty)$.

In this section, we consider a stabilization problem of the following nominal input-affine nonlinear system (without disturbances):

$$\dot{x} = f(x) + g(x) \cdot u, \tag{1}$$

where $x \in \mathbb{R}^n$ is a state vector, $u \in U \subset \mathbb{R}^m$ is an input vector, and U is an input constraint. In this paper, we suppose the following k-norm input constraint:

$$U = U_k^1 := \left\{ u \in \mathbb{R}^m \;\middle|\; \|u\|_k = \left(\sum_{i=1}^{m} |u_i|^k \right)^{\frac{1}{k}} < 1 \right\}. \tag{2}$$

Furthermore, we assume that $f : \mathbb{R}^n \to \mathbb{R}^n$, $g : \mathbb{R}^n \to \mathbb{R}^{n \times m}$ are continuous mappings, and $f(0) = 0$. Note that

⋆ This work was supported by Grant-in-Aid for Young Scientists(B) (19760288) and Grant-in-Aid for Special Purposes (19569004).

L_fV and L_gV denote $(\partial V/\partial x)\cdot f(x)$ and $(\partial V/\partial x)\cdot g(x)$ respectively.

Then, we introduce definitions of Control Lyapunov Function (CLF) and Small Control Property (SCP):

Definition 1. (CLF). A smooth proper [3] positive-definite function $V : X \to \mathbb{R}_{\geq 0}$ defined on a neighborhood of the origin $X \subset \mathbb{R}^n$ is said to be a local control Lyapunov function for system (1) if the condition

$$\inf_{u\in U}\{L_fV + L_gV\cdot u\} < 0 \tag{3}$$

is satisfied for all $x \in X\backslash\{0\}$. Moreover, $V(x)$ is said to be a control Lyapunov function (CLF) for system (1) if $V(x)$ is a function defined on the entire $\mathbb{R}^n$ and condition (3) is satisfied for all $x \in \mathbb{R}^n\backslash\{0\}$. □

Note that if $V : X \to \mathbb{R}_{\geq 0}$ is a local clf,

$$L_gV = 0 \quad \Longrightarrow \quad L_fV < 0, \quad \forall x \in X\backslash\{0\}. \tag{4}$$

Definition 2. (Small Control Property (SCP)). A (local) control Lyapunov function is said to satisfy Small Control Property (SCP) if for any $\varepsilon > 0$, there is a $\delta > 0$ such that

$$0 \neq \|x\| < \delta \quad \Longrightarrow \quad \exists\|u\| < \varepsilon \text{ s.t. } L_fV + L_gV\cdot u < 0. \tag{5}$$

□

In our previous papers [14, 15], the following results were obtained.

Lemma 1. We consider system (1) with input constraint (2). Let $V(x)$ be a local clf, $P(x)$ be a function defined by

$$P(x) = \frac{L_fV}{\|L_gV\|_{\frac{k}{k-1}}}, \tag{6}$$

and $a_1 > 0$ be the maximum such that

$$\begin{aligned}&\inf_{u\in U_k^1}\{L_fV + L_gV\cdot u\} < 0,\\ &\qquad \forall x \in W_1\backslash\{0\} := \{x|V(x) < a_1\}\backslash\{0\}.\end{aligned} \tag{7}$$

Then, the origin is asymptotically stabilizable in W_1, and

$$P(x) < 1, \quad \forall x \in \{x \in W_1|\ L_gV \neq 0\}. \tag{8}$$

□

Proposition 1. We consider system (1) with input constraint $u \in \bar{U}_k^1$, where $\bar{U}_k^1$ is the closure of U_k^1. Let $V(x)$ be a local clf. Then, input

$$u_i = \begin{cases} -\dfrac{|L_{g_i}V|^{\frac{1}{k-1}}}{\|L_gV\|_{\frac{k}{k-1}}^{\frac{1}{k-1}}}\operatorname{sgn}(L_{g_i}V) & (L_gV \neq 0)\\ 0 & (L_gV = 0)\end{cases} \quad (i = 1,\ldots,m) \tag{9}$$

minimizes the derivative $\dot{V}(x,u)$ for each x. □

Proof 1. For fixed x, note that finding an input that minimizes $\dot{V}(x,u)$ implies minimization of $L_gV\cdot u$. Hence we find an input $\bar{u}$ that minimizes $L_gV\cdot u$ in $\bar{U}_k^1$.

If $L_gV = 0$, $L_gV\cdot u = 0$ for any u. Therefore, $u = 0$ minimizes $L_gV\cdot u$.

We consider the case $L_gV \neq 0$. Consider a hyper-surface $Q : L_gV\cdot u = a_2$. Then, the considered problem is to find the minimum of a_2. Since $\bar{U}_k^1$ is a compact convex subspace, Q is tangent to $\bar{U}_k^1$ and $\|\bar{u}\|_k = 1$ when a_2 takes its minimum.

Therefore $\bar{u}$ satisfies the following conditions:

$$\sum_{i=1}^{m}|u_i|^k - 1 = 0 \tag{10}$$

and

$$\begin{aligned}&\left(|\bar{u}_1|^{k-1}\operatorname{sgn}(\bar{u}_1),\ldots,|\bar{u}_m|^{k-1}\operatorname{sgn}(\bar{u}_m)\right)\\ &\qquad = -a_3\left(L_{g_1}V,\ldots,L_{g_m}V\right),\end{aligned} \tag{11}$$

where $a_3 > 0$. By (10) and (11), input (9) is obtained. ■

Theorem 1. We consider system (1) with input constraint (2). Let $V(x)$ be a local clf, W_1 be a domain defined in Lemma 1, $P(x)$ be a function defined by (6), $c > 0$ and $q \geq 1$ be constants. Then, input

$$u_i = \begin{cases} -\dfrac{P + |P| + c\|L_gV\|_q}{2 + c\|L_gV\|_q}\cdot\dfrac{|L_{g_i}V|^{\frac{1}{k-1}}}{\|L_gV\|_{\frac{k}{k-1}}^{\frac{1}{k-1}}}\operatorname{sgn}(L_{g_i}V) & (L_gV \neq 0)\\ 0 & (L_gV = 0)\end{cases} \quad (i = 1,\ldots,m) \tag{12}$$

asymptotically stabilizes the origin in domain W_1. Moreover, input (12) is continuous on $W\backslash\{0\}$, and continuous at the origin if $V(x)$ has the SCP. □

3. RCLF AND ISS-CLF

Two major CLF relations for disturbance attenuation are "Robust Control Lyapunov Function (RCLF)" proposed by Freeman and Kokotović [1] and "Input-to-State Stabilizable Control Lyapunov Function (ISS-CLF)" proposed by Sontag and Wang [9] and Krstić and Li [7]. We introduce these CLFs in this section.

In this paper, we consider the following input and disturbance affine nonlinear control system:

$$\dot{x} = f(x) + g_1(x)\cdot d + g_2(x)\cdot u, \tag{13}$$

where $x \in X \subset \mathbb{R}^n$ is a state vector, $d \in D \subset \mathbb{R}^{m_1}$ is a disturbance vector, $u \in U \subset \mathbb{R}^{m_2}$ is an input vector, and U is a convex subspace containing the origin. We assume that $f : \mathbb{R}^n \to \mathbb{R}^n$, $g_1 : \mathbb{R}^n \to \mathbb{R}^{n\times m_1}$ and $g_2 : \mathbb{R}^n \to \mathbb{R}^{n\times m_2}$ are continuous mappings, and $f(0) = 0$.

Definition 3. (RCLF [1]). A smooth proper positive-definite function $V : X \to \mathbb{R}_{\geq 0}$ is called a robust control Lyapunov function (RCLF) for (13) if there exist $c_v \in \mathbb{R}_{\geq 0}$ such that

$$\begin{aligned}&\inf_{u\in U}\sup_{d\in D}\{L_fV + L_{g_1}V\cdot d + L_{g_2}V\cdot u\} < 0\\ &\qquad\qquad \forall x \in \{x|V(x) = c\}\end{aligned} \tag{14}$$

for all $c > c_v$. □

Definition 4. (ISS-CLF [7]). A smooth proper positive-definite function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is called an input-to-state stabilizable Lyapunov function (ISS-CLF) for (13) if there exists a K_∞ function ρ such that the following implication holds for all $x \neq 0$ and all $d \in \mathbb{R}^r$:

$$|x| \geq \rho(|d|) \Downarrow \inf_{u\in\mathbb{R}^m}\{L_fV + L_{g_1}V\cdot d + L_{g_2}V\cdot u\} < 0. \quad (15)$$

□

Function ρ plays important role in the following section due to the fact that function ρ characterizes an ISS gain [4] from disturbance input d to state x. The following is a definition of Small Control Property for ISS-CLF.

Definition 5. (Small Control Property for ISS-CLF). An ISS-CLF $V(x)$ is said to have an ISS-CLF-SCP if for all $x \neq 0$ and all $d \in \mathbb{R}^r$,

$$|x| \geq \rho(|d|) \Downarrow \inf_u L_fV + L_{g_1}V\cdot d + L_{g_2}V\cdot u < 0 \quad (16)$$

and, in addition for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$0 \neq \|x\| < \delta \Downarrow \exists\|u\| < \varepsilon \text{ s.t. } L_fV + L_{g_1}V\cdot d + L_{g_2}V\cdot u < 0. \quad (17)$$

□

Note that the condition (17) is weaker than the requirement in Krstić and Li [8].

We often use "local uniform Lagrange stability" defined as the following in the analysis using an RCLF [5]:

Definition 6. (Local Uniform Lagrange Stability). The origin of a system $\dot{x} = f(x,d)$ is said to be locally uniformly Lagrange stable if for each $R \in (0, R_m)$ there exists $S > 0$ such that each solution $x(\cdot)$ is continuable on $[t_0, +\infty)$ and

$$\|x_0\| < R \Rightarrow \|x(t)\| < S \quad \text{for each} \quad t \geq t_0. \quad (18)$$

If $R_m = \infty$, the system is called uniformly Lagrange stable.

Moreover, the origin of a control system $\dot{x} = f(x,u,d)$ is said to be locally uniformly Lagrange stabilizable if there exists a controller $u = k(x,t)$ such that the origin of the closed system $\dot{x} = f(x, k(x,t), d)$ is locally uniformly Lagrange stable. □

4. INPUT-TO-STATE STABILIZABLE ROBUST CONTROL LYAPUNOV FUNCTION (ISS-RCLF)

In the previous section, we introduced RCLF and ISS-CLF, however, these functions are not suitable for input constrained nonlinear systems. We address the reason in this section.

We also consider system (13) in this section. An ISS control problem considers an unbounded disturbance $d(t)$ [8]. However, we cannot stabilize the origin of (13) by a restricted input for all uniformly continuous L_2 disturbances in many cases.

Then, we assume the following restricted disturbance:

$$d \in U^1_{k_1} := \{d \,|\, \|d\|_{k_1} < 1\}, \quad (19)$$

where $k_1 > 1$ is a constant.

The problem to be solved in this paper is stated as the following:

Problem 1. Consider system (13) and assume disturbance constraint (19) and the following input constraint:

$$u \in U^1_{k_2} := \{u \,|\, \|u\|_{k_2} < 1\}, \quad (20)$$

where $k_2 > 1$ is a constant.

The problem is to construct a controller that stabilizes the origin for all disturbances in the sense of local uniform Lagrange stability, in addition, if $d \in L_2$ is uniformly continuous, the controller asymptotically stabilizes the origin. □

Note that "RCLF" only guarantees the existence of u such that the origin is locally Lagrange stable, and does not guarantee the existence of locally asymptotically stabilizing control u without disturbance. Thus, "RCLF" is not suitable for asymptotic stabilization of systems under uniformly continous L_2 disturbances, and "ISS-CLF" is not convenient for input and disturbance constrained nonlinear systems. Hence, we propose an ISS robust control Lyapunov function (ISS-RCLF) and an asymptotically stabilizable robust control Lyapunov function (AS-RCLF) that are inspired by ISS-CLF and RCLF:

Definition 7. (ISS-RCLF and AS-RCLF). Consider the following system:

$$\dot{x} = f(x) + g_1(x)\cdot d + g_2(x)\cdot u, \quad (21)$$

where $d \in U^1_{k_1}$, $u \in U^1_{k_2}$ and $f(0) = 0$.

Then, a smooth proper positive definite function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is called an AS-RCLF for (21) if the following are satisfied:

(1) there exist constants c_+ and c_- satisfies $c_+ \geq c_- \geq 0$ such that

$$\inf_{u\in U^1_{k_2}} \sup_{d\in U^1_{k_1}} \{L_fV + L_{g_1}V\cdot d + L_{g_2}V\cdot u\} < 0 \quad (22)$$

for all $x \in \{x \in \mathbb{R}^n | c_- \leq V(x) \leq c_+\}$.

(2) there exists a continuous (non-strictly) increasing function $\rho : [0,1) \to [0, c_-)$ satisfies $\lim_{r\to 1}\rho(r) = c_-$ such that the following implication holds for all $x \in \{x | V(x) < c_-\}$ and all $d \in \mathbb{R}^r$:

$$V(x) \geq \rho(\|d\|_{k_1}) \Downarrow \inf_{u\in\mathbb{R}^m}\{L_fV + L_{g_1}V\cdot d + L_{g_2}V\cdot u\} < 0. \quad (23)$$

If $c_+ = +\infty$, function V is called a global AS-RCLF. Moreover, function V is said to be an ISS-RCLF if ρ satisfies the following condition:

(3) ρ is strictly and satisfies $\rho(0) = 0$.

□

An AS-RCLF is not useful for controller design indeed. However, sometimes we must consider an AS-RCLF for a certain kind of a control problem. Thus, we introduced an AS-RCLF.

5. CONTINUOUS CONTROLLER DESIGN

In the previous sections, we proposed ISS-RCLF and AS-RCLF. In this section, we propose a controller design scheme with the ISS-RCLF and the AS-RCLF.

First, we show the asymptotic stabilizable domain with obtained AS-RCLF in the following theorem:

Theorem 2. Consider system (13) with disturbance constraint $d \in U^1_{k_1}$ and input constraint $u \in U^1_{k_2}$. Let $V(x)$ be an AS-RCLF and $P_3(x)$ be a function defined by

$$P_3(x) = \frac{L_f V + \rho^\dagger(V(x)) \|L_{g_1} V\|_{\frac{k_1}{k_1-1}}}{\|L_{g_2} V\|_{\frac{k_2}{k_2-1}}}, \tag{24}$$

where function $\rho^\dagger : \mathbb{R} \to \mathbb{R}_{>0}$ is defined as follows:

$$\rho^\dagger(x) := \begin{cases} 1 & (x \geq c_-) \\ \sup \rho^{-1}(x) & (x < c_-) \end{cases} \tag{25}$$

Then, c_+ satisfies the following equation:

$$P_3(x) < 1 \qquad \forall x \in W_3 := \{x | V(x) < c_+\}, \tag{26}$$

and the origin is locally uniform Lagrange stabilizable in W_3. Moreover, the origin of the system is stabilizable for all uniformly continuous $d \in L_2$. □

Then, we can show the following main theorem of the paper:

Theorem 3. Consider system (13) with disturbance constraint $d \in U^1_{k_1}$ and input constraint $u \in U^1_{k_2}$. Assume $d \in L_2$ is uniformly continuous, and let V be an AS-RCLF, W_3 be a domain defined in Theorem 2, $P_3(x)$ be a function defined in (24), $c > 0$ and $q \geq 1$ be constants. Then, input

$$u_i = \begin{cases} -\dfrac{P_3 + |P_3| + c\|L_{g_2}V\|_q}{2 + c\|L_{g_2}V\|_q} \cdot \dfrac{|L_{g_{2i}}V|^{\frac{1}{k-1}}}{\|L_{g_2}V\|^{\frac{1}{k-1}}_{\frac{k}{k-1}}} \operatorname{sgn}(L_{g_{2i}}V) & (L_{g_2}V \neq 0) \\ 0 & (L_{g_2}V = 0) \end{cases} \quad (i = 1, \ldots, m) \tag{27}$$

stabilizes the origin in domain W_3 in the sense of local uniform Lagrange stability and asymptotically stabilizes the origin if $d \in L_2$ is uniformly continuous. Moreover, if $V(x)$ is ISS-RCLF, input (27) is continuous on $W_3 \backslash \{0\}$, and it is also continuous at the origin if $V(x)$ has the ISS-CLF-SCP. □

Proof 2. When $x \in \{x | c_- < V(x) < c_+,\ L_{g_2}V \neq 0\}$,

$$u_i = -\left\{P_2 + \frac{c(1-P_2)\|L_{g_2}V\|_q}{2 + c\|L_{g_2}V\|_q}\right\} \frac{|L_{g_{2i}}V|^{\frac{1}{k_2-1}}}{\|L_{g_2}V\|^{\frac{1}{k_2-1}}_{\frac{k_2}{k_2-1}}} \cdot \operatorname{sgn}(L_{g_{2i}}V). \tag{28}$$

Note that $\inf_{d \in U^1_{k_1}} \dot{V}(x) < 0 \quad \forall x \in \{x | c_- < V(x) < c_+,\ L_{g_2}V = 0\}$, and we can obtain the following inequality:

$$\sup_{d \in U^1_{k_1}} \dot{V}(x) == \frac{c(P_2 - 1)\|L_{g_2}V\|_q \|L_{g_2}V\|_{\frac{k_2}{k_2-1}}}{2 + c\|L_{g_2}V\|_q} < 0 \tag{29}$$

$$\forall x \in \{x | c_- < V(x) < c_+\}. \tag{30}$$

This shows the origin of the system is locally uniformly Lagrange stable with Filippov solution.

By the same discussion, we can obtain $\dot{V}(x) < 0$ for all $x \in W_3 \backslash \{0\}$ when $\|d\|_{k_1} < \rho^\dagger(V(x))$ by using (27).

Note that $\|d\|_{k_1} \to 0$ as $t \to \infty$ if $d \in L_2$ is uniformly continuous. Therefore, input (27) asymptotically stabilizes the origin if $d \in L_2$ is uniformly continuous.

The continuity of (27) except at the origin is established if the following are proved:

(1) (27) is continuous on $\{x \in W_3 |\ L_{g_2}V \neq 0\}$.

(2) $\lim_{L_{g_2}V \to 0} \dfrac{P_3 + |P_3| + c\|L_g V\|_q}{2 + c\|L_g V\|_q} = 0$ except at the origin.

If $V(x)$ is an ISS-RCLF, it is obvious that the first condition is satisfied. Note that the following is true in W_3:

$$L_{g_2}V = 0 \Rightarrow L_f V + \rho^\dagger(x)\|L_{g_1}V\|_{\frac{k_1}{k_1-1}} < 0 \tag{31}$$

Then, $L_f V + \rho^\dagger(x)\|L_{g_1}V\|_{\frac{k_1}{k_1-1}} < 0$ in a small neighborhood of $x \in \{x \in W_1 |\ L_{g_2}V = 0,\ x \neq 0\}$. If $L_f V + \rho^\dagger(x)\|L_{g_1}V\|_{\frac{k_1}{k_1-1}} < 0$ and $L_g V \neq 0$,

$$\frac{P_3 + |P_3| + c\|L_{g_2}V\|_q}{2 + c\|L_{g_2}V\|_q} = \frac{c\|L_{g_2}V\|_q}{2 + c\|L_{g_2}V\|_q}.$$

Hence, the second condition is satisfied in W_3. Therefore, (27) is continuous on $W_3 \backslash \{0\}$.

If $L_{g_2}V = 0$, input constraint $u \in U^1_{k_2}$ is clearly satisfied. According to the assumption of $P_3 < 1$ in W_3, if $L_g V \neq 0$,

$$\|u\|_{k_2} = \frac{P_3 + |P_3| + c\|L_{g_2}V\|_q}{2 + c\|L_{g_2}V\|_q} < 1. \tag{32}$$

Therefore, input constraint $u \in U^1_{k_2}$ is satisfied.

If $V(x)$ has the ISS-CLF-SCP, there exists a $\delta < 1$ such that $\|L_{g_2}V\|_q < \delta$ and $L_f V + \rho^\dagger(V(x))\|L_{g_1}V\|_{\frac{k_1}{k_1-1}} < \delta\|L_{g_2}V\|_{\frac{k_2}{k_2-1}}$ in a small neighborhood of the origin, and $\delta \to 0$ as $x \to 0$. According to the direct calculation of (32) with the above conditions, we achieve $\lim_{\delta \to 0} \|u\|_{k_2} < \lim_{\delta \to 0}(2 + c)\delta = 0$. ■

Now, we can prove Theorem 2.

Proof 3. (Proof of Theorem 2). The controller (27) stabilizes the origin in the sense of local uniform Lagrange stability in W_3 and asymptotically stabilizes if $d \in L_2$ is uniformly continuous. ■

Thus we solved Problem 1. We proposed a continuous stabilizing controller in Theorem 3. Although the lack of smoothness of the controller does not become a practical problem in many cases, if $k_1 = k_2 = 2$, V is an ISS-RCLF and $\rho^\dagger$ is smooth, the following smooth controller based on Lin's controller [11] is also available:

$$u_i = \begin{cases} -\dfrac{P_3 + \sqrt{P_3^2 + \|L_{g_2}V\|_2^2}}{1 + \sqrt{1 + c\|L_{g_2}V\|_2^2}} \cdot \dfrac{L_{g_{2i}}V}{\|L_{g_2}V\|_2} & (L_{g_2}V \neq 0) \\ 0 & (L_{g_2}V = 0) \end{cases} \quad (i = 1, \ldots, m). \tag{33}$$

In Theorem 3, we can construct a continuous controller if V has the ISS-CLF-SCP. We can relax the assumption and obtain the following proposition:

Theorem 4. Consider system (13) with disturbance constraint $d \in U^1_{k_1}$ and input constraint $u \in U^1_{k_2}$. Let V be an ISS-RCLF and a local CLF having the SCP for the following nominal control system:

$$\dot{x} = f(x) + g_2(x)u. \tag{34}$$

Then, V has the ISS-CLF-SCP. □

Proof 4. By the SCP, there exists a δ such that $\|L_{g_2}V\| < \delta$ and

$$L_f V < \delta \|L_{g_2}V\|_{\frac{k_2}{k_2-1}} \tag{35}$$

in the neighborhood of the origin. This means there exists a strictly increasing function $\hat{\rho} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ satisfies $\rho(0) = 0$ such that

$$L_f V + \hat{\rho}(V(x))\|L_{g_1}V\|_{\frac{k_1}{k_1-1}} < \delta \|L_{g_2}V\|_{\frac{k_2}{k_2-1}}. \tag{36}$$

Therefore, V is an ISS-RCLF having the ISS-CLF-SCP.■

Remark 1. We can easily redesign function ρ. Note that a redesigned function $\hat{\rho}$ must satisfy the following equation:

$$\lim_{r \to 1} \hat{\rho} \leq c_+. \tag{37}$$

Otherwise, a disturbance may move the state to a set $\{x | V(x) \geq c_+\}$.

Remark 2. If there does not exist c_+, we cannot stabilize the system for all disturbance $d \in U^1_{k_1}$. However, if V is a local CLF, there must exist $\delta > 0$ such that we can stabilize the system with input $u \in U^1_{k_2}$ for all disturbance $d \in U^\delta_{k_1}$ by the discussion in this section. This means if we normalize the disturbance, we can apply controller (27).

Now, we can construct a controller that achieves local input-to-state stability. The following proposition for ISS gain [4] can be obtained straightforward:

Proposition 2. Consider system (13) with disturbance constraint $d \in U^1_{k_1}$ and input constraint $u \in U^1_{k_2}$. Assume $d \in L_2$ is uniformly continuous, and let V be an ISS-RCLF, W_3 be a domain defined in Theorem 2, $P_3(x)$ be a function defined by (24), $c > 0$ and $q \geq 1$ be constants. Assume V and $\|\cdot\|_{k_2}$ satisfies the following conditions:

$$\begin{aligned} \underline{\alpha}(\|x\|_2) \leq V(x) \leq \bar{\alpha}(\|x\|_2) \\ \underline{k} \cdot \|d\|_2 \leq \|d\|_{k_2} \leq \bar{k} \cdot \|d\|_2, \end{aligned} \tag{38}$$

where $\underline{\alpha}$ and $\bar{\alpha}$ are K_∞ functions, $\underline{k}$ and $\bar{k}$ are positive constants. Then, controller (27) guarantees the following local ISS gain γ of the closed-loop system in the neighborhood of the origin:

$$\gamma(s) = \underline{\alpha}^{-1} \circ \rho(\bar{k} \cdot s). \tag{39}$$

□

6. CONDITIONS FOR ISS-RCLF

We have already defined AS-RCLF and ISS-RCLF. There exists a natural question whether a smooth proper positive definite function V is an AS-RCLF. We discuss the topic in this section.

We can obtain the following lemmas by the same discussion as Proposition 1:

Lemma 2. We consider system (13) with input constraint $u \in \bar{U}^1_{k_2}$, and let $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ be a smooth function. Then, the following input minimizes the derivative $\dot{V}(x, u, d)$ for each x:

$$u_i = \begin{cases} -\dfrac{|L_{g_2 i}V|^{\frac{1}{k_2-1}}}{\|L_{g_2}V\|^{\frac{1}{k_2-1}}_{\frac{k_2}{k_2-1}}} & (L_{g_2}V \neq 0) \\ 0 & (L_{g_2}V = 0) \end{cases} \tag{40}$$

□

Lemma 3. We consider system (13) with disturbance constraint $d \in \bar{U}^1_{k_1}$, and let $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ be a smooth function. Then, the following disturbance maximizes the derivative $\dot{V}(x, u, d)$ for each x:

$$d_i = \begin{cases} \dfrac{|L_{g_1 i}V|^{\frac{1}{k_1-1}}}{\|L_{g_1}V\|^{\frac{1}{k_1-1}}_{\frac{k_1}{k_1-1}}} & (L_{g_1}V \neq 0) \\ 0 & (L_{g_1}V = 0) \end{cases}. \tag{41}$$

□

According to these lemmas, we can prove the following proposition.

Proposition 3. Consider system (13) with disturbance constraint $d \in U^1_{k_1}$ and input constraint $u \in U^1_{k_2}$. Let $V(x)$ be a proper function and $P_2(x)$ be a function defined by

$$P_2(x) = \frac{L_f V + \|L_{g_1}V\|_{\frac{k_1}{k_1-1}}}{\|L_{g_2}V\|_{\frac{k_2}{k_2-1}}}. \tag{42}$$

If there exists $c > 0$ such that $P_2(x) < 1$ for all $x \in \{x | V(x) = c\}$, V is an RCLF.

On the other hand, V is not an RCLF if for all $c > 0$, there exists $x \in \{x | V(x) = c\}$ such that $P_2(x) \geq 1$. □

Proof 5. By Lemma 2 and Lemma 3,

$$\begin{aligned} &\inf_{u \in U^1_{k_2}} \sup_{d \in U^1_{k_1}} \{L_f V + L_{g_1}V \cdot d + L_{g_2}V \cdot u\} \\ &= L_f V + \|L_{g_1}V\|_{\frac{k_1}{k_1-1}} - \|L_{g_2}V\|_{\frac{k_2}{k_2-1}}. \end{aligned} \tag{43}$$

According to the assumption that there exists $c > 0$ such that $P_2(x) < 1$ for all $x \in \{x | V(x) = c\}$, we have

$$\inf_{u \in U^1_{k_2}} \sup_{d \in U^1_{k_1}} \{L_f V + L_{g_1}V \cdot d + L_{g_2}V \cdot u\} < 0 \tag{44}$$

$$(\forall x \in \{x | V(x) = c\}). \tag{45}$$

Therefore, V is an RCLF for $X = \{x | V(x) \leq c\}$. Furthermore, V is a global RCLF if the supremum of c is ∞.

Otherwise, for all $c > 0$ there exist $x \in \{V(x) = c\}$ such that

$$\inf_{u \in U^1_{k_2}} \sup_{d \in U^1_{k_1}} \{L_f V + L_{g_1}V \cdot d + L_{g_2}V \cdot u\} \geq 0. \tag{46}$$

Thus V is not an RCLF. ■

Theorem 5. Consider system (13) with disturbance constraint $d \in U^1_{k_1}$ and input constraint $u \in U^1_{k_2}$. Let $V(x)$ be an RCLF. Assume $V(x)$ be a local CLF for the following nominal system:

$$\dot{x} = f(x) + g_2(x)u, \tag{47}$$

and $W_1 \cap W_2 \neq \phi$, where W_1 is defined in (7),

$$W_2 := \{x | V(x) \in C\}, \tag{48}$$

where

$$C := \{c | P_2 < 1 \ \forall x \in \{x | V(x) = c\}\}. \tag{49}$$

Then, $V(x)$ is an AS-RCLF, $c_- = \inf_{x \in W_1 \cap W_2} V(x)$ and c_+ is the supremum of $V(x)$ such that

$$P_2(x) < 1, \ \forall x \in \{x | c_- < V(x) < c_+\}. \tag{50}$$

□

Proof 6. First, we prove that Condition 1) in the definition of AS-RCLF is satisfied. According to the assumption that $V(x)$ is an RCLF and $W_1 \cap W_3 \neq \phi$,

$$\inf_{u \in U^1_{k_2}} \sup_{d \in U^1_{k_1}} \{L_f V + L_{g_1} V \cdot d + L_{g_2} V \cdot u\} < 0 \tag{51}$$

$\forall x \in W_2$. This implies the existence of c_- and c_+.

Then, we show condition 2). If $\|d\| = 0$,

$$\inf_{u \in U^2_{k_2}} \{L_f V + L_{g_1} V d + L_{g_2} V u\} < 0 \tag{52}$$

by the assumption. This means condition 2) is satisfied for $d = 0$. By the continuity of $L_{g_1} V$ and existence of c_-, there exists a function $\rho : [0, 1) \to [0, 1)$. ■

Remark 3. Note that V is not an AS-RCLF if $W_1 \cap W_2 = \phi$. This means the asymptotically stabilizable set is quite different from the uniformly Lagrange stabilizable set. Note that there exists a case in which the state cannot return to the origin when the disturbance is added.

The function ρ can be easily calculated. I will show the example in section VII. Note that we can obtain an ISS-RCLF by redesigning the obtained AS-RCLF as the following proposition:

Proposition 4. Consider system (13) with disturbance constraint $d \in U^1_{k_1}$ and input constraint $u \in U^1_{k_2}$. Let $V(x)$ be an AS-RCLF. Then, V is an ISS-RCLF with respect to function $\tilde{\rho}$ such that

$$\begin{aligned} \tilde{\rho}(0) &= 0 \\ \lim_{r \to 1} \tilde{\rho}(r) &\leq c_+ \\ \tilde{\rho}(r) &\geq \rho(r) \quad \forall r \in [0, 1). \end{aligned} \tag{53}$$

□

7. EXAMPLE

We proposed a controller in the previous section. In this section, we confirm the effectiveness of the proposed method by an example. We consider the following simple nonlinear system:

$$\dot{x} = x^3 - \frac{1}{2}x + \frac{1}{2}d + u, \tag{54}$$

where $x \in \mathbb{R}$, $d \in \mathbb{R}$ and $u \in \mathbb{R}$. Consider the following function:

$$V(x) = \frac{1}{2}x^2. \tag{55}$$

First, we check the existence of set W without the disturbance. In this example,

$$P = \frac{1}{2}|x|(2x^2 - 1), \tag{56}$$

and we obtain

$$W = \left\{ x \,\middle|\, V(x) < \frac{1}{72}\left(\left(108 - 6\sqrt{318}\right)^{\frac{1}{3}} + \left(6\left(18 + \sqrt{318}\right)\right)^{\frac{1}{3}}\right)^2 \right\}. \tag{57}$$

Then we calculate a set W_3. Note that

$$P_2 = \frac{1}{2}(|x|(2x^2 - 1) + 1), \tag{58}$$

and we obtain $c_- = 0$ and $c_+ = 1/2$. Then, $\rho(r) \equiv 0 \forall r < 1$. Now, we can construct the following controller for the system by using (27):

$$u_1 = -\frac{P_2 + |P_2| + |x|}{2 + |x|} \operatorname{sgn} x. \tag{59}$$

However, the controller is not continuous at the origin. Fig. 7 is a simulation result. The state converges to the origin, however, we can observe chattering in the input.

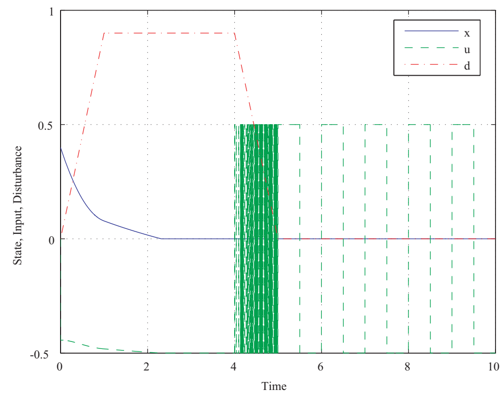

Fig. 1. Simulation Result: Discontinuous Control

Fortunately, V has an SCP without disturbances. Hence, we can construct a continuous controller via redesigning function ρ. We choose function ρ as the following:

$$\tilde{\rho}(d) = \frac{1}{2}d. \tag{60}$$

Note that $\tilde{\rho}$ satisfies $\tilde{\rho} \geq \rho$. In this case, we obtain

$$\tilde{\rho}^{\dagger}(V(x)) = \tilde{\rho}^{-1}(V(x)) = 2V(x) \tag{61}$$

and we can calculate P_3 as

$$P_3 = \frac{1}{2}\left(|x|(2x^2 - 1) + x^2\right). \tag{62}$$

Then, we obtain the following continuous controller by (27):

$$u_2 = -\frac{P_3 + |P_3| + |x|}{2 + |x|} \operatorname{sgn} x. \tag{63}$$

We show a simulation result in Fig. 7. The state converges to the origin, and the input also smoothly converges to 0.

Figure 7 illustrates a simulation result using Kidane's original controller [14]. This shows that a controller that does not mention disturbances cannot stabilize the origin.

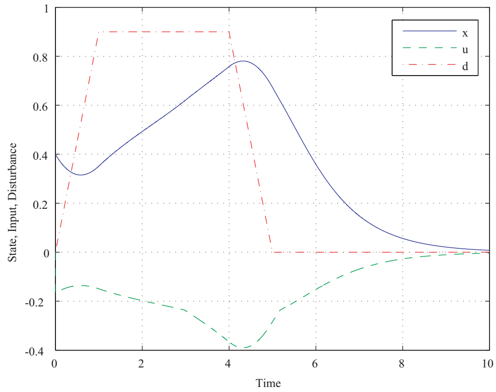

Fig. 2. Simulation Result: Continuous Disturbance Attenuation Control

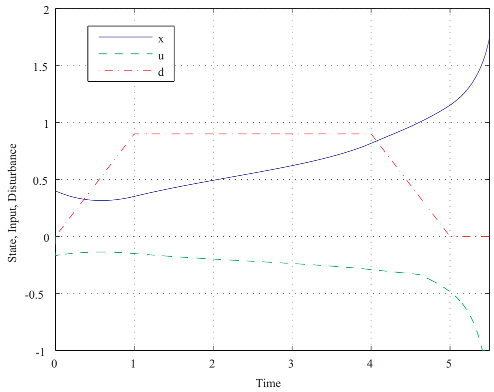

Fig. 3. Simulation Result: Control without Considering Disterbance

8. CONCLUSION

In this paper, we proposed an asymptotically stabilizable robust control funtion (AS-RCLF) for input constrained nonlinear systems.

We addressed some conditions for an AS-RCLF and when a proper function becomes an AS-RCLF. Moreover, we proposed a stabilizing controller for input and disturbance constrained nonlinear systems using AS-RCLF, which is continuous when an ISS-RCLF has an ISS-CLF small control property. Finally, we confirmed the effectiveness of the proposed method by computer simulation.

The inverse optimal control is very important in disturbance attenuation [8]. However, we could not mention “inverse optimality” in this paper, though the inverse optimal controller for input constrained systems has already been proposed [16]. The reason is there still exist many problems in applying Krstić's or Freeman's method to input constrained systems, and the challenge remains for future works.

REFERENCES

[1] R. A. Freeman and P. V. Kokotović, *Robust nonlinear control design*, Birkhäuser, Boston, 1996.

[2] M. Krstić and H. Deng, *Stabilization of nonlinear uncertain systems*, Springer, London, 1998.

[3] E. D. Sontag, *Mathematical Control Theory, Second Edition*, Springer, New York, 1998.

[4] A. Isidori, *Nonlinear Control Systems II*, Springer, London, 1999.

[5] A. Bacciotti and L. Rosier, *Liapunov Functions and Stability in Control Theory*, Springer, Berlin, 2005.

[6] M. M. Samir, *Nonlinear H2/H-infinity constrained feedback control: A practical design approach using neural networks*, ProQuest, 2005.

[7] M. Krstić and P. V. Kokotović, Modular Approach to Adaptive Nonlinear Stabilization, *Automatica*, vol. 32, no. 4, 1995, pp. 625–629.

[8] M. Kristić and Z.-H. Li, Inverse Optimal Design of Input-to-State Stabilizing Nonlinear Controllers, *IEEE Transactions on Automatic Control*, vol. 43, no. 3, 1998, pp. 336–350.

[9] E. D. Sontag and Y. Wang, On characterizations of input-to-state stability with respect to compact sets, *Proceedings of IFAC Nonlinear Control Systems Design Symposium (NOLCOS '95)*, 1995, pp. 226–231.

[10] M. Malisoff and E. D. Sontag, Universal formulas for feedback stabilization with respect to Minkowski balls, *Systems and Control Letters*, vol. 40, 2000, pp. 247–260.

[11] Y. Lin and E. D. Sontag, Control-Lyapunov universal formulas for rectricted inputs, *Control-Theory and Advanced Technology*, vol. 10, 1995, pp. 1981–2004.

[12] Y. Lin and E. D. Sontag, A universal formula for stabilization with bounded controls, *Systems and Control Letters*, vol. 13, 1991, pp. 393–397.

[13] D. Liberzon, E. D. Sontag and Y. Wang, Universal construction of feedback laws achieving ISS and integral-ISS disturbance attenuation, *Systems and Control Letters*, vol. 46, 2002, pp. 111–127.

[14] N. Kidane, H. Nakamura, Y. Yamashita and H. Nishitani, Controller for a nonlinear system with an input constraint by using a control Lyapunov function I, *16th IFAC World Congress*, 2005, Th-E03-TO/4.

[15] N. Kidane, H. Nakamura, Y. Yamashita and H. Nishitani, Controller for a nonlinear system with an input constraint by using a control Lyapunov function II, *16th IFAC World Congress*, 2005, Th-E03-TO/5.

[16] N. Nakamura, H. Nakamura, Y. Yamashita and H. Nishitani, Inverse Optimal Control for Nonlinear Systems with Input Constraints, *European Control Conference 2007 (ECC '07)*, 2007, to appear.

[17] H. Ito, State-dependent scaling problems and stability of interconnected iISS and ISS systems, *IEEE Transactions on Automatic Control*, vol. 51, no. 10, 2006, pp. 1626–1643.